GIBBS MEASURES FOR THE NON LINEAR HARMONIC OSCILLATOR

by

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Abstract. — We present some results of [**4**] concerning the nonlinear Schrödinger equation with harmonic potential. First we show how to construct a Gibbs measure for the nonlinear problem. Then we give some estimates which can be useful to show that the equation is almost surely well-posed on the support of the measure.

1. Introduction

In this paper we consider the following defocusing equation

$$\begin{cases} i\partial_t u + \partial_x^2 u - x^2 u = |u|^{k-1}u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}, \\ u(0,x) = f(x), \end{cases} \tag{1.1}$$

where $k \geq 3$ is an odd integer.

The equation (1.1) has been intensively studied since it is a model to describe the Bose-Einstein condensates. See e.g. R. Fukuizumi [**10**], K. Yajima - G. Zhang [**20**], R. Carles [**8**], for results related to this problem.

In this note, we present some of the results we obtained in [**4**]. In our work, we first construct a Gibbs measure ρ associated to the Hamiltonian equation (1.1). Then we show that there is a large set Σ of rough initial conditions leading to global solutions. Finally we prove that the measure ρ is invariant under the flow of (1.1) (which is well defined on Σ).

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The construction of the measure is quite straightforward, but the main difficulty is to prove a local existence theorem for (1.1), as this problem is $L^2(\mathbb{R})$ supercritical : we have to gain $1/2$ derivative (for k large), and to find a space which is stable by the almost surely well-defined flow, so that we can prove global existence.
Here we show in particular that the square of the free Schrödinger solution with initial condition in Σ is (almost surely) more regular than the solution itself. This can give an idea why a result on the nonlinear Schrödinger equation can be true.

In the following, H will stand for the operator $H = -\partial_x^2 + x^2$. The operator H has a self-adjoint extension on $L^2(\mathbb{R})$ (still denoted by H) and has eigenfunctions $\big(h_n\big)_{n\geq 1}$ which form an Hilbertian basis of $L^2(\mathbb{R})$ and satisfy $Hh_n = \lambda_n^2 h_n$, $n \geq 0$, with $\lambda_n = \sqrt{2n+1} \longrightarrow +\infty$, when $n \longrightarrow +\infty$.
For $1 \leq p \leq +\infty$ and $s \in \mathbb{R}$, we define the space $\mathcal{W}^{s,p}(\mathbb{R})$ via the norm

$$\|u\|_{\mathcal{W}^{s,p}(\mathbb{R})} = \|H^{s/2}u\|_{L^p(\mathbb{R})}\,.$$

In the case $p=2$ we write $\mathcal{W}^{s,2}(\mathbb{R}) = \mathcal{H}^s(\mathbb{R})$ and if

$$u = \sum_{n=0}^{\infty} c_n h_n \quad \text{we have} \quad \|u\|^2_{\mathcal{H}^s} = \sum_{n=0}^{\infty} \lambda_n^{2s} |c_n|^2\,.$$

Let $(\Omega, \mathcal{F}, \mathbf{p})$ be a probability space and $\big(g_n(\omega)\big)_{n\geq 0}$ a sequence of independent complex normalised gaussians, $g_n \in \mathcal{N}_{\mathbb{C}}(0,1)$.

1.1. Hamiltonian formulation. —

The equations (1.1) has the following Hamiltonian

$$J(u) = \frac{1}{2}\int_{-\infty}^{\infty} |H^{1/2}u(x)|^2\,\mathrm{d}x + \frac{1}{k+1}\int_{-\infty}^{\infty} |u(x)|^{k+1}\,\mathrm{d}x.$$

Write $u = \sum_{n=0}^{\infty} c_n h_n$, then in the coordinates $c = (c_n)$ the Hamiltonian reads

$$J(c) = \frac{1}{2}\sum_{n=0}^{\infty} \lambda_n^2 |c_n|^2 + \frac{1}{k+1}\int_{-\infty}^{\infty} \Big|\sum_{n=0}^{\infty} c_n h_n(x)\Big|^{k+1}\,\mathrm{d}x.$$

Let us define the complex vector space E_N by $E_N = \mathrm{span}(h_0, h_1, \cdots, h_N)$. Then we introduce the spectral projector Π_N on E_N by

$$\Pi_N\Big(\sum_{n=0}^{\infty} c_n h_n\Big) = \sum_{n=0}^{N} c_n h_n\,.$$

Let $\chi_0 \in \mathcal{C}_0^\infty(-1,1)$, $0 \le \chi \le 1$ so that $\chi_0 = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$ and let S_N be the operator

$$S_N\Big(\sum_{n=0}^{\infty} c_n h_n\Big) = \sum_{n=0}^{\infty} \chi_0(\frac{n}{N}) c_n h_n\,.$$

It is clear that $\|S_N\|_{L^2\to L^2} \le \|\Pi_N\|_{L^2\to L^2}$ and we have

$$S_N\,\Pi_N = \Pi_N\,S_N = S_N, \quad \text{and} \quad S_N^* = S_N.$$

In fact, S_N is a smooth version of Π_N, and this operator is needed for technical reasons. In particular we use that for $1 \le p \le +\infty$, $S_N : L^p \longrightarrow L^p$ is continuous.

1.2. Definition of the Gibbs measure. —

Now write $c_n = a_n + ib_n$. For $N \ge 1$, consider the probability measure on $\mathbb{R}^{2(N+1)}$ defined by

$$\mathrm{d}\mu_N = d_N \prod_{n=0}^{N} e^{-\frac{\lambda_n^2}{2}(a_n^2+b_n^2)} \mathrm{d}a_n \mathrm{d}b_n,$$

where d_N is such that

$$\frac{1}{d_N} = \prod_{n=0}^{N} \int_{\mathbb{R}^2} e^{-\frac{\lambda_n^2}{2}(a_n^2+b_n^2)} \mathrm{d}a_n \mathrm{d}b_n = (2\pi)^{N+1} \prod_{n=0}^{N} \frac{1}{\lambda_n^2} = (2\pi)^{N+1} \prod_{n=0}^{N} \frac{1}{2n+1}\,.$$

The measure μ_N defines a measure on E_N via the map

$$(a_n, b_n)_{n=0}^N \longmapsto \sum_{n=0}^{N} (a_n + ib_n) h_n,$$

which will still be denoted by μ_N. Then μ_N may be seen as the distribution of the E_N valued random variable

$$\omega \longmapsto \sum_{n=0}^{N} \frac{\sqrt{2}}{\lambda_n} g_n(\omega) h_n(x) \equiv \varphi_N(\omega, x),$$

where $(g_n)_{n=0}^N$ is a system of independent, centered, L^2 normalised complex gaussians.

Let $\sigma > 0$. Then (φ_N) is a Cauchy sequence in $L^2(\Omega; \mathcal{H}^{-\sigma}(\mathbb{R}))$ which defines

$$\varphi(\omega, x) = \sum_{n=0}^{\infty} \frac{\sqrt{2}}{\lambda_n} g_n(\omega) h_n(x), \tag{1.2}$$

as the limit of (φ_N). Indeed, the map $\omega \longmapsto \displaystyle\sum_{n=0}^{\infty} \frac{\sqrt{2}}{\lambda_n} g_n(\omega) h_n(x)$ defines a (Gaussian) measure on $\mathcal{H}^{-\sigma}(\mathbb{R})$ which will be denoted by μ.

Now, we define the following Gibbs measure on E_N

$$\mathrm{d}\tilde{\rho}_N(u) = \exp\Big(-\frac{1}{k+1}\|S_N u\|^{k+1}_{L^{k+1}(\mathbb{R})}\Big)\mathrm{d}\mu_N(u).$$

If A is a Borel set of $\mathcal{H}^{-\sigma}(\mathbb{R})$ then $A \cap E_N$ is a Borel set of E_N. Now, in each of the two previous cases, we define ρ_N which is the natural extension of $\tilde{\rho}_N$ to $\mathcal{H}^{-\sigma}(\mathbb{R})$, equipped with the Borel sigma algebra $\mathcal{B}$. More precisely for every $A \in \mathcal{B}$ which is a Borel set of $\mathcal{H}^{-\sigma}(\mathbb{R})$, we set

$$\rho_N(A) \equiv \tilde{\rho}_N(A \cap E_N)\,. \tag{1.3}$$

1.3. Statement of the main results. —

We have the following statement defining the Gibbs measure associated to the equation (1.1).

Theorem 1.1. —
Let $k \geq 3$. We define the Gibbs measure by

$$d\rho(u) = \exp\Big(-\frac{1}{k+1}\|u\|^{k+1}_{L^{k+1}(\mathbb{R})}\Big)\,d\mu(u),$$

and the measure is nontrivial. Moreover the sequence ρ_N converges weakly to ρ as N tends to infinity.

By (1.2), the measure ρ is supported in $\displaystyle\bigcap_{\sigma>0} \mathcal{H}^{-\sigma}(\mathbb{R})$.

There is a large literature on the construction of Gibbs measures for dispersive equations. See e.g. J. Bourgain [**2, 3**], P. Zhidkov [**22**] N. Tzvetkov [**19, 18, 17**], N. Burq-N. Tzvetkov [**5, 7**], T. Oh [**12, 13**], and references therein.

Remark 1.2. — In [**4**], we also construct a Gibbs measure for the cubic focusing equation

$$i\partial_t u + \partial_x^2 u - |x|^2 u = -|u|^2 u,$$

but in this case, the task is much harder, as the weight $\exp(\frac{1}{4}\|u\|^4_{L^4})$ doesn't belong to $L^1(\mathrm{d}\mu)$.

The free Schrödinger group of (1.1) enjoys Strichartz estimates. Therefore, the deterministic problem (1.1) is well-posed in $\mathcal{H}^s(\mathbb{R})$ for $s \geq \max(0, \frac{1}{2} - \frac{2}{k-1})$ (see [**11**]). Let $k \geq 5$. For $s < \frac{1}{2} - \frac{2}{k-1}$, the problem is $\mathcal{H}^s(\mathbb{R})$-supercritical, and by [**9, 1, 15**], it is ill-posed : there is a loss of regularity in the Sobolev scale ; in particular the equation can not be solved with a usual fixed point argument.
However, as the support of ρ lies in $\bigcap\limits_{\sigma>0} \mathcal{H}^{-\sigma}(\mathbb{R})$, we have to the solve (1.1) for rough initial conditions, and this will be done with stochastic methods. Indeed we can combine Theorem 1.1 with a local existence theory for (1.1) to obtain a global existence result for any $k \geq 3$. Then we are able to show that ρ is invariant.

Theorem 1.3. — *Let $k \geq 3$ be an odd integer. The Gibbs measure ρ is invariant under the ρ a.s. well-defined global in time flow $\Phi(t)$ of (1.1). More precisely :*
(i) There exists a set $\Sigma \subset \bigcap_{\sigma>0} \mathcal{H}^{-\sigma}(\mathbb{R})$ of full ρ measure and $s < \frac{1}{2}$ so that for every $f \in \Sigma$ the equation (1.1) with initial condition $u(0) = f$ has a unique global solution in the class

$$u(t, \cdot) = e^{-itH} f + \mathcal{C}\big(\mathbb{R}; \mathcal{H}^s(\mathbb{R})\big) \bigcap L^4_{loc}\big(\mathbb{R}; \mathcal{W}^{s,\infty}(\mathbb{R})\big).$$

Moreover, for all $\sigma > 0$ and $t \in \mathbb{R}$

$$\|u(t, \cdot)\|_{\mathcal{H}^{-\sigma}(\mathbb{R})} \leq C\big(\Lambda(f, \sigma) + \ln\big(2 + |t|\big)^{\frac{1}{2}}\big),$$

and the constant $\Lambda(f, \sigma)$ satisfies the probabilistic bound

$$\mathbf{p}\big(\omega \in \Omega \ : \ \Lambda(f, \sigma) > \lambda\big) \leq C e^{-c\lambda^2}.$$

(ii) For any ρ measurable set $A \subset \Sigma$, for any $t \in \mathbb{R}$, $\rho(A) = \rho\big(\Phi(t)(A)\big)$.

Remark 1.4. — By Yajima-Zhang [**20**], the equation (1.1) is locally well-posed in $L^2(\mathbb{R})$ when $k = 3, 5$. Then, by the conservation of the L^2 norm, we infer that the solutions are global in time.

The starting point of this work was the following observation : Let φ as in (1.2), then for all $t \in \mathbb{R}$ and $\theta < \frac{1}{2}$

$$\big(\mathrm{e}^{-itH}\varphi\big)^2 \in \mathcal{H}^\theta(\mathbb{R}), \quad a.s. \tag{1.4}$$

Recall that $\varphi \in L^2(\Omega; \mathcal{H}^{-\sigma}(\mathbb{R}))$ and for almost all $\omega \in \Omega$, $\varphi(\omega, \cdot) \notin L^2(\mathbb{R})$. Hence there is a gain of $1/2$ derivative for the square of φ (see Propositions 3.3 and 3.2 for quantitative results). This property is a consequence of good bilinear estimates on the Hermite functions proved by P.Gérard (see [**4**] for the proof).

However in [**4**], we found a new proof of Theorem 1.3, which only relies on the usual linear smoothing effect and a stochastic improvement of it.

Notations. — *In this paper c, C denote constants the value of which may change from line to line. These constants will always be universal, or uniformly bounded with respect to the other parameters.*
We denote by $\mathbb{N}$ the set of the non negative integers, and $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$.
The notation L^p_T stands for $L^p(-T,T)$, whereas $L^q = L^q(\mathbb{R})$, $L^p_T L^q = L^p(-T,T;L^q(\mathbb{R}))$, and $\mathcal{H}^s = \mathcal{H}^s(\mathbb{R})$.
In all the paper $\lambda_n = \sqrt{2n+1}$, so that λ_n^2 is the $(n+1)$th eigenvalue of the operator H. We use this notation to avoid square roots.

2. Proof of Theorem 1.1

First we recall the following Gaussian bound, which is one of the key points in the study of our random series. See e.g. [**6**, Lemma 4.2.] for a proof.

Lemma 2.1. — *Let $\big(g_n(\omega)\big)_{n\geq 0} \in \mathcal{N}_{\mathbb{C}}(0,1)$ be independent, complex, L^2-normalised gaussians. Then there exists $C>0$ such that for all $r\geq 2$ and $(c_n)\in l^2(\mathbb{N})$*

$$\|\sum_{n\geq 0} g_n(\omega)\, c_n\|_{L^r(\Omega)} \leq C\sqrt{r}\Big(\sum_{n\geq 0}|c_n|^2\Big)^{\frac{1}{2}}.$$

We will need the following particular case of the bounds on the eigenfunctions (h_n), proved by K. Yajima and G. Zhang [**21**]. For every $p\geq 4$ there exists $C(p)$ such that for every $n\geq 0$,

$$\|h_n\|_{L^p(\mathbb{R})} \leq C(p)\lambda_n^{-\frac{1}{6}}\,.$$

Thanks to these two ingredients, following [**6**], we can prove

Lemma 2.2. — *Fix $p\in[4,\infty)$ and $s\in[0,1/6)$. Then*

$$\exists\, C>0, \exists\, c>0,\ \forall \lambda\geq 1, \forall N\geq 1,\qquad \mu\big(\,u\in\mathcal{H}^{-\sigma}\,:\,\|S_N u\|_{\mathcal{W}^{s,p}(\mathbb{R})}>\lambda\,\big)\leq Ce^{-c\lambda^2}\,. \tag{2.1}$$

Moreover there exists $\beta(s)>0$ such that

$$\exists\, C>0, \exists\, c>0,\ \forall \lambda\geq 1, \forall N\geq N_0\geq 1,\qquad \mu\big(\,u\in\mathcal{H}^{-\sigma}\,:\,\|S_N u - S_{N_0}u\|_{\mathcal{W}^{s,p}(\mathbb{R})}>\lambda\,\big)\leq Ce^{-cN_0^{\beta(s)}\lambda^2}\,. \tag{2.2}$$

The assertion (2.1) shows in particular that $\|u\|_{L^4(\mathbb{R})}$ is μ almost surely finite. Therefore, the measure ρ defined in Theorem 1.1 is nontrivial.
By (2.2), $\|S_N u\|_{L^{k+1}(\mathbb{R})}$ converges to $\|u\|_{L^{k+1}(\mathbb{R})}$ with respect to the measure μ. Thus when $n \longrightarrow +\infty$,

$$\exp\Big(-\frac{1}{k+1}\|S_N u\|^{k+1}_{L^{k+1}(\mathbb{R})}\Big) \longrightarrow \exp\Big(-\frac{1}{k+1}\|u\|^{k+1}_{L^{k+1}(\mathbb{R})}\Big),$$

with respect to the measure μ. It is then easy to prove the weak convergence of $\mathrm{d}\rho_N$ to $\mathrm{d}\rho$.

3. The gain of $\frac{1}{2}$ derivative

In order to prove Theorem 1.3, we first have to develop a local well-posedness theory for (1.1) for data f in the support of the measure ρ. However, as we mentioned in the introduction, this problem is supercritical : we have to gain $\frac{1}{2} - \frac{2}{k-1}$ derivative, i.e. almost $\frac{1}{2}$ when k is large. In (2.1) we can see that we already have gained $\frac{1}{6}$ derivative in a probabilistic sense (indeed we even gain $\frac{1}{4}$ derivative with this method, using more precise bounds on Hermite functions, see e.g. [**14**]).
We present here the two tools needed to almost reach $\frac{1}{2}$ derivative.

3.1. Linear smoothing effect. —

We have the following statement

Lemma 3.1 **(Stochastic smoothing effect)**. — *Let $0 < s < \sigma < \frac{1}{2}$ and $q \geq 2$. Then there exist $C, c > 0$ so that for all $\lambda > 0$, $N \geq 1$ and $0 \leq T \leq 2\pi$*

$$\rho_N\big(\, u \in \mathcal{H}^{-\sigma} \;:\; \Big\| \frac{1}{\langle x\rangle^{\sigma}}\, H^{\frac{s}{2}}\, e^{-itH} u \Big\|_{L^q_T L^2(\mathbb{R})} > \lambda\,\big) \leq C e^{-c\lambda^2}.$$

This result is an improvement of the well-known deterministic smoothing effect, as we can take $q \geq 2$ as large as we want. Notice also that we have slightly modified the indexes, so that the weight has a rate (strictly) less than $\frac{1}{2}$. The proof is quite simple, using Lemma 2.1 and local estimates of the Hermite functions.

3.2. Bilinear smoothing effect. —

We now state a bilinear estimate on Hermite functions. There exists $C > 0$ so that for all $0 \leq \theta \leq 1$ and $n, m \in \mathbb{N}$

$$\|h_n\, h_m\|_{\mathcal{H}^{\theta}(\mathbb{R})} \leq C \max(n,m)^{-\frac{1}{4}+\frac{\theta}{2}} \Big(\log\big(\min(n,m)+1\big)\Big)^{\frac{1}{2}}. \tag{3.1}$$

The estimate for $\theta = 0$ is proved by P. Gérard. The case $\theta = 1$ is then obtained thanks to the recurrence formula of the Hermite functions, and the general case follows by interpolation. See [**4**] for the proof.

With (3.1) we can then prove the following large deviation estimate.

Proposition 3.2. — *Let $0 \leq \theta < \frac{1}{2}$, $0 \leq T \leq 2\pi$ and $q \geq 2$. Then there exist $c, C > 0$ so that for all $\lambda > 0$ and $0 \leq T \leq 2\pi$*

$$\rho_N\big(\, u \in \mathcal{H}^{-\sigma} \;:\; \big\|\big(e^{-itH}u\big)^2\big\|_{L^q_T\mathcal{H}^\theta(\mathbb{R})} > \lambda\,\big) \leq Ce^{-c\lambda},$$

and

$$\rho_N\big(\, u \in \mathcal{H}^{-\sigma} \;:\; \big\|\big|e^{-itH}u\big|^2\big\|_{L^q_T\mathcal{H}^\theta(\mathbb{R})} > \lambda\,\big) \leq Ce^{-c\lambda}.$$

Recall the notation (1.2) of $\varphi(\cdot, \omega)$. We prove the following result, and Proposition 3.2 will follow by the Bienaymé-Tchebychev inequality.

Proposition 3.3. — *Let $\theta < \frac{1}{2}$, then there exists $C > 0$ so that for all $2 \leq q \leq r$ and $0 \leq T \leq 2\pi$*

$$\big\|\big(\, e^{-itH}\varphi\,\big)^2\,\big\|_{L^r(\Omega)L^q_T\mathcal{H}^\theta(\mathbb{R})} \leq C\, r\, T^{\frac{1}{q}}, \tag{3.2}$$

and

$$\big\|\big|\, e^{-itH}\varphi\,\big|^2\,\big\|_{L^r(\Omega)L^q_T\mathcal{H}^\theta(\mathbb{R})} \leq C\, r\, T^{\frac{1}{q}}. \tag{3.3}$$

Proof. — We only prove (3.2), the proof of (3.3) is similar. To avoid too many subscripts, in the proof we will write $\mathrm{e}^{-itH}\varphi = u$. We make the decomposition $u = \displaystyle\sum_{N\geq 0} u_N$ with $u_N(t,x) = \displaystyle\sum_{2^N-1\leq n\leq 2(2^N-1)} \alpha_n(t)h_n(x)g_n(\omega)$, where $\alpha_n(t) = \dfrac{\sqrt{2}}{\sqrt{2n+1}}\mathrm{e}^{-i(2n+1)t}$.

Denote by $\Lambda = H^{\frac{1}{2}}$. Let $0 \leq \theta < \frac{1}{2}$. Then by Cauchy-Schwarz, for all $\varepsilon > 0$

$$\begin{aligned}
\big|\Lambda^\theta\big(u^2\big)\big| &= \big|\sum_{N,M\geq 0}\Lambda^\theta\big(u_N\,u_M\big)\big| \\
&= \big|\sum_{N,M\geq 0}\max\big(2^N,2^M\big)^{-\varepsilon}\max\big(2^N,2^M\big)^{\varepsilon}\Lambda^\theta\big(u_N\,u_M\big)\big| \\
&\leq C\Big(\sum_{N,M\geq 0}\max\big(2^N,2^M\big)^{2\varepsilon}|\Lambda^\theta\big(u_N\,u_M\big)|^2\Big)^{\frac{1}{2}}.
\end{aligned} \tag{3.4}$$

By the definition of u_N, we have

$$\Lambda^\theta\big(u_N\,u_M\big) = \sum_{\substack{2^N-1\leq n\leq 2(2^N-1)\\ 2^M-1\leq m\leq 2(2^M-1)}} \alpha_n\,\alpha_m\,g_n\,g_m\,\Lambda^\theta\big(h_n\,h_m\big),$$

therefore by the second order Wiener chaos estimates (see e.g. [**17, 16**]), there exists $C>0$ such that for all $r\geq 2$

$$\|\Lambda^\theta\big(u_N\, u_M\big)\|_{L^r(\Omega)}\leq Cr\,\|\Lambda^\theta\big(u_N\, u_M\big)\|_{L^2(\Omega)}. \tag{3.5}$$

Then by (3.4), Minkowski and (3.5)

$$\begin{aligned}\|\Lambda^\theta\big(u^2\big)\|_{L^r(\Omega)} &\leq C\Big(\sum_{N,M\geq 0}\max\big(2^N,2^M\big)^{2\varepsilon}\|\Lambda^\theta\big(u_N\, u_M\big)\|^2_{L^r(\Omega)}\Big)^{\frac12}\\ &\leq Cr\Big(\sum_{N,M\geq 0}\max\big(2^N,2^M\big)^{2\varepsilon}\|\Lambda^\theta\big(u_N\, u_M\big)\|^2_{L^2(\Omega)}\Big)^{\frac12}. \end{aligned}\tag{3.6}$$

We now estimate $\|\Lambda^\theta\big(u_N\, u_M\big)\|_{L^2(\Omega)}$. We make the expansion

$$|\Lambda^\theta\big(u_N\, u_M\big)|^2= \sum_{\substack{2^N-1\leq n_1,n_2\leq 2(2^N-1)\\ 2^M-1\leq m_1,m_2\leq 2(2^M-1)}}\alpha_{n_1}\,\overline{\alpha}_{n_2}\,\alpha_{m_1}\,\overline{\alpha_{m_2}}\,g_{n_1}\,\overline{g_{n_2}}\,g_{m_1}\,\overline{g_{m_2}}\,\Lambda^\theta\big(h_{n_1}\,h_{m_1}\big)\,\overline{\Lambda^\theta\big(h_{n_2}\,h_{m_2}\big)}.$$

The r.v. g_n are centred and independent, hence $\mathbb{E}\big[\,g_{n_1}\,\overline{g_{n_2}}\,g_{m_1}\,\overline{g_{m_2}}\,\big]=0$, unless ($n_1=n_2$ and $m_1=m_2$) or ($n_1=m_2$ and $n_2=m_1$). This implies that

$$\mathbb{E}\big[\,|\Lambda^\theta\big(u_N\, u_M\big)|^2\,\big]\leq C\sum_{\substack{2^N-1\leq n\leq 2(2^N-1)\\ 2^M-1\leq m\leq 2(2^M-1)}}|\alpha_n|^2|\alpha_m|^2|\Lambda^\theta\big(h_n\,h_m\big)|^2. \tag{3.7}$$

We integrate (3.7) in x and by (3.1) we deduce

$$\begin{aligned}\mathbb{E}\big[\,\|u_N\,u_M\|^2_{\mathcal{H}^\theta(\mathbb{R})}\,\big] &\leq C\sum_{\substack{2^N-1\leq n\leq 2(2^N-1)\\ 2^M-1\leq m\leq 2(2^M-1)}}|\alpha_n|^2|\alpha_m|^2\int_{\mathbb{R}}|\Lambda^\theta\big(h_n\,h_m\big)|^2\mathrm{d}x\\ &\leq C\sum_{\substack{2^N-1\leq n\leq 2(2^N-1)\\ 2^M-1\leq m\leq 2(2^M-1)}}\min(N,M)\max\big(2^N,2^M\big)^{-\frac12+\theta}|\alpha_n|^2|\alpha_m|^2.\end{aligned}\tag{3.8}$$

By (3.6) and (3.8), an integration in x and Minkowski yields

$$\begin{aligned}\|\Lambda^\theta\big(u^2\big)\|^2_{L^r(\Omega)L^2(\mathbb{R})} &\leq \|\Lambda^\theta\big(u^2\big)\|^2_{L^2(\mathbb{R})L^r(\Omega)}\\ &\leq Cr^2\sum_{N,M\geq 0}\sum_{\substack{2^N-1\leq n\leq 2(2^N-1)\\ 2^M-1\leq m\leq 2(2^M-1)}}\max\big(2^N,2^M\big)^{-\frac12+\theta+2\varepsilon}|\alpha_n|^2|\alpha_m|^2\\ &\leq Cr^2\Big(\sum_{n\geq 0}|\langle n\rangle^{-\frac18+\frac{\theta}{4}+\frac{\varepsilon}{2}}\alpha_n|^2\Big)^2.\end{aligned}$$

Lastly use that $|\alpha_n| \leq \langle n \rangle^{-\frac{1}{2}}$ to deduce

$$\|\Lambda^{\theta}(u^2)\|_{L^r(\Omega)L^2(\mathbb{R})} = \|u^2\|_{L^r(\Omega)\mathcal{H}^{\theta}(\mathbb{R})} \leq Cr, \tag{3.9}$$

for $\varepsilon > 0$ small enough, with $C > 0$ independent of $t \in \mathbb{R}$.
Finally, let $2 \leq q \leq r$ and $0 \leq T \leq 1$, then by Minkowski and (3.9) we conclude that

$$\|u^2\|_{L^r(\Omega)L^q_T\mathcal{H}^{\theta}(\mathbb{R})} \leq \|u^2\|_{L^q_T L^r(\Omega)\mathcal{H}^{\theta}(\mathbb{R})} \leq CrT^{\frac{1}{q}},$$

which was the claim (3.2). □

We refer to [**4**, Section 7] for the local existence theory for (1.1) with initial conditions of the form (1.2). The main tool is Proposition 3.1, which shows that we regain almost $\frac{1}{2}$ derivative at the price of a power $\langle x \rangle^{\frac{1}{2}}$.

4. Global existence for (1.1)

We now give some ideas of the proof the global existence part of Theorem 1.3.
We introduce the following finite dimensional approximation of (1.1)

$$(i\partial_t - H)u = S_N\big(|S_N u|^{k-1} S_N u\big), \quad u(0,x) = S_N(u(0,x)) \in E_N, \tag{4.1}$$

which is an ordinary differential equation.
For $u \in E_N$, write $u = \displaystyle\sum_{n=0}^{N} c_n h_n$, then we can check that the equation (4.1) is a Hamiltonian ODE, with Hamiltonian

$$J(c_0, \overline{c_0}, \cdots, c_N, \overline{c_N}) = \frac{1}{2}\sum_{n=0}^{N} \lambda_n^2 |c_n|^2 + \frac{1}{k+1}\int_{-\infty}^{\infty} \Big| S_N\Big(\sum_{n=0}^{N} c_n h_n(x)\Big)\Big|^{k+1} \mathrm{d}x.$$

In particular, we deduce that (4.1) has a well-defined global flow Φ_N, and thanks to Liouville's theorem, we can state

Proposition 4.1. — *The measure $\widetilde{\rho}_N$ as defined in the introduction is invariant under the flow Φ_N of* (4.1).

We can now adapt the strategy of [**5, 7**] (which uses ideas of Bourgain) to show that there are many "good" initial conditions in the support of $\widetilde{\rho}_N$ leading the solutions of (4.1) with bounds independent of N. In some sense, the invariant measure $\widetilde{\rho}_N$ plays the role of a Lyapunov function and gives a large time control of the solutions.
Thanks to limiting arguments, we then prove almost sure global existence for (1.1).

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