

EFFECTIVE DYNAMICS OF AN ELECTRON COUPLED TO AN EXTERNAL POTENTIAL IN NON-RELATIVISTIC QED

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ABSTRACT. In the framework of non-relativistic QED, we show that the renormalized mass of the electron (after having taken into account radiative corrections) appears as the kinematic mass in its response to an external potential force. Specifically, we study the dynamics of an electron in a slowly varying external potential and with slowly varying initial conditions and prove that, for a long time, it is accurately described by an associated effective dynamics of a Schrödinger electron in the same external potential and for the same initial data, with a kinetic energy operator determined by the renormalized dispersion law of the translation-invariant QED model.

1. INTRODUCTION

In this paper we show that the renormalized mass of the electron, taking into account radiative corrections due to its interaction with the quantized electromagnetic field, and the kinematic mass appearing in its response to a slowly varying external potential force are identical. Our analysis is carried out within the standard framework of non-relativistic quantum electrodynamics (QED). The renormalized electron mass, m_{ren} , is defined as the inverse curvature at zero momentum of the energy (dispersion law), $E(p)$, of a dressed electron as a function of its momentum p (no external potentials are present), i.e., $m_{\text{ren}} = E''(0)^{-1}$, while the kinematic mass of the electron enters the (effective) dynamical equations when it moves under the influence of an external potential force.

Our starting point is the dynamics generated by the Hamiltonian, H^V , describing a non-relativistic electron interacting with the quantized electromagnetic field and moving under the influence of a slowly varying potential, V_ϵ . We consider the time evolution of dressed one-electron states parametrized by wave functions $u_0^\epsilon \in H^1(\mathbb{R}^3)$, with $\|u_0^\epsilon\|_{L^2} = 1$ and $\|\nabla u_0^\epsilon\|_{L^2} \leq \epsilon$, and prove that their evolution is accurately approximated, during a long interval of time, by an effective Schrödinger dynamics generated by the one-particle Schrödinger operator

$$H_{\text{eff}} := E(-i\nabla_x) + V_\epsilon(x), \quad (1.1)$$

with kinetic energy given by the dispersion law $E(p)$. This result is in line with the general idea that any kind physical dynamics is an effective dynamics that can ultimately be derived from a more fundamental theory. While results of similar nature have been proven for quantum-mechanical particles interacting with *massive* bosons, [26], ours is the first result covering the physically more interesting situation of electrons interacting with *massless* bosons (photons) and revealing effects of radiative corrections to the electron mass. An interesting result on the effective dynamics of two heavy particles interacting via exchange of massless bosons has previously been obtained in [27].

In the usual model of non-relativistic QED, the Hilbert space of states of a system consisting of a single electron and arbitrarily many photons (described in the Coulomb gauge) is given by

$$\mathcal{H} := L^2(\mathbb{R}^3) \otimes \mathfrak{F}, \quad (1.2)$$

where $L^2(\mathbb{R}^3)$ is the Hilbert space of square-integrable wave functions describing the electron degrees of freedom, (electron spin is neglected for notational convenience). The space \mathfrak{F} is the Fock space of physical states of photons,

$$\mathfrak{F} = \bigoplus_{n \geq 0} \mathfrak{F}_n.$$

Here $\mathfrak{F}_n := \text{Sym}(L^2(\mathbb{R}^3 \times \{+, -\}))^{\otimes n}$ denotes the physical Hilbert space of states of n photons. The Hamiltonian acting on the space \mathcal{H} is given by the expression

$$H^V = H + V_\epsilon \otimes \mathbf{1}_f, \quad (1.3)$$

where H is the generator of the dynamics of a single, freely moving non-relativistic electron minimally coupled to the quantized electromagnetic field, i.e.,

$$H := \frac{1}{2}(-i\nabla_x \otimes \mathbf{1}_f + \sqrt{\alpha}A(x))^2 + \mathbf{1}_{el} \otimes H_f, \quad (1.4)$$

and where $V_\epsilon(x) := V(\epsilon x)$ is a slowly varying potential, with $\epsilon > 0$ small; its precise properties are formulated in Theorem 1.1 below. Furthermore,

$$A(x) := \sum_\lambda \int_{|k| \leq 1} \frac{dk}{|k|^{1/2}} \{ \epsilon_\lambda(k) e^{ikx} \otimes a_\lambda(k) + h.c. \} \quad (1.5)$$

denotes the quantized electromagnetic vector potential in the Coulomb gauge with an ultraviolet cutoff imposed, $|k| \leq 1$, and

$$H_f = \sum_\lambda \int dk |k| a_\lambda^*(k) a_\lambda(k) \quad (1.6)$$

is the photon Hamiltonian. In Eqs. (1.5) and (1.6), $a_\lambda^*(k)$, $a_\lambda(k)$ are the usual photon creation- and annihilation operators, $\lambda = \pm$ indicates photon helicity, and $\epsilon_\lambda(k)$ is a polarization vector perpendicular to k corresponding to helicity λ .

We observe that the Hamiltonian H is translation-invariant, in the sense that H commutes with translations, $T_y : \Psi(x) \rightarrow e^{iy \cdot P_f} \Psi(x + y)$, for $y \in \mathbb{R}^3$, where $P_f = \sum_\lambda \int dk k a_\lambda^*(k) a_\lambda(k)$ is the momentum operator of the quantized radiation field. Hence H commutes with the total momentum operator

$$P_{tot} := -i\nabla_x \otimes \mathbf{1}_f + \mathbf{1}_{el} \otimes P_f, \quad (1.7)$$

of the electron and the photon field: $[H, P_{tot}] = 0$. It follows that H can be decomposed as a direct integral

$$U H U^{-1} = \int_{\mathbb{R}^3}^{\oplus} H(p) dp, \quad (1.8)$$

of fiber operators, $H(p)$, over the spectrum of P_{tot} , where $H(p)$ is defined on the fiber space $\mathcal{H}_p \cong \mathfrak{F}$ in the direct integral decomposition, $\mathcal{H} \cong \int_{\mathbb{R}^3}^{\oplus} dp \mathcal{H}_p$, of \mathcal{H} . The operator $U : \mathcal{H} \rightarrow \int_{\mathbb{R}^3}^{\oplus} dp \mathcal{H}_p$ is a generalized Fourier transform defined on smooth, rapidly decaying functions,

$$(U\Psi)(p) := (F e^{iP_f \cdot x} \Psi)(p) = \int_{\mathbb{R}^3} e^{-i(p - P_f) \cdot x} \Psi(x) dx, \quad (1.9)$$

where F is the standard Fourier transform for Hilbert space-valued functions,

$$(F\Psi)(p) = \int_{\mathbb{R}^3} e^{-ip \cdot x} \Psi(x) dx.$$

For smooth, rapidly decaying vector-valued functions $\Phi(p) \in \mathcal{H}$, its inverse is given by

$$(U^{-1}\Phi)(x) := e^{-iP_f \cdot x} (F^{-1}\Phi)(x) = \int_{\mathbb{R}^3} e^{ix \cdot (p - P_f)} \Phi(p) dp. \quad (1.10)$$

We note that

$$(UH\Psi)(p) = H(p)(U\Psi)(p) \quad , \quad (UP_{tot}\psi)(p) = p(U\psi)(p). \quad (1.11)$$

Since U is the composition of two unitary operators, $e^{iP_f \cdot x}$ and the standard Fourier transform F , it is unitary, too, and Eq. (1.10) defines its inverse.

We define creation- and annihilation operators, $b_\lambda^*(k)$ and $b_\lambda(k)$, on the fiber spaces \mathcal{H}_p by

$$b_\lambda(k) := U e^{ikx} a_\lambda(k) U^{-1} \quad , \quad b_\lambda^*(k) = U e^{-ikx} a_\lambda^*(k) U^{-1}, \quad (1.12)$$

i.e.,

$$(U e^{ikx} a_\lambda(k) \Psi)(p) = b_\lambda(k)(U\Psi)(p) \quad , \quad (U e^{-ikx} a_\lambda^*(k) \Psi)(p) = b_\lambda^*(k)(U\Psi)(p), \quad (1.13)$$

for $\Psi \in \mathcal{H}$. Obviously, the operator-valued distributions $b_\lambda(k)$ and $b_\lambda^*(k)$ commute with P_{tot} . Thus, the operators $b_\lambda^{(*)}(f) := \int b_\lambda^{(*)}(k) \hat{f}(k) dk$ map the fiber spaces \mathcal{H}_p to themselves, for any test function f . The fact that these operators satisfy the usual canonical commutation relations is obvious. The Fock space constructed from the operators $b_\lambda^{(*)}(f)$, $f \in L^2(\mathbb{R}^3 \times \{+, -\})$, and the vacuum vector Ω is denoted by \mathfrak{F}^b .

From abstract theory, the fiber operators $H(p)$, $p \in \mathbb{R}^3$, are nonnegative self-adjoint operators acting on $\mathcal{H}_p \cong \mathfrak{F}^b$. Their explicit form is determined in the next section. We define $E(p) = \inf \text{spec} H(p)$, for all $p \in \mathbb{R}^3$, and

$$\mathcal{S} := \left\{ p \in \mathbb{R}^3 \mid |p| \leq \frac{1}{3} \right\}. \quad (1.14)$$

Making use of approximate ground states, $\Phi^\rho(p)$, $\rho > 0$, (dressed by a cloud of soft photons with frequencies below ρ) of the operators $H(p)$, which will be defined in (2.14), we introduce a family of maps $\mathcal{J}_0^\rho : L^2(\mathbb{R}^3) \mapsto \mathcal{H}$, from the space $L^2(\mathbb{R}^3)$ of square-integrable one-particle wave functions, u , to a subspace of *dressed one-electron states*, $\hat{u} \Phi^\rho$, as

$$\mathcal{J}_0^\rho(u)(x) := (U^{-1} \chi_{S_\mu} \hat{u} \Phi^\rho)(x) = \int dp \hat{u}(p) e^{ix \cdot (p - P_f)} \chi_{S_\mu}(p) \Phi^\rho(p), \quad (1.15)$$

where χ_{S_μ} is a smooth approximate characteristic function of the set $S_\mu := (1 - \mu)\mathcal{S} \subset \mathcal{S} \subset \mathbb{R}^3$, ($0 < \mu < 1$).

In this paper we study the time evolution of one-electron states, $\mathcal{J}_0^\rho(u_0^\epsilon)$, where u_0^ϵ is a slowly varying one-particle wave function, dressed by an *infrared cloud* of photons with frequencies below ρ . More precisely, we study solutions of the Schrödinger equation

$$i\partial_t \Psi(t) = H^V \Psi(t), \quad \text{with } \Psi(0) = \mathcal{J}_0^\rho(u_0^\epsilon). \quad (1.16)$$

The key idea is to relate the solution $\Psi(t) = e^{-itH^V} \mathcal{J}_0^\rho(u_0^\epsilon)$ of this Schrödinger equation to the solution of the Schrödinger equation

$$i\partial_t u_t^\epsilon = H_{\text{eff}} u_t^\epsilon, \quad \text{with } u_{t=0}^\epsilon = u_0^\epsilon, \quad (1.17)$$

corresponding to the one-particle Schrödinger operator (1.1), where $E(p)$ has been defined above. We consider the comparison state

$$\mathcal{J}_0^\rho(u_t^\epsilon) \in \mathcal{H}, \quad (1.18)$$

where $u_t^\epsilon := e^{-itH_{\text{eff}}}u_0^\epsilon$ is the solution of (1.17), and show that $\Psi(t)$ remains close to $\mathcal{J}_0^\rho(u_t^\epsilon)$, for a long time. The choice of initial data satisfying

$$\|u_0^\epsilon\|_{L^2(\mathbb{R}^3)} = 1 \quad \text{and} \quad \|\nabla u_0^\epsilon\|_{L^2(\mathbb{R}^3)} \leq \epsilon^\kappa, \quad 0 \leq \kappa < \frac{1}{3}, \quad (1.19)$$

guarantees that \widehat{u}_t^ϵ remains concentrated in \mathcal{S} during the time scales relevant for this problem, provided the support of \widehat{u}_0^ϵ is contained in \mathcal{S} .

Theorem 1.1. *Let $0 < \epsilon < 1/3$, $0 \leq \kappa < 1/3$ and assume that $u_0^\epsilon \in L^2(\mathbb{R}^3)$ obeys (1.19). Assume, furthermore, that $V \in L^\infty(\mathbb{R}^3; \mathbb{R})$ is such that $\widehat{V} \in L^1(\mathbb{R}^3)$ and that \widehat{V} is supported in the unit ball,*

$$\text{supp}(\widehat{V}) \subset \{k \in \mathbb{R}^3 \mid |k| \leq 1\}. \quad (1.20)$$

Let $0 < \delta < 2(\frac{1}{3} - \kappa)$, and choose $\rho = \rho_\epsilon := \epsilon^{\frac{2}{3} - \delta}$.

Then there exists $\alpha_\delta > 0$ such that, for all $0 \leq \alpha \leq \alpha_\delta$, the bound

$$\|e^{-itH^V} \mathcal{J}_0^{\rho_\epsilon}(u_0^\epsilon) - \mathcal{J}_0^{\rho_\epsilon}(e^{-itH_{\text{eff}}}u_0^\epsilon)\|_{\mathcal{H}} \leq C_\delta \left(\epsilon^{\frac{1}{3} - \frac{\delta}{2} + \kappa} t + \epsilon^{\frac{4}{3} - \frac{\delta}{2}} t^2 \right), \quad (1.21)$$

holds for all times $t \geq 0$. In particular, for all $0 \leq t \leq \epsilon^{-2/3}$, we have that

$$\|e^{-itH^V} \mathcal{J}_0^{\rho_\epsilon}(u_0^\epsilon) - \mathcal{J}_0^{\rho_\epsilon}(e^{-itH_{\text{eff}}}u_0^\epsilon)\|_{\mathcal{H}} \leq C_\delta \epsilon^{\frac{1}{3} - \frac{\delta}{2} + \kappa} t. \quad (1.22)$$

Remark 1.2. *Theorem 1.1 implies that, for all $\delta' > 0$ such that $\delta' < \frac{1}{3} - \frac{\delta}{2} + \kappa$*

$$\|e^{-itH^V} \mathcal{J}_0^{\rho_\epsilon}(u_0^\epsilon) - \mathcal{J}_0^{\rho_\epsilon}(e^{-itH_{\text{eff}}}u_0^\epsilon)\|_{\mathcal{H}} \leq C_\delta \epsilon^{\delta'} \quad (1.23)$$

holds for all times t with $0 \leq t \leq \epsilon^{-(\frac{1}{3} - \frac{\delta}{2} + \kappa) + \delta'}$.

Remark 1.3. *The initial conditions in Theorem 1.1 are chosen such that the initial momentum is $O(\epsilon^\kappa)$. The conditions on the external potential imply that the expected force, and, thus, the acceleration, is of order $O(\epsilon)$. Hence, at time t , the momentum is of order $O(\epsilon^\kappa) + O(\epsilon t)$, and therefore the action, $E(p)t - E(0)t \approx \frac{1}{2m_{\text{ren}}}p^2t$, is of order $O(\epsilon^{2\kappa}t) + O(\epsilon^2t^3)$. Hence, if $\frac{1}{3} - \kappa > \frac{\delta}{2}$ and $t \leq \epsilon^{-1 + \kappa}$, then this term is much larger than the error term in Eq. (1.21).*

Remark 1.4. *To make the previous remark more precise, we define the operator $\widetilde{H}_{\text{eff}} := E(0) + V(\epsilon x)$, and consider the difference between $e^{-itH_{\text{eff}}}$ and $e^{-it\widetilde{H}_{\text{eff}}}$. We write $e^{-itH_{\text{eff}}} - e^{-it\widetilde{H}_{\text{eff}}}$ as the integral of a derivative,*

$$e^{-itH_{\text{eff}}} - e^{-it\widetilde{H}_{\text{eff}}} = -i \int_0^t ds e^{-i(t-s)H_{\text{eff}}} (E(p) - E(0)) e^{-is\widetilde{H}_{\text{eff}}}, \quad (1.24)$$

and use that $E'(0) = 0$ so that $cp^2 < E(p) - E(0) = \frac{1}{2m_{\text{ren}}}p^2(1 + o(1)) < Cp^2$ (see Proposition 2.1, below). Then, using

$$e^{is\widetilde{H}_{\text{eff}}}p^2e^{-is\widetilde{H}_{\text{eff}}} = p^2 + 2\epsilon p \cdot (\nabla V)(\epsilon x)s + \epsilon^2 \Delta V(\epsilon x)s^2, \quad (1.25)$$

we find that

$$e^{-itH_{\text{eff}}} - e^{-it\widetilde{H}_{\text{eff}}} = A_t + O(\epsilon t^2 p) + O(\epsilon^2 t^3), \quad (1.26)$$

where $A_t = O(tp^2)$. Adding the second and third term on the r.h.s. of (1.26) to the error estimated by (1.22), we observe that

$$O(t\epsilon^{\frac{1}{3} - \frac{\delta}{2} + \kappa}) + O(\epsilon t^2 p) + O(\epsilon^2 t^3) = O\left[t(\epsilon^{\frac{1}{3} - \frac{\delta}{2} + \kappa} + \epsilon^{\frac{1}{3} + \kappa} + \epsilon^{\frac{2}{3}})\right] = O(t\epsilon^{\frac{1}{3} - \frac{\delta}{2} + \kappa}), \quad (1.27)$$

provided that $0 < t \leq \epsilon^{-2/3}$. Assuming that A_t is not only bounded above by $O(tp^2)$, but is actually of order

$$\|A_t u_0^\epsilon\| \geq C t \epsilon^{2\kappa}, \quad (1.28)$$

with $C \equiv C(u_0^\epsilon, m_{\text{ren}}) > 0$ depending on the initial data and on the renormalized mass m_{ren} , we can compare this contribution to (1.27) and observe that

$$\frac{\|e^{-itH^V} \mathcal{J}_0^{\rho_\epsilon}(u_0^\epsilon) - \mathcal{J}_0^{\rho_\epsilon}(A_t u_0^\epsilon)\|_{\mathcal{H}}}{\|A_t u_0^\epsilon\|} \leq O(\epsilon^{\frac{1}{3} - \frac{\delta}{2} - \kappa}) \|A_t u_0^\epsilon\|, \quad (1.29)$$

provided $\epsilon^{-2\kappa} \leq t \leq \epsilon^{-2/3}$. Thus our estimate allows us to separate the main contribution of the dynamics from the error terms on a suitable time scale.

1.1. Outline of proof strategy. To prove Theorem 1.1, we introduce an *infrared regularized* version of the model defined by (1.3), (1.4), obtained by restricting the integration domain in the quantized electromagnetic vector potential (1.5) to the region $\{\sigma \leq |k| \leq 1\}$, for an arbitrary infrared cutoff $\sigma > 0$. Thereby, we obtain infrared regularized Hamiltonians H_σ^V and H_σ , as well as an infrared regularized family of maps \mathcal{J}_σ^ρ corresponding to \mathcal{J}_0^ρ .

We note that, unlike $H(p)$, the infrared cut-off fiber Hamiltonian $H_\sigma(p)$ has a ground state $\Psi_\sigma(p) \in \mathcal{H}_p \cong \mathfrak{F}$, for every $p \in \mathcal{S}$ and for $\sigma > 0$, but $\Psi_\sigma(p)$ does not possess a limit in $\mathcal{H}_p \cong \mathfrak{F}$, as $\sigma \searrow 0$. In particular, we expect that the number of photons in the state $\Psi_\sigma(p)$ diverges, as $\sigma \searrow 0$, (thus the lack of convergence of $\Psi_\sigma(p)$ in \mathfrak{F}). This is a well-known aspect of the *infrared problem* in QED, [8, 9, 10, 11, 22]. It is remedied by applying a dressing transformation, $W_{\nabla E_\sigma(p)}^{\sigma, \rho}$, defined in (2.14), below, to $\Psi_\sigma(p)$, where $E_\sigma(p) = \inf \text{spec} H_\sigma(p)$. The resulting vector, $\Phi_\sigma^\rho(p) := W_{\nabla E_\sigma(p)}^\rho \Psi_\sigma(p)$, describes an *infraparticle* (or *dressed electron*) state containing infrared photons with frequencies in $[\sigma, \rho]$. As $\sigma \searrow 0$, the limit

$$\Phi^\rho(p) = \lim_{\sigma \rightarrow 0} \Phi_\sigma^\rho(p) \quad (1.30)$$

exists in \mathfrak{F} , see Proposition 2.2. This allows us to construct the map \mathcal{J}_0^ρ as the limit of the maps \mathcal{J}_σ^ρ , as $\sigma \searrow 0$. Note that, while $\Psi_\sigma(p)$ does not converge in \mathfrak{F} , we have that $\lim_{\sigma \searrow 0} E_\sigma(p) = E(p)$.

We note that $\Phi_\sigma^\rho(p)$ is the ground state eigenvector of the fiber Hamiltonian

$$K_\sigma^\rho(p) := W_{\nabla E_\sigma(p)}^{\sigma, \rho} H_\sigma(p) (W_{\nabla E_\sigma(p)}^{\sigma, \rho})^* \quad (1.31)$$

which is obtained by applying to $H_\sigma(p)$ the *Bogoliubov transformation* corresponding to the dressing transformation $W_{\nabla E_\sigma(p)}^{\sigma, \rho}$.

In Theorem 2.3, below, we prove that an estimate similar to (1.21) is satisfied for the infrared regularized model; namely,

$$\|e^{-itH_\sigma^V} \mathcal{J}_\sigma^\rho(u_0^\epsilon) - \mathcal{J}_\sigma^\rho(e^{-itH_{\text{eff}, \sigma}} u_0^\epsilon)\|_{\mathcal{H}} \leq C_\delta (1 + \ln(\rho^{-1})) \epsilon^{\frac{2}{3} - \delta} t + C \alpha^{\frac{1}{2}} \rho^{\frac{1}{2}} t (\epsilon^\kappa + \epsilon t), \quad (1.32)$$

holds uniformly in the infrared cutoff σ and the cut-off $\rho > \sigma$. This result crucially uses the regularity properties of the dressed electron states $\Phi_\sigma^\rho(p)$, which allow us to take advantage of the fact that V_ϵ is slowly varying. An additional key ingredient is the bound $\|(H_\sigma(p) - K_\sigma^\rho(p)) \Phi_\sigma^\rho(p)\|_{\mathfrak{F}} \leq C \alpha^{\frac{1}{2}} \rho^{\frac{1}{2}} |p|$, for $p \in \mathcal{S}$, proven in Appendix A. In (1.32) we take $\rho = \rho_\epsilon := \epsilon^{\frac{2}{3} - \delta}$ and absorb $\ln(\rho^{-1})$ into $\epsilon^{\frac{2}{3} - \delta}$.

In Section 3, we control the limit $\sigma \searrow 0$, thus concluding the proof of Theorem 1.1. This requires control of the radiation emitted by the electron due to its acceleration in the external potential V_ϵ , in the limit $\sigma \searrow 0$.

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2. INFRARED CUT-OFF AND CONSTRUCTION OF $\Phi^\rho(p)$

As noted in the introduction, we analyze the original dynamics by first imposing an infrared (IR) cut-off, and controlling the dynamics generated by the resulting Hamiltonian. Thus, we define the IR regularized Hamiltonian

$$H_\sigma^V = H_\sigma + V_\epsilon(x) \otimes \mathbf{1}_f, \quad (2.1)$$

where

$$H_\sigma := \frac{1}{2}(-i\nabla_x \otimes \mathbf{1}_f + \sqrt{\alpha}A_\sigma(x))^2 + \mathbf{1}_{el} \otimes H_f \quad (2.2)$$

is the generator of the dynamics of a single, freely moving non-relativistic electron minimally coupled to the electromagnetic radiation field. In (2.2),

$$A_\sigma(x) = \sum_\lambda \int_{\sigma \leq |k| \leq 1} \frac{dk}{|k|^{1/2}} \{ \epsilon_\lambda(k) e^{ikx} \otimes a_\lambda(k) + h.c. \} \quad (2.3)$$

denotes the quantized electromagnetic vector potential with an infrared and ultraviolet cutoff corresponding to $\sigma \leq |k| \leq 1$. Since $V \in L^\infty(\mathbb{R}^3)$ is a bounded operator, $D(H_\sigma^V) = D(H_\sigma) = D(-\Delta_x \otimes \mathbf{1}_f + \mathbf{1}_{el} \otimes H_f)$.

The Hamiltonian H_σ is also translation invariant and, similarly to H , can be represented as the fiber integral

$$UH_\sigma U^{-1} = \int_{\mathbb{R}^3}^\oplus H_\sigma(p) dp, \quad (2.4)$$

over the spectrum of P_{tot} , defined on the fiber integral $\int^\oplus dp \mathcal{H}_p$, with fibers $\mathcal{H}_p \cong \mathfrak{F}^b$. The decomposition (2.4) is equivalent to

$$(UH_\sigma \Psi)(p) = H_\sigma(p)(U\Psi)(p). \quad (2.5)$$

Again, by abstract theory, the fiber Hamiltonians $H_\sigma(p)$, $p \in \mathbb{R}^3$, are self-adjoint operators on $\mathcal{H}_p \cong \mathfrak{F}^b$. Written in terms of the creation- and annihilation operators on the fiber space, they are given by

$$H_\sigma(p) = \frac{1}{2}(p - P_f^b - \sqrt{\alpha}A_\sigma^b)^2 + H_f^b \quad (2.6)$$

where

$$H_f^b := \sum_{\lambda} \int dk |k| b_{\lambda}^*(k) b_{\lambda}(k) \quad , \quad P_f^b := \sum_{\lambda} \int dk k b_{\lambda}^*(k) b_{\lambda}(k) \quad (2.7)$$

and

$$A_{\sigma}^b := \sum_{\lambda} \int_{\sigma \leq |k| \leq 1} \frac{dk}{|k|^{1/2}} \{ \epsilon_{\lambda}(k) b_{\lambda}(k) + h.c. \}. \quad (2.8)$$

Henceforth, we will drop the superscripts "b" from the notation.

While $H(p)$ has a ground state only for $p = 0$, it is proven in [2, 6] that, for $p \in \mathcal{S} := \{p \in \mathbb{R}^3 \mid |p| \leq 1/3\}$ and $\sigma > 0$, $H_{\sigma}(p)$ has a non-degenerate (fiber) ground state. This motivates the introduction of the cut-off. Properties of the fiber ground state energy, $E_{\sigma}(p) = \inf \text{spec} H_{\sigma}(p)$, are given in the following proposition proven in [2, 6, 9, 10]:

Proposition 2.1. *The infimum of the spectrum of the fiber Hamiltonian,*

$$E_{\sigma}(p) = \inf \text{spec} H_{\sigma}(p), \quad (2.9)$$

satisfies:

- (1) For any $\sigma > 0$, $E_{\sigma} \in C^2(\mathcal{S})$, and for all $p \in \mathcal{S} = \{p \in \mathbb{R}^3 \mid |p| \leq \frac{1}{3}\}$, $E_{\sigma}(p)$ is a simple eigenvalue.
- (2) There exist $\alpha_0 > 0$ and $c < \infty$ such that, for any $p \in \mathcal{S}$, $0 < \alpha \leq \alpha_0$, and $\sigma \geq 0$, we have that

$$|\nabla_p E_{\sigma}(p) - p| \leq c \alpha |p|, \quad \text{and} \quad 1 - c \alpha \leq \partial_{|p|}^2 E_{\sigma}(p) \leq 1. \quad (2.10)$$

- (3) The following limit exists in $C^2(\mathcal{S})$

$$\lim_{\sigma \searrow 0} E_{\sigma}(\cdot) = E(\cdot). \quad (2.11)$$

We let $\Psi_{\sigma}(p) \in \mathfrak{F}$, with $\|\Psi_{\sigma}(p)\|_{\mathfrak{F}} = 1$, denote the normalized fiber ground state corresponding to $E_{\sigma}(p)$,

$$H_{\sigma}(p) \Psi_{\sigma}(p) = E_{\sigma}(p) \Psi_{\sigma}(p), \quad (2.12)$$

for $p \in \mathcal{S}$. For $0 < \sigma < \rho \leq 1$ and $p \in \mathcal{S}$, we introduce the Weyl operators

$$W_{\nabla E_{\sigma}(p)}^{\sigma, \rho} := \exp \left[\alpha^{\frac{1}{2}} \sum_{\lambda} \int_{\sigma \leq |k| \leq \rho} dk \frac{\nabla E_{\sigma}(p) \cdot \epsilon_{\lambda}(k) b_{\lambda}(k) - h.c.}{|k|^{1/2} (|k| - \nabla E_{\sigma}(p) \cdot k)} \right], \quad (2.13)$$

with $\nabla E_{\sigma}(p) \equiv \nabla_p E_{\sigma}(p)$, which are unitary on \mathfrak{F} , for $\sigma > 0$. Moreover, we define dressed electron states

$$\Phi_{\sigma}^{\rho}(p) := W_{\nabla E_{\sigma}(p)}^{\sigma, \rho} \Psi_{\sigma}(p). \quad (2.14)$$

For $p \in \mathcal{S}$, we define the Bogoliubov-transformed fiber Hamiltonians

$$K_{\sigma}^{\rho}(p) := W_{\nabla E_{\sigma}(p)}^{\sigma, \rho} H_{\sigma}(p) (W_{\nabla E_{\sigma}(p)}^{\sigma, \rho})^*. \quad (2.15)$$

It is convenient to define $K_{\sigma}^{\rho}(p) := H_{\sigma}(p)$, for $p \in \mathbb{R}^3 \setminus \mathcal{S}$.

The dressed electron states $\Phi_\sigma^\rho(p)$, for $p \in \mathcal{S}$, are the ground states of the Bogoliubov-transformed fiber Hamiltonians $K_\sigma^\rho(p)$, defined in (2.15), i.e.,

$$K_\sigma^\rho(p) \Phi_\sigma^\rho(p) = E_\sigma(p) \Phi_\sigma^\rho(p). \quad (2.16)$$

The properties of these states are described in the following proposition

Proposition 2.2. *For any $p \in \mathcal{S}$, $0 < \rho \leq 1$, and for sufficiently small values of the finestructure constant $0 < \alpha \ll 1$, the ground state eigenvector $\Phi_\sigma^\rho(p)$ satisfies:*

(1) *The strong limit*

$$\Phi^\rho(p) := \lim_{\sigma \rightarrow 0} \Phi_\sigma^\rho(p) \quad (2.17)$$

exists in \mathfrak{F} .

(2) *For $\theta < \frac{2}{3}$, the vectors $\Phi_\sigma^\rho(p)$ are θ -Hölder continuous in p ,*

$$\sup_{p, q \in \mathcal{S}} \frac{\|\Phi_\sigma^\rho(p) - \Phi_\sigma^\rho(q)\|}{|p - q|^\theta} \leq C(\theta) < \infty, \quad (2.18)$$

uniformly in σ and ρ , with $0 \leq \sigma < \rho \leq 1$.

The proof of θ -Hölder continuity for $\theta < \frac{2}{3}$ is given in Section 5; (see also [9, 10, 22] for earlier results covering the range $\theta < \frac{1}{4}$, in the case where $\rho = 1$).

For arbitrary $u \in L^2(\mathbb{R}^3)$ (with Fourier transform denoted by \widehat{u}), we define the linear map

$$\mathcal{J}_\sigma^\rho : u \mapsto \int_{\mathcal{S}} dp \widehat{u}(p) e^{ix(p-P_f)} \chi_{\mathcal{S}_\mu}(p) \Phi_\sigma^\rho(p), \quad (2.19)$$

where x is the electron position, $\chi_{\mathcal{S}_\mu}$ is a smooth approximate characteristic function of the set

$$\mathcal{S}_\mu := (1 - \mu) \mathcal{S} \subset \mathcal{S} \subset \mathbb{R}^3, \quad (2.20)$$

and $0 < \mu < 1$. Note that $\mathcal{J}_\sigma^\rho : L^2(\mathbb{R}^3) \rightarrow \mathcal{M} \subset \mathcal{H}$, where

$$\mathcal{M} := \left\{ \int_{\mathbb{R}^3} dp \widehat{u}(p) e^{ix(p-P_f)} \chi_{\mathcal{S}_\mu}(p) \Phi_\sigma^\rho(p) \mid u \in L^2(\mathbb{R}^3) \right\}, \quad (2.21)$$

the subspace of vectors in \mathcal{H} supported on the one-particle shell of the operator $\int_{\mathcal{S}}^\oplus dp K_\sigma^\rho(p)$. We also note that in (2.21) we do not require that $\text{supp}(\widehat{u}) \subset \mathcal{S}_\mu$; instead, we cutoff \widehat{u} outside the region \mathcal{S}_μ by multiplying it by $\chi_{\mathcal{S}_\mu}$.

Furthermore, we introduce the one-particle Schrödinger operator

$$H_{\text{eff}, \sigma} := E_{\text{eff}, \sigma}(-i\nabla_x) + V_\epsilon(x), \quad (2.22)$$

where $(t, x) \in \mathbb{R} \times \mathbb{R}^3$. Here, the kinetic energy operator is defined by

$$E_{\text{eff}, \sigma}(p) := E_\sigma(p), \quad (2.23)$$

for all $p \in \mathbb{R}^3$. Note that the restriction of $E_{\text{eff}, \sigma}$ to \mathcal{S} is twice continuously differentiable, $E_{\text{eff}, \sigma}|_{\mathcal{S}} \in C^2(\mathcal{S})$.

As a first step towards proving Theorem 1.1, we prove the following result.

Theorem 2.3. *Under the conditions of Theorem 1.1, there exists $\alpha_\delta > 0$ such that, for all $0 \leq \alpha \leq \alpha_\delta$, the bound (1.32) holds uniformly in the infrared cutoff $\sigma > 0$ and the cutoff $\rho > \sigma$.*

Proof. Our proof makes crucial use of the properties of the fiber ground state energy $E_\sigma(p)$ and of the corresponding dressed electron states $\Phi_\sigma^\rho(p)$, for $p \in \mathcal{S}$, given in Propositions 2.1 and 2.2 above. We define the operator K_σ^ρ acting on \mathcal{H} ,

$$K_\sigma^\rho := \int^\oplus K_\sigma^\rho(p) dp, \quad (2.24)$$

and the perturbed operator $K_\sigma^V := K_\sigma^\rho + V_\epsilon$. Note that the operator K_σ^ρ has the property that

$$K_\sigma^\rho \mathcal{J}_\sigma^\rho = \mathcal{J}_\sigma^\rho E_{\text{eff},\sigma}(-i\nabla). \quad (2.25)$$

We write the difference on the LHS of (1.32) as the integral of a derivative, substitute $H_\sigma^V \rightarrow H_\sigma^V - K_\sigma^V + K_\sigma^V$ inside the integral and group terms suitably to obtain

$$\begin{aligned} e^{-itH_\sigma^V} \mathcal{J}_\sigma^\rho(u_0^\epsilon) - \mathcal{J}_\sigma^\rho(e^{-itH_{\text{eff},\sigma}} u_0^\epsilon) &= -i e^{-itH_\sigma^V} \int_0^t ds e^{isH_\sigma^V} (H_\sigma^V \mathcal{J}_\sigma^\rho(u_s^\epsilon) - \mathcal{J}_\sigma^\rho(H_{\text{eff}} u_s^\epsilon)) \\ &=: \phi^1(t) + \phi^2(t), \end{aligned} \quad (2.26)$$

where $u_s^\epsilon := e^{-isH_{\text{eff},\sigma}} u_0^\epsilon$ and

$$\phi^1(t) := -i e^{-itH_\sigma^V} \int_0^t ds e^{isH_\sigma^V} (H_\sigma - K_\sigma^\rho) \mathcal{J}_\sigma^\rho(u_0^\epsilon), \quad (2.27)$$

where we have used the cancelation of V in $H_\sigma^V - K_\sigma^V = H_\sigma - K_\sigma^\rho$, and

$$\phi^2(t) := -i e^{-itH_\sigma^V} \int_0^t ds e^{isH_\sigma^V} (K_\sigma^V \mathcal{J}_\sigma^\rho(u_s^\epsilon) - \mathcal{J}_\sigma^\rho(H_{\text{eff}} u_s^\epsilon)).$$

The first term on the r.h.s. accounts for the radiation of infrared photons due to the motion of the dressed electron in the external potential, while the second term accounts for the influence of the external potential V_ϵ on the full QED dynamics $\Psi(t) = e^{-itH_\sigma^V} \mathcal{J}_\sigma^\rho(u_0^\epsilon)$, as compared to the effective Schrödinger evolution $e^{-itH_{\text{eff},\sigma}} u_0^\epsilon$.

Using the fiber integral decomposition, we obtain

$$\|\phi^1(t)\|_{\mathcal{H}} \leq \sup_{p \in \mathcal{S}} \left\{ \frac{1}{|p|} \|(H_\sigma - K_\sigma^\rho)(p) \Phi_\sigma^\rho(p)\|_{\mathfrak{F}} \right\} \int_0^t \|\nabla u_s^\epsilon\|_{L^2(\mathbb{R}^3)} ds. \quad (2.28)$$

In Appendix A we prove the following key result:

$$\sup_{p \in \mathcal{S}} \left\{ \frac{1}{|p|} \|(H_\sigma - K_\sigma^\rho)(p) \Phi_\sigma^\rho(p)\|_{\mathfrak{F}} \right\} \leq C \alpha^{\frac{1}{2}} \rho^{\frac{1}{2}}, \quad (2.29)$$

uniformly in $\sigma \geq 0$. Furthermore, we have the estimate

$$\int_0^t \|\nabla u_s^\epsilon\|_{L^2(\mathbb{R}^3)} ds \leq C t (\epsilon^\kappa + \epsilon t), \quad (2.30)$$

as shown below in (2.36)–(2.38), by using the condition $\|\nabla u_0^\epsilon\|_{L^2(\mathbb{R}^3)} \leq \epsilon^\kappa$ on u_0^ϵ , and the fact that the potential V satisfies (1.20). We obtain

$$\|\phi^1(t)\|_{\mathcal{H}} \leq C t (\epsilon^\kappa + \epsilon t) \alpha^{\frac{1}{2}} \rho^{\frac{1}{2}}, \quad (2.31)$$

which yields the second contribution to the r.h.s. of (1.32).

For the second term on the r.h.s. of (2.26), using the fiber decomposition and the equation $K_\sigma^\rho(p) \Phi_\sigma^\rho(p) = E_\sigma(p) \Phi_\sigma^\rho(p)$, we have that

$$\phi^2(t) = -i e^{-itH_\sigma^V} \int_0^t ds e^{isH_\sigma^V} (V_\epsilon \mathcal{J}_\sigma^\rho(u_s^\epsilon) - \mathcal{J}_\sigma^\rho(V_\epsilon u_s^\epsilon)). \quad (2.32)$$

Let $\|\Phi\|_{C^\theta(\mathcal{S})} := \sup_{p,q \in \mathcal{S}} \frac{\|\Phi(p) - \Phi(q)\|}{|p - q|^\theta}$.

In (2.39)–(2.46) below, we prove an estimate of the form

$$\|\phi^2(t)\|_{\mathcal{H}} \leq tC \|\widehat{|\nabla|^\theta V_\epsilon}\|_{L^1(\mathbb{R}^3)} (1 + \|\Phi_\sigma^\rho\|_{C^\theta(\mathcal{S})}), \quad (2.33)$$

for $\theta < \frac{2}{3}$. The key point here is that the θ -Hölder continuity of the fiber ground state $\Phi_\sigma^\rho(p)$ enables us to gain a θ derivative of the potential, yielding $\|\widehat{|\nabla|^\theta V_\epsilon}\|_{L^1(\mathbb{R}^3)} \leq C\epsilon^\theta$. Using the θ -Hölder continuity of $\Phi_\sigma^\rho(\cdot)$, which holds uniformly in σ , with $0 < \sigma < \rho$, and that

$$\|\widehat{|\nabla|^\theta V}\|_{L^1(\mathbb{R}^3)} \leq \gamma, \quad \text{where } \gamma := \|\widehat{V}(k)\|_{L^1} < \infty, \quad (2.34)$$

(see (1.20)) and using $\|\Phi_\sigma^\rho\|_{C^\theta(\mathcal{S})} \leq C_\delta (1 + \ln(\rho^{-1}))$, which we prove in Proposition 5.4, we arrive at

$$\|\phi^2(t)\|_{\mathcal{H}} \leq C_\delta t \epsilon^\theta (1 + \ln(\rho^{-1})), \quad (2.35)$$

which yields the first term on the RHS of (1.32). \square

Proof of (2.30). To verify (2.30), a simple calculation shows that

$$\nabla u_s^\epsilon = e^{-isH_{\text{eff},\sigma}} \nabla u_0^\epsilon - i \int_0^s dv e^{-ivH_{\text{eff},\sigma}} \nabla V_\epsilon(x) e^{-i(s-v)H_{\text{eff},\sigma}} u_0^\epsilon. \quad (2.36)$$

Using that $\|\nabla u_0^\epsilon\|_{L^2} \leq \epsilon^\kappa$, and that

$$\|\nabla V_\epsilon\|_{L^\infty} = \|\widehat{|\nabla V_\epsilon|}\|_{L^1} \leq \gamma \epsilon, \quad (2.37)$$

we conclude that

$$\|\nabla u_s^\epsilon\|_{L^2} \leq C(\epsilon^\kappa + \epsilon s), \quad (2.38)$$

and thus, (2.30). \square

Proof of (2.33). In what follows we use the notation

$$(U\Psi)(p) = \widehat{\Psi}(p) \quad \text{and} \quad (U^{-1}\Phi)(x) = \Phi^\vee(x).$$

We define

$$\psi_s := V_\epsilon \mathcal{J}_\sigma^\rho(u_s^\epsilon) - \mathcal{J}_\sigma^\rho(V_\epsilon u_s^\epsilon). \quad (2.39)$$

Using the definition of \mathcal{J}_σ^ρ and computing the Fourier transform, we find that

$$\widehat{\psi}_s(p) = \int_{\mathbb{R}^3} dq \widehat{V}_\epsilon(p - q) \widehat{u}(s, q) (\chi_{S_\mu}(q) \Phi_\sigma^\rho(q) - \chi_{S_\mu}(p) \Phi_\sigma^\rho(p)). \quad (2.40)$$

By relations (2.32) and (2.39) and the unitarity of the generalized Fourier transform we have that

$$\|\phi^2(t)\|_{\mathcal{H}} \leq \int_0^t ds \|\psi_s\|_{L_x^2 \otimes \mathfrak{F}} = \int_0^t ds \|\widehat{\psi}_s\|_{L_p^2 \otimes \mathfrak{F}}. \quad (2.41)$$

It is important to note that, for any function $f \in L^2(\mathbb{R}^3)$ with $\text{supp}(f) \subset \mathcal{S}_\mu$,

$$\text{supp}(\widehat{V}_\epsilon * f) \subset \mathcal{S}, \quad (2.42)$$

for $\epsilon \leq \mu/3$, since we are assuming $\text{supp}(\widehat{V}) \subset \{k \mid |k| \leq 1\}$, so that $\text{supp}(\widehat{V}_\epsilon) \subset \{k \mid |k| \leq \epsilon\}$. Since the term in the integrand given by $(\widehat{u}_s^\epsilon \chi_{\mathcal{S}_\mu} \Phi_\sigma^\rho)(q)$ is supported in $q \in \mathcal{S}_\mu$, so that, by (2.42), its convolution with \widehat{V}_ϵ has support in \mathcal{S} , we find

$$\widehat{\psi}_s(p) = \mathbf{1}_{\mathcal{S}}(p) \int_{\mathbb{R}^3} dq \widehat{V}_\epsilon(p-q) \widehat{u}(s,q) (\chi_{\mathcal{S}_\mu}(q) \Phi_\sigma^\rho(q) - \chi_{\mathcal{S}_\mu}(p) \Phi_\sigma^\rho(p)), \quad (2.43)$$

for $\epsilon \leq \mu/3$, where $\mathbf{1}_{\mathcal{S}}$ is the characteristic function of the set \mathcal{S} . Inserting $|p-q|^\theta |p-q|^{-\theta} = \mathbf{1}$ into (2.43), using the definition of $|\nabla|^\theta$ by its Fourier transform and using that, since $\chi_{\mathcal{S}_\mu}$ is a smooth function,

$$\sup_{p,q \in \mathcal{S}} |p-q|^{-\theta} \|(\chi_{\mathcal{S}_\mu}(q) \Phi_\sigma^\rho(q) - \chi_{\mathcal{S}_\mu}(p) \Phi_\sigma^\rho(p))\|_{\mathfrak{F}} \leq C(1 + \|\Phi_\sigma^\rho\|_{C^\theta(\mathcal{S})}), \quad (2.44)$$

we obtain the bound $\|\widehat{\psi}_s\|_{L_x^2 \otimes \mathfrak{F}} \leq C(1 + \|\Phi_\sigma^\rho\|_{C^\theta(\mathcal{S})}) \|\mathbf{1}_{\mathcal{S}} \widehat{u}_s^\epsilon * |\widehat{\nabla}|^\theta \widehat{V}_\epsilon\|_{L^2(\mathcal{S})}$. Next, using Young's inequality, $\|f * g\|_{L^r} \leq \|f\|_{L^1} \|g\|_{L^r}$, we find that

$$\|\widehat{\psi}_s\|_{L_x^2 \otimes \mathfrak{F}} \leq C(1 + \|\Phi_\sigma^\rho\|_{C^\theta(\mathcal{S})}) \|\widehat{\nabla}|^\theta \widehat{V}_\epsilon\|_{L^1(\mathbb{R}^3)} \sup_{s \in [0,t]} \|\mathbf{1}_{\mathcal{S}} \widehat{u}_s^\epsilon\|_{L^2(\mathbb{R}^3)}. \quad (2.45)$$

Finally, observing that

$$\|\mathbf{1}_{\mathcal{S}} \widehat{u}_s^\epsilon\|_{L^2(\mathbb{R}^3)} \leq \|\widehat{u}_s^\epsilon\|_{L^2(\mathbb{R}^3)} = \|u_s^\epsilon\|_{L^2(\mathbb{R}^3)} = \|u_0^\epsilon\|_{L^2(\mathbb{R}^3)} = 1, \quad (2.46)$$

by unitarity of $e^{-itH_{\text{eff},\sigma}}$, and using (2.41), we arrive at (2.33). \square

3. THE LIMIT $\sigma \searrow 0$

In this section we remove the infrared cut-off from the evolution.

Proposition 3.1. *Under the conditions of Theorem 2.3, the strong limits*

$$s - \lim_{\sigma \searrow 0} e^{-itH_\sigma^V} \mathcal{J}_\sigma^\rho(u_0^\epsilon) = e^{-itH^V} \mathcal{J}_0^\rho(u_0^\epsilon) \quad (3.1)$$

and

$$s - \lim_{\sigma \searrow 0} \mathcal{J}_\sigma^\rho(e^{-itH_{\text{eff},\sigma}} u_0^\epsilon) = \mathcal{J}_0^\rho(e^{-itH_{\text{eff}}} u_0^\epsilon) \quad (3.2)$$

exist, for arbitrary $|t| < \infty$.

Proof. We write

$$e^{-itH_\sigma^V} \mathcal{J}_\sigma^\rho(u_0^\epsilon) - e^{-itH^V} \mathcal{J}_0^\rho(u_0^\epsilon) = (e^{-itH_\sigma^V} - e^{-itH^V}) \mathcal{J}_0^\rho(u_0^\epsilon) + e^{-itH_\sigma^V} (\mathcal{J}_\sigma^\rho - \mathcal{J}_0^\rho)(u_0^\epsilon). \quad (3.3)$$

Clearly,

$$\left\| e^{-itH_\sigma^V} (\mathcal{J}_\sigma^\rho - \mathcal{J}_0^\rho)(u_0^\epsilon) \right\| = \|(\mathcal{J}_\sigma^\rho - \mathcal{J}_0^\rho)(u_0^\epsilon)\| \leq \|u_0^\epsilon\|_{L^2} \sup_{p \in \mathcal{S}_\mu} \|\Phi_\sigma^\rho(p) - \Phi^\rho(p)\|_{\mathcal{F}}.$$

Thus,

$$\lim_{\sigma \searrow 0} \left\| e^{-itH_\sigma^V} (\mathcal{J}_\sigma^\rho - \mathcal{J}_0^\rho)(u_0^\epsilon) \right\| = 0,$$

follows from Proposition 5.1.

Next, we discuss the first term on the right side of (3.3). In order to prove that it converges to 0, as $\sigma \searrow 0$, it suffices to show that H_σ^V converges to H^V in the norm resolvent sense; (see [24, Theorem VIII.21]), i.e.,

$$\lim_{\sigma \searrow 0} \|(H_\sigma^V + i)^{-1} - (H^V + i)^{-1}\| = 0.$$

From the second resolvent equation and the fact that $\|(H_\sigma^V + i)^{-1}\| \leq 1$, it follows that

$$\|(H_\sigma^V + i)^{-1} - (H^V + i)^{-1}\| = \|(H^V + i)^{-1} Q_\sigma (H_\sigma^V + i)^{-1}\|, \quad (3.4)$$

where

$$Q_\sigma := H^V - H_\sigma^V = \alpha^{\frac{1}{2}} A_{<\sigma}(x) \cdot v_\sigma + \frac{\alpha}{2} (A_{<\sigma}(x))^2,$$

and

$$v_\sigma := -i\nabla_x + \alpha^{\frac{1}{2}} A_\sigma(x)$$

is the velocity operator. Here $A_\sigma(x)$ is defined in (2.3), and

$$A_{<\sigma}(x) := \sum_\lambda \int_{|k| \leq \sigma} \frac{dk}{|k|^{1/2}} \{ \epsilon_\lambda(k) e^{-ikx} \otimes a_\lambda(k) + h.c. \}. \quad (3.5)$$

In order to estimate the norm of $Q_\sigma (H^V + i)^{-1}$, we use the following well-known lemma.

Lemma 3.2. *Let $f, g \in L^2(\mathbb{R}^3 \times \{+, -\}; \mathcal{B}(\mathcal{H}_{\text{el}}))$ be operator-valued functions such that $\|(1 + |k|^{-1})^{1/2} f\|, \|(1 + |k|^{-1})^{1/2} g\| < \infty$. Then*

$$\|a^\#(f)(H_f + 1)^{-\frac{1}{2}}\| \leq \|(1 + |k|^{-1})^{\frac{1}{2}} f\|_{L^2}, \quad (3.6)$$

$$\|a^\#(f)a^\#(g)(H_f + 1)^{-1}\| \leq \|(1 + |k|^{-1})^{\frac{1}{2}} f\|_{L^2} \|(1 + |k|^{-1})^{\frac{1}{2}} g\|_{L^2}, \quad (3.7)$$

where $a^\#$ stands for a or a^* .

In particular, using the Kato-Rellich theorem, one easily shows that, for α small enough, $D(H^V) = D(-\Delta_x \otimes I + I \otimes H_f) \subset D(H_f)$. Thus, we have that

$$\|(H_f + 1)(H^V + i)^{-1}\| \leq C,$$

which when combined with Lemma 3.2 yields

$$\left\| \frac{\alpha}{2} (A_{<\sigma}(x))^2 (H^V + i)^{-1} \right\| \leq C \alpha \sigma. \quad (3.8)$$

Likewise one verifies that

$$\left\| \alpha^{\frac{1}{2}} A_{<\sigma}(x) \cdot v_\sigma (H^V + i)^{-1} \right\| \leq C \alpha^{\frac{1}{2}} \sigma^{\frac{1}{2}}, \quad (3.9)$$

since $0 \leq v_\sigma^2 \leq H^V + \|V\|_{L^\infty}$ is bounded relative to H^V . Estimates (3.8) and (3.9) yield

$$\|Q_\sigma (H^V + i)^{-1}\| \leq C \alpha^{\frac{1}{2}} \sigma^{\frac{1}{2}}.$$

By (3.4), we have shown that H_σ^V converges to H^V , as $\sigma \searrow 0$, in the norm resolvent sense. \square

4. PROOF OF THEOREM 1.1

In this section, we prove the bound in Theorem 1.1, which compares the full dynamics to the effective dynamics for the system without infrared cutoff. We have that

$$\begin{aligned} & \| e^{-itH^V} \mathcal{J}_0^\rho(u_0^\epsilon) - \mathcal{J}_0^\rho(e^{-itH_{\text{eff}}} u_0^\epsilon) \|_{\mathcal{H}} \\ & \leq \| e^{-itH_\sigma^V} \mathcal{J}_\sigma^\rho(u_0^\epsilon) - \mathcal{J}_\sigma^\rho(e^{-itH_{\text{eff},\sigma}} u_0^\epsilon) \|_{\mathcal{H}} \\ & \quad + \| e^{-itH_\sigma^V} \mathcal{J}_\sigma^\rho(u_0^\epsilon) - e^{-itH^V} \mathcal{J}_0^\rho(u_0^\epsilon) \|_{\mathcal{H}} \\ & \quad + \| \mathcal{J}_\sigma^\rho(e^{-itH_{\text{eff},\sigma}} u_0^\epsilon) - \mathcal{J}_0^\rho(e^{-itH_{\text{eff}}} u_0^\epsilon) \|_{\mathcal{H}}, \end{aligned} \quad (4.1)$$

for any t and $0 < \sigma < \rho \leq 1$. It follows from Theorem 2.3 that the first term on the r.s. of the inequality sign is bounded by $C_\delta (1 + \ln(\rho^{-1})) \epsilon^{\frac{2}{3}-\delta} t + C \alpha^{\frac{1}{2}} \rho^{\frac{1}{2}} t (\epsilon^\kappa + \epsilon t)$, uniformly in $\sigma > 0$.

From Proposition 3.1, it follows that the second and third term on the r.s. converge to zero, as $\sigma \searrow 0$. By taking σ to zero, we thus conclude that

$$\| e^{-itH^V} \mathcal{J}_0^\rho(u_0^\epsilon) - \mathcal{J}_0^\rho(e^{-itH_{\text{eff}}} u_0^\epsilon) \|_{\mathcal{H}} \leq C_\delta (1 + \ln(\rho^{-1})) \epsilon^{\frac{2}{3}-\delta} t + C \alpha^{\frac{1}{2}} \rho^{\frac{1}{2}} t (\epsilon^\kappa + \epsilon t). \quad (4.2)$$

Due to our choice $\rho = \epsilon^{\frac{2}{3}-\delta}$, this concludes the proof of Theorem 1.1. We note that in the inequality (1.21), the logarithmic term $\ln(\rho_\epsilon^{-1})$ has been absorbed by an arbitrary small shift of δ , which we do not keep track of notationally. \square

5. HÖLDER CONTINUITY OF THE GROUND STATE

We recall that $\Phi_\sigma^\rho(p)$ denotes a normalized ground state of the Bogoliubov transformed fiber Hamiltonian $K_\sigma^\rho(p) = W_{\nabla E_\sigma(p)}^\rho H_\sigma(p) (W_{\nabla E_\sigma(p)}^\rho)^*$, with infrared cutoff $\sigma > 0$ (see (2.15)). Our aim in this appendix is to prove that, for a suitable choice of the vectors $\Phi_\sigma^\rho(p)$, the map $p \mapsto \Phi_\sigma^\rho(p)$ is θ -Hölder continuous, for $\theta < 2/3$.

For $\rho = 1$, we set

$$\Phi_\sigma(p) := \Phi_\sigma^1(p), \quad K_\sigma(p) := K_\sigma^1(p). \quad (5.1)$$

We remark that

$$K_\sigma^\rho(p) = (W_{\nabla E_\sigma(p)}^{\rho,1})^* K_\sigma(p) W_{\nabla E_\sigma(p)}^{\rho,1}, \quad \Phi_\sigma^\rho(p) = (W_{\nabla E_\sigma(p)}^{\rho,1})^* \Phi_\sigma(p), \quad (5.2)$$

where we recall that $W_{\nabla E_\sigma(p)}^{\rho,1}$ is defined in (2.13).

Letting

$$\mathfrak{F}_\sigma := \bigoplus_{n \geq 0} \text{Sym}(L^2(\{k \in \mathbb{R}^3, |k| \geq \sigma\} \times \{+, -\}))^{\otimes n} \quad (5.3)$$

denote the Fock space of photons of energies $\geq \sigma$, and identifying \mathfrak{F}_σ with a subspace of \mathfrak{F} , we observe that $K_\sigma(p)$ leaves \mathfrak{F}_σ invariant. Let $\tilde{K}_\sigma(p)$ denote the restriction of $K_\sigma(p)$ to \mathfrak{F}_σ . An important property, proven in [3, 10, 13], is that there is an energy gap of size $\eta\sigma$, where $\eta > 0$ is

uniform in $\sigma \searrow 0$, in the spectrum of $\tilde{K}_\sigma(p)$ above the ground state energy $E_\sigma(p)$. Moreover, one can choose

$$\Phi_\sigma(p) = \tilde{\Phi}_\sigma(p) \otimes \Omega_{<\sigma}, \quad (5.4)$$

in the representation $\mathfrak{F} \simeq \mathfrak{F}_\sigma \otimes \mathfrak{F}_{<\sigma}$, where

$$\mathfrak{F}_{<\sigma} := \bigoplus_{n \geq 0} \text{Sym}(L^2(\{k \in \mathbb{R}^3, |k| \leq \sigma\} \times \{+, -\}))^{\otimes n}. \quad (5.5)$$

Now, let Ω_σ denote the vacuum sector in \mathfrak{F}_σ and $\tilde{\Pi}_\sigma(p)$ be the rank-one projection onto the eigenspace associated with $E_\sigma(p) = \inf \text{spec}(\tilde{K}_\sigma(p))$. By [10, 13],

$$\|\tilde{\Pi}_\sigma(p)\Omega_\sigma\| \geq \frac{1}{3}, \quad (5.6)$$

for arbitrary $\sigma > 0$ and $|p| \leq 1/3$ provided that α is chosen sufficiently small. Then $\tilde{\Phi}_\sigma(p)$ can be chosen in the following way:

$$\tilde{\Phi}_\sigma(p) = \frac{\tilde{\Pi}_\sigma(p)\Omega_\sigma}{\|\tilde{\Pi}_\sigma(p)\Omega_\sigma\|}. \quad (5.7)$$

Let N denote the number operator,

$$N = \sum_\lambda \int dk b_\lambda^*(k) b_\lambda(k). \quad (5.8)$$

The following proposition has been proven in [8, 10, 13].

Proposition 5.1. *For $\alpha \ll 1$ and $|p| \leq 1/3$, there exists a normalized vector $\Phi(p)$ in the Fock space \mathfrak{F} such that $\Phi_\sigma(p) \rightarrow \Phi(p)$, strongly, as $\sigma \rightarrow 0$. The following bound holds,*

$$\|N^{\frac{1}{2}}\Phi_\sigma(p)\| \leq C < \infty, \quad (5.9)$$

uniformly in $\sigma \geq 0$. Moreover, For all $\delta > 0$, there exists $\alpha_\delta > 0$ and $C_\delta < \infty$ such that, for all $0 \leq \alpha \leq \alpha_\delta$, $0 \leq \sigma' < \sigma \leq 1$ and $|p| \leq 1/3$,

$$\|\Phi_\sigma(p) - \Phi_{\sigma'}(p)\| \leq C_\delta \alpha^{\frac{1}{4}} \sigma^{1-\delta}, \quad (5.10)$$

$$|\nabla E_\sigma(p) - \nabla E_{\sigma'}(p)| \leq C_\delta \alpha^{\frac{1}{4}} \sigma^{1-\delta}. \quad (5.11)$$

As a consequence, we show the following corollary.

Corollary 5.2. *Let $0 < \rho < 1$. For all $\delta > 0$, there exists $\alpha_\delta > 0$ such that, for all $0 \leq \alpha \leq \alpha_\delta$ and $|p| \leq 1/3$, there exists a vector $\Phi^\rho(p)$ in the Fock space such that $\Phi_\sigma^\rho(p) \rightarrow \Phi^\rho(p)$, strongly, as $\sigma \rightarrow 0$. Moreover, there exists a constant $C_\delta < \infty$ such that, for all $0 \leq \alpha \leq \alpha_\delta$, $0 \leq \sigma' < \sigma \leq 1$ and $|p| \leq 1/3$,*

$$\|\Phi_\sigma^\rho(p) - \Phi_{\sigma'}^\rho(p)\| \leq C_\delta \alpha^{\frac{1}{4}} \sigma^{1-\delta} (1 + \ln(\rho^{-1})). \quad (5.12)$$

Proof. Using (5.2), we split

$$\Phi_\sigma^\rho(p) - \Phi_{\sigma'}^\rho(p) = \left((W_{\nabla E_\sigma(p)}^{\rho,1})^* - (W_{\nabla E_{\sigma'}(p)}^{\rho,1})^* \right) \Phi_\sigma(p) + (W_{\nabla E_{\sigma'}(p)}^{\rho,1})^* (\Phi_\sigma(p) - \Phi_{\sigma'}(p)). \quad (5.13)$$

By Proposition 5.1 and unitarity of $W_{\nabla E_\sigma(p)}^{\rho,1}$, the second term is estimated as

$$\left\| (W_{\nabla E_{\sigma'}(p)}^{\rho,1})^* (\Phi_\sigma(p) - \Phi_{\sigma'}(p)) \right\| \leq C_\delta \alpha^{\frac{1}{4}} \sigma^{1-\delta}. \quad (5.14)$$

The first term in the right side of (5.13) is estimated as

$$\begin{aligned} \left\| \left((W_{\nabla E_\sigma(p)}^{\rho,1})^* - (W_{\nabla E_{\sigma'}(p)}^{\rho,1})^* \right) \Phi_\sigma(p) \right\| &= \left\| \left(\mathbf{1} - W_{\nabla E_\sigma(p)}^{\rho,1} (W_{\nabla E_{\sigma'}(p)}^{\rho,1})^* \right) \Phi_\sigma(p) \right\| \\ &\leq \|B(\rho) \Phi_\sigma(p)\|, \end{aligned} \quad (5.15)$$

by unitarity of $W_{\nabla E_\sigma(p)}^{\rho,1}$ and the spectral theorem, where

$$B(\rho) := \alpha^{\frac{1}{2}} \sum_\lambda \int_{\rho \leq |k| \leq 1} dk \left(\frac{\nabla E_\sigma(p) \cdot \epsilon_\lambda(k) b_\lambda(k) - h.c.}{|k|^{1/2} (|k| - \nabla E_\sigma(p) \cdot k)} - \frac{\nabla E_{\sigma'}(p) \cdot \epsilon_\lambda(k) b_\lambda(k) - h.c.}{|k|^{1/2} (|k| - \nabla E_{\sigma'}(p) \cdot k)} \right). \quad (5.16)$$

To estimate $\|B(\rho) \Phi_\sigma(p)\|$, we use the well known fact that, for any $f \in L^2(\mathbb{R}^3 \times \{+, -\})$,

$$\|a^\#(f)(N+1)^{-\frac{1}{2}}\| \leq \sqrt{2} \|f\|_{L^2}. \quad (5.17)$$

Clearly,

$$\begin{aligned} &\frac{\nabla E_\sigma(p) \cdot \epsilon_\lambda(k)}{|k|^{1/2} (|k| - \nabla E_\sigma(p) \cdot k)} - \frac{\nabla E_{\sigma'}(p) \cdot \epsilon_\lambda(k)}{|k|^{1/2} (|k| - \nabla E_{\sigma'}(p) \cdot k)} \\ &= \frac{(\nabla E_\sigma(p) - \nabla E_{\sigma'}(p)) \cdot \epsilon_\lambda(k)}{|k|^{1/2} (|k| - \nabla E_\sigma(p) \cdot k)} + \frac{\nabla E_{\sigma'}(p) \cdot \epsilon_\lambda(k)}{|k|^{1/2} (|k| - \nabla E_\sigma(p) \cdot k)} \frac{(\nabla E_\sigma(p) - \nabla E_{\sigma'}(p)) \cdot k}{(|k| - \nabla E_{\sigma'}(p) \cdot k)}. \end{aligned} \quad (5.18)$$

Hence, by (5.11) and the facts that $|\nabla E_\sigma(p)|, |\nabla E_{\sigma'}(p)| \leq 1/2$ for α small enough (see Proposition 2.1 (2)), we obtain

$$\left| \frac{\nabla E_\sigma(p) \cdot \epsilon_\lambda(k)}{|k|^{1/2} (|k| - \nabla E_\sigma(p) \cdot k)} - \frac{\nabla E_{\sigma'}(p) \cdot \epsilon_\lambda(k)}{|k|^{1/2} (|k| - \nabla E_{\sigma'}(p) \cdot k)} \right| \leq \frac{2C_\delta \alpha^{\frac{1}{4}} \sigma^{1-\delta}}{|k|^{\frac{3}{2}}}. \quad (5.19)$$

Thus, (5.16) and (5.17) yield that

$$\begin{aligned} \|B(\rho) \Phi_\sigma(p)\| &\leq 2CC_\delta \alpha^{\frac{3}{4}} \sigma^{1-\delta} \left\| \frac{\mathbf{1}_{\rho \leq |k| \leq 1}(|k|)}{|k|^{\frac{3}{2}}} \right\|_{L_k^2} \|(N+1)^{\frac{1}{2}} \Phi_\sigma(p)\| \\ &\leq 2CC_\delta \alpha^{\frac{3}{4}} \sigma^{1-\delta} \ln(\rho^{-1}). \end{aligned} \quad (5.20)$$

where we used (5.9) in the last inequality. Together with (5.13) – (5.15), this concludes the proof of Corollary 5.2. \square

The following result follows from [10, 13] (it is also a consequence of (2.10) in Proposition 2.1 (2)).

Proposition 5.3. *There exist $\alpha_c > 0$ and $C > 0$ such that, for all $0 \leq \alpha \leq \alpha_c$ and p, p' satisfying $|p| \leq 1/3, |p'| \leq 1/3$,*

$$|\nabla E_\sigma(p) - \nabla E_\sigma(p')| \leq C |p - p'|, \quad (5.21)$$

uniformly in $\sigma > 0$.

We now prove the following proposition.

Proposition 5.4. *Let $0 < \rho < 1$. For all $\delta > 0$, there exist $\alpha_\delta > 0$ and $C_\delta < \infty$ such that, for all $0 \leq \alpha \leq \alpha_\delta$, $\sigma > 0$ and $p, k \in \mathbb{R}^3$ satisfying $|p| \leq 1/3$, $|p+k| \leq 1/3$,*

$$\|\Phi_\sigma^\rho(p+k) - \Phi_\sigma^\rho(p)\| \leq C_\delta (1 + \ln(\rho^{-1})) |k|^{\frac{2}{3}-\delta}. \quad (5.22)$$

Proof.

Step 1. We first prove that, for all $0 < \sigma < \rho \leq 1$,

$$\|\Phi_\sigma^\rho(p+k) - \Phi_\sigma^\rho(p)\| \leq C |k| \sigma^{-\frac{1}{2}}. \quad (5.23)$$

We decompose

$$\begin{aligned} \Phi_\sigma^\rho(p+k) - \Phi_\sigma^\rho(p) &= (W_{\nabla E_\sigma(p+k)}^{\rho,1})^* \Phi_\sigma(p+k) - (W_{\nabla E_\sigma(p)}^{\rho,1})^* \Phi_\sigma(p) \\ &= \left((W_{\nabla E_\sigma(p+k)}^{\rho,1})^* - (W_{\nabla E_\sigma(p)}^{\rho,1})^* \right) \Phi_\sigma(p) \\ &\quad + (W_{\nabla E_\sigma(p+k)}^{\rho,1})^* (\Phi_\sigma(p+k) - \Phi_\sigma(p)). \end{aligned} \quad (5.24)$$

To estimate the first term in the right side of (5.24), we proceed as in the proof of Corollary 5.2. Namely, we have that

$$\begin{aligned} \left\| \left((W_{\nabla E_\sigma(p+k)}^{\rho,1})^* - (W_{\nabla E_\sigma(p)}^{\rho,1})^* \right) \Phi_\sigma(p) \right\| &= \left\| \left(\mathbf{1} - W_{\nabla E_\sigma(p+k)}^{\rho,1} (W_{\nabla E_\sigma(p)}^{\rho,1})^* \right) \Phi_\sigma(p) \right\| \\ &\leq \|C(\rho) \Phi_\sigma(p)\|, \end{aligned} \quad (5.25)$$

by the spectral theorem, where

$$C(\rho) := \alpha^{\frac{1}{2}} \sum_\lambda \int_{\rho \leq |\tilde{k}| \leq 1} d\tilde{k} \left(\frac{\nabla E_\sigma(p+k) \cdot \epsilon_\lambda(\tilde{k}) b_\lambda(\tilde{k}) - h.c.}{|\tilde{k}|^{1/2} (|\tilde{k}| - \nabla E_\sigma(p+k) \cdot \tilde{k})} - \frac{\nabla E_\sigma(p) \cdot \epsilon_\lambda(\tilde{k}) b_\lambda(\tilde{k}) - h.c.}{|\tilde{k}|^{1/2} (|\tilde{k}| - \nabla E_\sigma(p) \cdot \tilde{k})} \right).$$

Using Proposition 5.3, one verifies that

$$\left| \frac{\nabla E_\sigma(p+k) \cdot \epsilon_\lambda(\tilde{k})}{|\tilde{k}|^{1/2} (|\tilde{k}| - \nabla E_\sigma(p+k) \cdot \tilde{k})} - \frac{\nabla E_\sigma(p) \cdot \epsilon_\lambda(\tilde{k})}{|\tilde{k}|^{1/2} (|\tilde{k}| - \nabla E_\sigma(p) \cdot \tilde{k})} \right| \leq \frac{C |k|}{|\tilde{k}|^{\frac{3}{2}}}. \quad (5.26)$$

Hence (5.17) implies that

$$\begin{aligned} \|C(\rho) \Phi_\sigma(p)\| &\leq C |k| \left\| \frac{\mathbf{1}_{\rho \leq |\tilde{k}| \leq 1}(\tilde{k})}{|\tilde{k}|^{\frac{3}{2}}} \right\|_{L_k^2} \|(N+1)^{\frac{1}{2}} \Phi_\sigma(p)\| \\ &\leq C |k| \ln(\rho^{-1}), \end{aligned} \quad (5.27)$$

where we used (5.9) in the last inequality. Equations (5.25) and (5.27) yield

$$\begin{aligned} \left\| \left((W_{\nabla E_\sigma(p+k)}^{\rho,1})^* - (W_{\nabla E_\sigma(p)}^{\rho,1})^* \right) \Phi_\sigma(p) \right\| &= \left\| \left(\mathbf{1} - W_{\nabla E_\sigma(p+k)}^{\rho,1} (W_{\nabla E_\sigma(p)}^{\rho,1})^* \right) \Phi_\sigma(p) \right\| \\ &\leq C |k| \ln(\rho^{-1}). \end{aligned} \quad (5.28)$$

It remains to estimate the second term in the right side of (5.24). By unitarity of $W_{\nabla E_\sigma(p+k)}^{\rho,1}$, it suffices to estimate

Using (5.6) and the relation

$$\begin{aligned}
\|(\tilde{\Pi}_\sigma(p) - \tilde{\Pi}_\sigma(p+k))\varphi\|^2 &= \langle \varphi, (\tilde{\Pi}_\sigma(p+k) + \tilde{\Pi}_\sigma(p) - \tilde{\Pi}_\sigma(p)\tilde{\Pi}_\sigma(p+k) - \tilde{\Pi}_\sigma(p+k)\tilde{\Pi}_\sigma(p))\varphi \rangle \\
&= \langle \varphi, (\tilde{\Pi}_\sigma^\perp(p+k)\tilde{\Pi}_\sigma(p) + \tilde{\Pi}_\sigma^\perp(p)\tilde{\Pi}_\sigma(p+k))\varphi \rangle, \\
&= \langle \varphi, (\tilde{\Pi}_\sigma(p)\tilde{\Pi}_\sigma^\perp(p+k)\tilde{\Pi}_\sigma(p) + \tilde{\Pi}_\sigma^\perp(p)\tilde{\Pi}_\sigma(p+k)\tilde{\Pi}_\sigma^\perp(p))\varphi \rangle, \\
&= \|\tilde{\Pi}_\sigma^\perp(p+k)\tilde{\Pi}_\sigma(p)\varphi\|^2 + \|\tilde{\Pi}_\sigma(p+k)\tilde{\Pi}_\sigma^\perp(p)\varphi\|^2,
\end{aligned}$$

for any $\varphi \in \mathfrak{F}_\sigma$, where $\tilde{\Pi}_\sigma^\perp(p) := I - \tilde{\Pi}_\sigma(p)$, we obtain that

$$\begin{aligned}
\|\Phi_\sigma(p+k) - \Phi_\sigma(p)\| &= \|\tilde{\Phi}_\sigma(p+k) - \tilde{\Phi}_\sigma(p)\| \\
&\leq \frac{2}{\|\tilde{\Pi}_\sigma(p)\Omega_\sigma\|} \|(\tilde{\Pi}_\sigma(p) - \tilde{\Pi}_\sigma(p+k))\Omega_\sigma\| \\
&\leq 6\|\tilde{\Pi}_\sigma(p) - \tilde{\Pi}_\sigma(p+k)\| \\
&\leq 6(\|\tilde{\Pi}_\sigma^\perp(p+k)\tilde{\Pi}_\sigma(p)\| + \|\tilde{\Pi}_\sigma^\perp(p)\tilde{\Pi}_\sigma(p+k)\|) \\
&\leq 6(\|\tilde{\Pi}_\sigma^\perp(p+k)\tilde{\Phi}_\sigma(p)\| + \|\tilde{\Pi}_\sigma^\perp(p)\tilde{\Phi}_\sigma(p+k)\|). \tag{5.29}
\end{aligned}$$

Since there is an energy gap of size $\eta\sigma$ above $E_\sigma(p+k)$ in the spectrum of the operator $\tilde{K}_\sigma(p+k)$, we can estimate

$$\tilde{\Pi}_\sigma^\perp(p+k) \leq \frac{1}{\eta\sigma} (\tilde{K}_\sigma(p+k) - E_\sigma(p+k)),$$

and hence

$$\|\tilde{\Pi}_\sigma^\perp(p+k)\tilde{\Phi}_\sigma(p)\| \leq \frac{2}{\eta^{1/2}\sigma^{1/2}} \|(\tilde{K}_\sigma(p+k) - E_\sigma(p+k))^{1/2}\tilde{\Phi}_\sigma(p)\|. \tag{5.30}$$

We have by (2.15), (5.1), the definition after (5.3) and (2.6)

$$\tilde{K}_\sigma(p+k) = \tilde{K}_\sigma(p) + k \cdot \nabla_p \tilde{K}_\sigma(p) + k^2/2. \tag{5.31}$$

Using this expansion and the Feynman-Hellman formula,

$$\langle \tilde{\Phi}_\sigma(p), \nabla_p \tilde{K}_\sigma(p) \tilde{\Phi}_\sigma(p) \rangle = \nabla E_\sigma(p), \tag{5.32}$$

together with the mean-value theorem and Proposition 5.3, we have that (see also [7, Lemma 3.6])

$$\begin{aligned}
&\|(\tilde{K}_\sigma(p+k) - E_\sigma(p+k))^{1/2}\tilde{\Phi}_\sigma(p)\|^2 \\
&= \langle \tilde{\Phi}_\sigma(p), (\tilde{K}_\sigma(p+k) - E_\sigma(p+k))\tilde{\Phi}_\sigma(p) \rangle \\
&= \langle \tilde{\Phi}_\sigma(p), (\tilde{K}_\sigma(p) + k \cdot (\nabla_p \tilde{K}_\sigma(p)) + k^2/2 - E_\sigma(p+k))\tilde{\Phi}_\sigma(p) \rangle \\
&= E_\sigma(p) - E_\sigma(p+k) + k \cdot (\nabla_p E_\sigma(p)) + k^2/2 \\
&= \frac{1}{2}k^2 + \int_0^1 k \cdot [\nabla_p E_\sigma(p) - \nabla_p E_\sigma(p + \tau k)] d\tau \\
&\leq C k^2. \tag{5.33}
\end{aligned}$$

Hence,

$$\|(\tilde{K}_\sigma(p+k) - E_\sigma(p+k))^{1/2}\tilde{\Phi}_\sigma(p)\| \leq C |k|. \tag{5.34}$$

Combining (5.30) and (5.34), we obtain that

$$\|\tilde{\Pi}_\sigma^\perp(p+k)\tilde{\Phi}_\sigma(p)\| \leq C|k|\sigma^{-\frac{1}{2}}. \quad (5.35)$$

Proceeding in the same way, it follows likewise that

$$\|\tilde{\Pi}_\sigma^\perp(p)\tilde{\Phi}_\sigma(p+k)\| \leq C|k|\sigma^{-\frac{1}{2}}, \quad (5.36)$$

and hence, by (5.29), (5.23) follows.

Step 2. We now prove that $\|\Phi_\sigma^\rho(p+k) - \Phi_\sigma^\rho(p)\| \leq C_\delta |k|^{2/3-\delta}$ (with $C_\delta < \infty$ for $\delta > 0$).

Suppose first that $\sigma \geq |k|^{2/3}$. Then by Step 1, we have that

$$\|\Phi_\sigma^\rho(p+k) - \Phi_\sigma^\rho(p)\| \leq C|k||k|^{-\frac{1}{3}} = C|k|^{\frac{2}{3}}. \quad (5.37)$$

Conversely, assume that $\sigma \leq |k|^{2/3}$. We write

$$\begin{aligned} \|\Phi_\sigma^\rho(p+k) - \Phi_\sigma^\rho(p)\| &\leq \|\Phi_\sigma^\rho(p+k) - \Phi^\rho(p+k)\| + \|\Phi^\rho(p+k) - \Phi_{|k|^{2/3}}^\rho(p+k)\| \\ &\quad + \|\Phi_\sigma^\rho(p) - \Phi^\rho(p)\| + \|\Phi^\rho(p) - \Phi_{|k|^{2/3}}^\rho(p)\| \\ &\quad + \|\Phi_{|k|^{2/3}}^\rho(p+k) - \Phi_{|k|^{2/3}}^\rho(p)\|. \end{aligned} \quad (5.38)$$

By Corollary 5.2, the first two lines are bounded by

$$\begin{aligned} &\|\Phi_\sigma^\rho(p+k) - \Phi^\rho(p+k)\| + \|\Phi^\rho(p+k) - \Phi_{|k|^{2/3}}^\rho(p+k)\| \\ &+ \|\Phi_\sigma^\rho(p) - \Phi^\rho(p)\| + \|\Phi^\rho(p) - \Phi_{|k|^{2/3}}^\rho(p)\| \\ &\leq C_\delta \alpha^{\frac{1}{4}} (1 + \ln(\rho^{-1})) |k|^{\frac{2}{3}(1-\delta)}, \end{aligned} \quad (5.39)$$

whereas by Step 1, the last term is bounded by $C|k|^{2/3}$. Setting $\delta' = 2\delta/3$ and changing notations concludes the proof of the proposition. \square

APPENDIX A. PROOF OF ESTIMATE (2.29)

In this Appendix, we prove (2.29). It asserts that

$$\|(K_\sigma^\rho(p) - H_\sigma(p))\Phi_\sigma^\rho(p)\|_{\mathfrak{F}} \leq C \alpha^{\frac{1}{2}} \rho^{\frac{1}{2}} |p|, \quad (A.1)$$

for all $p \in \mathcal{S}$, for a constant $C < \infty$ independent of α , σ , and ρ , where $0 < \sigma < \rho \leq 1$.

To begin with, let

$$v_\lambda^\sharp(k) := \alpha^{\frac{1}{2}} \mathbf{1}_{\sigma \leq |k| \leq \rho}(|k|) \frac{\nabla E_\sigma(p) \cdot \epsilon_\lambda^\sharp(k)}{|k|^{1/2}(|k| - \nabla E_\sigma(p) \cdot k)}, \quad (A.2)$$

(scalar-valued) and

$$w_\lambda^\sharp(k) := \alpha^{\frac{1}{2}} \mathbf{1}_{\sigma \leq |k| \leq 1}(|k|) \frac{\epsilon_\lambda^\sharp(k)}{|k|^{1/2}} \quad (A.3)$$

(vector-valued). We note that

$$|v_\lambda(k)| \leq C \alpha^{\frac{1}{2}} |p| \frac{\mathbf{1}_{\sigma \leq |k| \leq \rho}(|k|)}{|k|^{\frac{3}{2}}} \quad (\text{A.4})$$

and

$$|w_\lambda(k)| \leq C \alpha^{\frac{1}{2}} \frac{\mathbf{1}_{\sigma \leq |k| \leq 1}(|k|)}{|k|^{\frac{1}{2}}} \quad (\text{A.5})$$

where we have used that $|\nabla E_\sigma(p)| \leq C|p|$, uniformly in the infrared cutoff $0 \leq \sigma \leq 1$.

Using that

$$W_{\nabla E_\sigma(p)}^{\sigma, \rho} b_\lambda^\sharp(k) (W_{\nabla E_\sigma(p)}^{\sigma, \rho})^* = b_\lambda^\sharp(k) + v_\lambda^\sharp(k), \quad (\text{A.6})$$

a straightforward calculation yields

$$\begin{aligned} K_\sigma^\rho(p) - H_\sigma(p) &= W_{\nabla E_\sigma(p)}^{\sigma, \rho} H_\sigma(p) (W_{\nabla E_\sigma(p)}^{\sigma, \rho})^* - H_\sigma(p) \\ &= 2V(p) \cdot (\nabla_p H_\sigma(p)) + V^2(p) + Y(p), \end{aligned} \quad (\text{A.7})$$

where

$$\nabla_p H_\sigma(p) = p - P_f - \alpha^{\frac{1}{2}} A_\sigma, \quad (\text{A.8})$$

with

$$A_\sigma = \sum_\lambda (b_\lambda(w_\lambda) + b_\lambda^*(w_\lambda)), \quad (\text{A.9})$$

and

$$V(p) := \sum_\lambda \left[b_\lambda(kv_\lambda) + b_\lambda^*(kv_\lambda) + 2\text{Re}(w_\lambda, v_\lambda) + (v_\lambda, kv_\lambda) \right], \quad (\text{A.10})$$

(vector-valued operator) and

$$Y(p) := \sum_\lambda \left[b_\lambda((k^2 + |k|)v_\lambda) + b_\lambda^*((k^2 + |k|)v_\lambda) + (v_\lambda, |k|v_\lambda) + 2\text{Re}(k \cdot w_\lambda, v_\lambda) \right], \quad (\text{A.11})$$

(scalar-valued operator). Note that both $V(p)$ and $Y(p)$ are proportional to $|\nabla E_\sigma(p)|$ since all terms are of first or higher order in v_λ (which is proportional to $|\nabla E_\sigma(p)| \leq C|p|$).

Indeed, using Lemma 3.2 and (A.4), we observe that

$$\begin{aligned} \|V(p)(H_f + 1)^{-1/2}\| &\leq 2\|(|k| + |k|^2)^{\frac{1}{2}} v_\lambda\|_{L^2} + \| |k|^{\frac{1}{2}} v_\lambda \|_{L^2}^2 + \|w_\lambda v_\lambda\|_{L^1} \\ &\leq C \alpha^{1/2} |p| \rho^{1/2}, \end{aligned} \quad (\text{A.12})$$

and similarly

$$\|V(p)^2(H_f + 1)^{-1}\| \leq C \alpha |p|^2 \rho, \quad (\text{A.13})$$

$$\|Y(p)(H_f + 1)^{-1/2}\| \leq C \alpha^{1/2} |p| \rho. \quad (\text{A.14})$$

Next we note that for any normalized vector $\Phi \in D(H(p))$, we have the estimate

$$\begin{aligned} \left\| \left(\frac{1}{2}(p - P_f)^2 + H_f + 1 \right) \Phi \right\| &\leq \left\| (H_\sigma(p) + 1) \Phi \right\| + \alpha^{1/2} \left\| A_\sigma \cdot \nabla H_\sigma(p) \Phi \right\| + \alpha \left\| A_\sigma^2 \Phi \right\| \\ &\leq \left\| (H_\sigma(p) + 1) \Phi \right\| + C \alpha^{1/2} \left\| (H_f + 1)^{1/2} \nabla H_\sigma(p) \Phi \right\| \\ &\quad + C \alpha \left\| (H_f + 1) \Phi \right\|. \end{aligned} \quad (\text{A.15})$$

Since furthermore P_f and H_f commute, we have that $(H_f + 1)^2 \leq (\frac{1}{2}(p - P_f)^2 + H_f + 1)^2$ and hence

$$\left\| (H_f + 1) \Phi \right\| \leq 2 \left\| (H_\sigma(p) + 1) \Phi \right\| + C \alpha^{1/2} \left\| (H_f + 1)^{1/2} \nabla H_\sigma(p) \Phi \right\|, \quad (\text{A.16})$$

provided $\alpha > 0$ is sufficiently small. Now, we observe that

$$\left\| [H_f, \nabla H_\sigma(p)] (H_f + 1)^{-1/2} \right\| = \alpha^{1/2} \left\| [H_f, A_\sigma] (H_f + 1)^{-1/2} \right\| \leq C \alpha^{1/2}, \quad (\text{A.17})$$

which implies that

$$\begin{aligned} \left\| (H_f + 1)^{1/2} \nabla H_\sigma(p) \Phi \right\|^2 &= \left\langle \nabla H_\sigma(p) \Phi, (H_f + 1) \nabla H_\sigma(p) \Phi \right\rangle \\ &= \left\langle \nabla H_\sigma(p)^2 \Phi, (H_f + 1) \Phi \right\rangle - \left\langle [H_f, \nabla H_\sigma(p)] \Phi, \nabla H_\sigma(p) \Phi \right\rangle \\ &\leq C \left\| H_\sigma(p) \Phi \right\| \left\| (H_f + 1) \Phi \right\| + C \alpha^{1/2} \left\| (H_f + 1)^{1/2} \nabla H_\sigma(p) \Phi \right\|. \end{aligned} \quad (\text{A.18})$$

Hence, for sufficiently small $\alpha > 0$, we have that

$$\left\| (H_f + 1)^{1/2} \nabla H_\sigma(p) \Phi \right\| \leq C \left\| H_\sigma(p) \Phi \right\|^{1/2} \left\| (H_f + 1) \Phi \right\|^{1/2}. \quad (\text{A.19})$$

Inserting this estimate into (A.16), we obtain for all normalized Φ that

$$\left\| (H_f + 1) \Phi \right\| \leq C \left\| (H_\sigma(p) + 1) \Phi \right\|, \quad (\text{A.20})$$

and, additionally using (A.19), that

$$\left\| (H_f + 1)^{1/2} \nabla H_\sigma(p) \Phi \right\| \leq C \left\| (H_\sigma(p) + 1) \Phi \right\|, \quad (\text{A.21})$$

provided $\alpha > 0$ is sufficiently small.

We arrive at the assertion by applying Estimates (A.12), (A.13), (A.14), (A.20), and (A.21),

$$\begin{aligned} \left\| (K_\sigma^\rho(p) - H_\sigma(p)) \Phi_\sigma^\rho(p) \right\| &\leq 2 \left\| V(p) \cdot \nabla_p H_\sigma(p) \Phi_\sigma^\rho(p) \right\| + \left\| V(p)^2 \Phi_\sigma^\rho(p) \right\| + \left\| Y(p) \Phi_\sigma^\rho(p) \right\| \\ &\leq 2 \left\| V(p) (H_f + 1)^{-1/2} \right\| \left\| (H_f + 1)^{1/2} \nabla_p H_\sigma(p) \Phi_\sigma^\rho(p) \right\| \\ &\quad + \left\| V(p)^2 (H_f + 1)^{-1} \right\| \left\| (H_f + 1) \Phi_\sigma^\rho(p) \right\| \\ &\quad + \left\| Y(p) (H_f + 1)^{-1/2} \right\| \left\| (H_f + 1) \Phi_\sigma^\rho(p) \right\| \\ &\leq C \alpha^{1/2} |p| \rho^{1/2} \left\| (H_\sigma(p) + 1) \Phi_\sigma^\rho(p) \right\| \\ &\leq C' \alpha^{1/2} |p| \rho^{1/2}, \end{aligned} \quad (\text{A.22})$$

which is Inequality (A.1) or (2.29), respectively. \square

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