

Weakness of the topological twin building of a measure for the cone topology

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Abstract

Masures are generalizations of Bruhat-Tits buildings introduced by Gaussent and Rousseau in order to study Kac-Moody groups over valued fields. A measure admits a building at infinity $\text{Ch}(\partial\Delta)$, which is a twin building. Ciobotaru, Mühlherr and Rousseau equipped $\text{Ch}(\partial\Delta)$ with a topology called the cone topology. They proved that this equips $\text{Ch}(\partial\Delta)$ with a structure of weak topological twin building in the definition of Hartnick, Köhl and Mars. In this note, we prove however that unless G is reductive, $\text{Ch}(\partial\Delta)$ is not a topological twin building.

1 Introduction

Bruhat-Tits buildings are an important tool in the study of reductive groups over non-archimedean local fields. Kac-Moody groups are interesting generalizations of reductive groups. In order to have an analogous tool in the Kac-Moody frameworks, Gaussent and Rousseau defined measures in [GR08] (where they are called hovels). Thanks to the works of Charignon and Rousseau, a measure is now associated to each almost-split Kac-Moody group over a non-archimedean valued field, see [Cha10], [Rou11], [Rou16] and [Rou17]. Rousseau gave an axiomatic definition of measures in [Rou11] and we simplified it in [Héb17].

Let G be a split Kac-Moody group over a non-archimedean local field and Δ be its measure. As in the Bruhat-Tits theory, Δ is a union of apartments, all isomorphic to a standard one \mathbb{A} . The group G acts on Δ and the apartments of Δ are the $g.\mathbb{A}$ for $g \in G$. Let us describe \mathbb{A} . There is still a Weyl group W^v and a system Φ of roots but contrary to the reductive case, W^v and Φ are infinite except if G is reductive. One can consider Φ as a set of linear forms on a finite dimensional affine space \mathbb{A} and we can define a fundamental chamber C_f^v . The Tits cone is the set $\mathcal{T} = \bigcup_{w \in W^v} \overline{w.C_f^v}$. When G is reductive, $\mathcal{T} = \mathbb{A}$ but when G is not reductive, \mathcal{T} is a proper convex cone of \mathbb{A} . If $\epsilon \in \{-, +\}$, one defines $\partial\Delta_\epsilon$ to be the set of $g.\epsilon\xi$ such that $g \in G$ and ξ is the direction of a ray included in $\epsilon\overset{\circ}{\mathcal{T}}$. Then $\partial\Delta_+$ and $\partial\Delta_-$ are buildings of type W^v . Moreover, $\partial\Delta_+$ and $\partial\Delta_-$ are naturally twinned and one obtains a twin building $\partial\Delta$ at infinity of Δ . This twin building coincides with the twin building of G .

If G is reductive, then Δ is a Bruhat-Tits building. It is thus equipped with a structure of CAT(0)-space. This enables in particular to define a topology - the cone topology - on the set of chambers $\partial\Delta$ of the building at infinity $\partial\Delta$. When G is no more reductive, no structure of CAT(0)-space on Δ is known. However, motivated by the study of actions of

groups on Δ , Ciobotaru and Rousseau generalized the definition of the cone topology to the frameworks of mesures, see [CR15].

It seems that topological twin buildings were first studied and axiomatized by Kramer in [Kra02]. In [HKM13], Hartnick, Köhl and Mars propose an (a priori non equivalent) axiomatization of topological twin buildings. For them a topological twin building is a twin building satisfying the axioms TTB1 to TTB4 of Section 3 of [HKM13]. In [CMR17], which contains the results of [CR15], Ciobotaru, Mühlherr and Rousseau study the properties of the cone topology. They prove that it satisfies the axioms TTB1, TTB2 and TTB4. Thus $\partial\Delta$ is a weak topological building in the terminology of [HKM13]. They also prove that it satisfies additional axioms and variations, see 3.4 of [CMR17]. The aim of this note is to prove that when G is not reductive, then TTB3 is not satisfied and thus $\partial\Delta$ is not a topological twin building (see Proposition 3.1).

Actually we do not limit our study to mesures associated to Kac-Moody groups: for us a mesure is a set satisfying the axioms of [Héb17] (which are equivalent to the axioms of [Rou11]) and whose apartments are associated to a root generating system (and thus to a Kac-Moody matrix). We do not assume that there exists a group acting strongly transitively on it. In order to simplify the notation, we consider only semi-discrete mesures, which means that if M is a wall of \mathbb{A} , the set of walls parallel to it is discrete. When the mesure is associated to a Kac-Moody group over a valued field, this is equivalent to assuming that the valuation is discrete. This hypothesis can be suppressed with minor changes.

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2 Measure and cone topology

In this section we give the definition of abstract mesures as defined in [Rou11]. We use the axiomatic of [Héb17]. We only define semi-discrete mesures, see [Rou11], [Rou17] or [Héb17] for a definition in a full generality.

2.1 Vectorial apartment

2.1.1 Root generating system

A Kac-Moody matrix (or generalized Cartan matrix) is a square matrix $A = (a_{i,j})_{i,j \in I}$ with integers coefficients, indexed by a finite set I and satisfying:

1. $\forall i \in I, a_{i,i} = 2$
2. $\forall (i,j) \in I^2 | i \neq j, a_{i,j} \leq 0$
3. $\forall (i,j) \in I^2, a_{i,j} = 0 \Leftrightarrow a_{j,i} = 0$.

A **root generating system** is a 5-tuple $\mathcal{S} = (A, X, Y, (\alpha_i)_{i \in I}, (\alpha_i^\vee)_{i \in I})$ made of a Kac-Moody matrix A indexed by I , of two dual free \mathbb{Z} -modules X (of characters) and Y (of cocharacters) of finite rank $\text{rk}(X)$, a family $(\alpha_i)_{i \in I}$ (of simple roots) in X and a family $(\alpha_i^\vee)_{i \in I}$ (of simple coroots) in Y . They have to satisfy the following compatibility condition:

$a_{i,j} = \alpha_j(\alpha_i^\vee)$ for all $i, j \in I$. We also suppose that the family $(\alpha_i)_{i \in I}$ is free in X and that the family $(\alpha_i^\vee)_{i \in I}$ is free in Y .

We now fix a Kac-Moody matrix A and a root generating system with matrix A .

Let $\mathbb{A} = Y \otimes \mathbb{R}$. Every element of X induces a linear form on \mathbb{A} . We will consider X as a subset of the dual \mathbb{A}^* of \mathbb{A} : the α_i 's, $i \in I$ are viewed as linear forms on \mathbb{A} . For $i \in I$, we define an involution r_i of \mathbb{A} by $r_i(v) = v - \alpha_i(v)\alpha_i^\vee$ for all $v \in \mathbb{A}$. Its space of fixed points is $\ker \alpha_i$. The subgroup of $\text{GL}(\mathbb{A})$ generated by the α_i 's for $i \in I$ is denoted by W^v and is called the Weyl group of \mathcal{S} . The system $(W^v, \{r_i | i \in I\})$ is a Coxeter system.

Let $Q_{\mathbb{Z}}^\vee = \bigoplus_{i \in I} \mathbb{Z}\alpha_i^\vee$ be the **coroot lattice of \mathbb{A}** .

One defines an action of the group W^v on \mathbb{A}^* by the following way: if $x \in \mathbb{A}$, $w \in W^v$ and $\alpha \in \mathbb{A}^*$ then $(w.\alpha)(x) = \alpha(w^{-1}.x)$. Let $\Phi = \{w.\alpha_i | (w, i) \in W^v \times I\}$, Φ is the set of **real roots**. Then $\Phi \subset Q$, where $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ is the **root lattice of \mathbb{A}** . Let $W = Q^\vee \rtimes W^v \subset \text{GA}(\mathbb{A})$ be the **affine Weyl group** of \mathcal{S} , where $\text{GA}(\mathbb{A})$ is the group of affine isomorphisms of \mathbb{A} .

2.2 Vectorial faces and Tits cone

Define $C_f^v = \{v \in \mathbb{A} | \alpha_i(v) > 0, \forall i \in I\}$. We call it the **fundamental chamber**. For $J \subset I$, one sets $F^v(J) = \{v \in \mathbb{A} | \alpha_i(v) = 0 \forall i \in J, \alpha_i(v) > 0 \forall i \in I \setminus J\}$. Then the closure $\overline{C_f^v}$ of C_f^v is the union of the $F^v(J)$ for $J \subset I$. The **positive** (resp. **negative**) **vectorial faces** are the sets $w.F^v(J)$ (resp. $-w.F^v(J)$) for $w \in W^v$ and $J \subset I$. A **vectorial face** is either a positive vectorial face or a negative vectorial face. We call **positive chamber** (resp. **negative**) every cone of the shape $w.C_f^v$ for some $w \in W^v$ (resp. $-w.C_f^v$). For all $x \in C_f^v$ and for all $w \in W^v$, $w.x = x$ implies that $w = 1$. In particular the action of w on the positive chambers is simply transitive. The **Tits cone** \mathcal{T} is defined by $\mathcal{T} = \bigcup_{w \in W^v} w.\overline{C_f^v}$. We also consider the negative cone $-\mathcal{T}$. We define a W^v -invariant relation \leq on \mathbb{A} by: $\forall (x, y) \in \mathbb{A}^2$, $x \leq y \Leftrightarrow y - x \in \mathcal{T}$. This relation is called the **Tits preorder**.

A vectorial face is **spherical** if its fixer in W^v is finite.

2.3 Measure

2.3.1 Filters

Definition 2.1. *A filter in a set E is a nonempty set F of nonempty subsets of E such that, for all subsets S, S' of E , if $S, S' \in F$ then $S \cap S' \in F$ and, if $S' \subset S$, with $S' \in F$ then $S \in F$.*

If F is a filter in a set E , and E' is a subset of E , one says that F contains E' if every element of F contains E' . If E' is nonempty, the set $F_{E'}$ of subsets of E containing E' is a filter. By abuse of language, we will sometimes say that E' is a filter by identifying $F_{E'}$ and E' . If F is a filter in E , its closure \overline{F} (resp. its convex hull) is the filter of subsets of E containing the closure (resp. the convex hull) of some element of F . A filter F is said to be contained in another filter F' : $F \subset F'$ (resp. in a subset Z in E : $F \subset Z$) if and only if any set in F' (resp. if Z) is in F .

If $x \in \mathbb{A}$ and Ω is a subset of \mathbb{A} containing x in its closure, then the **germ** of Ω in x is the filter $\text{germ}_x(\Omega)$ of subsets of \mathbb{A} containing a neighborhood in Ω of x .

A **sector** in \mathbb{A} is a set of the shape $\mathfrak{s} = x + C^v$ with $C^v = \pm w.C_f^v$ for some $x \in \mathbb{A}$ and $w \in W^v$. The point x is its **base point** and C^v is its **direction**. The intersection of two sectors of the same direction is a sector of the same direction.

The **sector-germ** of a sector $\mathfrak{s} = x + C^v$ is the filter \mathfrak{S} of subsets of \mathbb{A} containing an \mathbb{A} -translate of \mathfrak{s} . It only depends on the direction C^v .

A **sector-face** in \mathbb{A} is a set of the shape $\mathfrak{s} = x + F^v$ with $F^v = \pm w.F^v(J)$ for some $x \in \mathbb{A}$, $w \in W^v$ and some $J \subset I$. The **the germ at infinity** or simply the **germ** of a sector-face $\mathfrak{s} = x + F^v$ is the filter \mathfrak{S} of subsets of \mathbb{A} containing $u + x + F^v$ for some $u \in \overline{F^v}$.

A ray Ξ with base point x and containing $y \neq x$ (or the interval $]x, y] = [x, y] \setminus \{x\}$ or $[x, y]$) is called **preordered** if $x \leq y$ or $y \leq x$ and **generic** if $y - x \in \pm \overline{\mathcal{T}}$, the interior of $\pm \mathcal{T}$.

Let Ξ be a generic ray. The **germ** $germ_\infty(\Xi)$ of Ξ is the set of subsets E of \mathbb{A} such that $\Xi \setminus E$ is bounded.

2.3.2 Definitions of walls, enclosures, faces and related notions

Enclosure A hyperplane of the form $\alpha^{-1}(\{k\})$ with $\alpha \in \Phi$ and $k \in \mathbb{Z}$ is called a **wall**. A half-space of \mathbb{A} delimited by a wall is a **half-apartment**. If $\alpha \in \Phi$ and $k \in \mathbb{Z}$, one sets $M(\alpha, k) = \{x \in \mathbb{A} \mid \alpha(x) + k = 0\}$, $D(\alpha, k) = \{x \in \mathbb{A} \mid \alpha(x) + k \geq 0\}$ and $D^\circ(\alpha, k) = \overset{\circ}{D}(\alpha, k)$.

A set $P \subset \mathbb{A}$ is said to be **enclosed** if there exist $n \in \mathbb{N}$ and half-apartments D_1, \dots, D_n such that $P = \bigcap_{i=1}^n D_i$.

If \mathcal{X} is a filter of \mathbb{A} , its **enclosure** is the filter $\text{cl}(\mathcal{X})$ defined as follows. A set E is in $\text{cl}(\mathcal{X})$ if and only if there exists an enclosed set $E' \subset E$ such that E' is enclosed and $E' \in \mathcal{X}$.

Faces A **local face** F^ℓ in \mathbb{A} is a filter associated to a point $x \in \mathbb{A}$, its **vertex** and a vectorial face $F^v \subset \mathbb{A}$, its **direction**. It is defined by $F^\ell = germ_x(x + F^v)$ and we denote it by $F^\ell(x, F^v)$. A **face** F in \mathbb{A} is a filter associated to a point $x \in \mathbb{A}$ and a vectorial face $F^v \subset \mathbb{A}$. More precisely, a subset S of \mathbb{A} is an element of the face $F = F(x, F^v)$ if and only if it contains a finite intersection of half-apartments or open half-apartments containing $F^\ell(x, F^v)$.

There is an order on the (local) faces: if $F \subset \overline{F'}$ we say that “ F is a face of F' ” or “ F' contains F ” or “ F' dominates F ”. The dimension of a face F is the smallest dimension of an affine space generated by some $S \in F$. Such an affine space is unique and is called its support.

As W^v stabilizes Φ , any element of W^v permutes the sets of the shape $D(\alpha, k)$ where α runs over Φ and $k \in \mathbb{Z}$. Thus W permutes the enclosures, faces, ... of \mathbb{A} .

A **chamber** (or alcove) is a maximal face, i.e a face $F^\ell(x, \pm w.C_f^v)$ for $x \in \mathbb{A}$ and $w \in W^v$.

A **panel** is a spherical face maximal among faces that are not chambers or equivalently a spherical face of dimension $\dim \mathbb{A} - 1$.

Chimneys A **chimney** in \mathbb{A} is associated to a face $F = F(x, F_0^v)$ and to a vectorial face F^v ; it is the filter $\mathfrak{r}(F, F^v) = \text{cl}(F + F^v)$. The face F is a **basis** of the chimney and the vectorial face F^v its direction. A chimney is **splayed** if F^v is spherical.

A **shortening** of a chimney $\mathfrak{r}(F, F^v)$, with $F = F(x, F_0^v)$ is a chimney of the shape $\mathfrak{r}(F(x + \xi, F_0^v), F^v)$ for some $\xi \in \overline{F^v}$. The **germ** of a chimney \mathfrak{r} is the filter of subsets of \mathbb{A} containing a shortening of \mathfrak{r} .

2.3.3 Measure

An apartment of type \mathbb{A} is a set A with a nonempty set $\text{Isom}(\mathbb{A}, A)$ of bijections (called Weyl-isomorphisms) such that if $f_0 \in \text{Isom}(\mathbb{A}, A)$ then $f \in \text{Isom}(\mathbb{A}, A)$ if and only if, there exists $w \in W$ satisfying $f = f_0 \circ w$. We will say isomorphism instead of Weyl-isomorphism in the sequel. An isomorphism between two apartments $\phi : A \rightarrow A'$ is a bijection such that ($f \in \text{Isom}(\mathbb{A}, A)$ if, and only if, $\phi \circ f \in \text{Isom}(\mathbb{A}, A')$). We extend all the notions that are

preserved by W to each apartment. Thus sectors, enclosures, faces and chimneys are well defined in any apartment of type \mathbb{A} .

If A, A' are apartments, $\phi : A \rightarrow A'$ is an isomorphism of apartments and $E \subset A \cap A'$, the notation $\phi : A \xrightarrow{E} A'$ means that ϕ fixes E .

Definition 2.2. *A **measure of type** \mathbb{A} is a set \mathcal{I} endowed with a covering \mathcal{A} of subsets called apartments such that:*

(MA i): *Any $A \in \mathcal{A}$ admits a structure of an apartment of type \mathbb{A} .*

(MA ii): *if two apartments A, A' contain a generic ray, then $A \cap A'$ is enclosed and there exists an isomorphism $\phi : A \xrightarrow{A \cap A'} A'$.*

(MA iii): *if \mathfrak{R} is the germ of a splayed chimney and if F is a face or a germ of a chimney, then there exists an apartment containing \mathfrak{R} and F .*

In this definition, one says that an apartment contains a germ of a filter if it contains at least one element of this germ. One says that a map fixes a germ if it fixes at least one element of this germ.

The measure is said to be **thick** if for each panel of Δ , there exists at least three chambers dominating it.

When Δ is associated to an affine Kac-Moody group, one can replace (MA ii) by

(MA af ii): *if A, A' are two apartments then $A \cap A'$ is enclosed and there exists an isomorphism $\phi : A \xrightarrow{A \cap A'} A'$.*

2.4 Twin building at infinity and cone topology

2.4.1 Buildings at infinity

We now define the twin building of Δ , see also Section 3 of [Rou11] or 2.4.2 of [CMR17].

Let F and F' (resp. ξ and ξ') be two spherical sector-faces (resp. two germs of generic rays) of Δ . By (MA iii), there exists an apartment A containing their germs \mathfrak{F} and \mathfrak{F}' (resp. ξ and ξ'). One says that F and F' (resp. ξ and ξ') are **parallel** if there exists a vectorial face F_A^v (resp. a generic ray Ξ_A) of A such that $\mathfrak{F} = \text{germ}_\infty(x + F_A^v)$ and $\mathfrak{F}' = \text{germ}_\infty(x' + F_A^v)$ for some $x, x' \in A$ (resp. $\xi = \text{germ}_\infty(x + \Xi_A)$ and $\xi' = \text{germ}_\infty(x' + \Xi_A)$). This does not depend on the choice of A (this is a consequence of (MA ii)). Parallelism is an equivalence relation. The parallelism class of a sector-face germ \mathfrak{F} is denoted \mathfrak{F}^∞ and is called the **direction of \mathfrak{F}** . The parallelism class of a generic ray is called its **direction**.

Let $\partial\Delta_+$ (resp. $\partial\Delta_-$) be the set of directions of positive (resp. negative) spherical sector-faces of Δ . Then $\partial\Delta_+$ and $\partial\Delta_-$ are disjoint unless Δ is a Bruhat-Tits building (which happens if and only if W^v is finite), in which case $\partial\Delta_+ = \partial\Delta_-$. One sets $\partial\Delta = \partial\Delta_+ \cup \partial\Delta_-$.

Let $\epsilon \in \{-, +\}$. One defines an order on $\partial\Delta_\epsilon$ as follows. Let $\mathfrak{F}_1^\infty, \mathfrak{F}_2^\infty \in \partial\Delta_\epsilon$. Let A be an apartment containing $\mathfrak{F}_1^\infty, \mathfrak{F}_2^\infty$. Then one says that \mathfrak{F}_1^∞ dominates \mathfrak{F}_2^∞ if there exists sector-faces F_1, F_2 of A such that F_1 dominates F_2 and whose directions are \mathfrak{F}_1^∞ and \mathfrak{F}_2^∞ .

If A is an apartment, one denotes by ∂A_ϵ the set of directions of sector-faces of sign ϵ and one sets $\partial A = \partial A_+ \cup \partial A_-$. Let \mathcal{A} be the set of apartments of Δ and $\partial\mathcal{A} = \{\partial A \mid A \in \mathcal{A}\}$. The elements of $\partial\mathcal{A}$ are called **apartments at infinity**. Then by Théorème 3.4 of [Rou11], $\partial\Delta_\epsilon$ equipped with its system of apartments and the domination relation is a combinatorial building of type W^v (see 2.4.1 of [Rém02] for a definition, it is however not really a simplicial complex, as we consider only directions of spherical sector-faces).

If $\epsilon \in \{-, +, \emptyset\}$, one denotes by $\text{Ch}(\partial\Delta)_\epsilon \subset \partial\Delta_\epsilon$ the set of directions of sectors. The elements of $\text{Ch}(\partial\Delta)$ are called **chambers at infinity**.

2.4.2 W^v -distance and codistance

Let $\epsilon \in \{-, +\}$. Let $(\mathfrak{F}_1^\infty, \mathfrak{F}_2^\infty)$ in $(\text{Ch}(\partial\Delta)_\epsilon)^2$ (resp. in $(\text{Ch}(\partial\Delta)_\epsilon \times \text{Ch}(\partial\Delta)_{-\epsilon}) \cup (\text{Ch}(\partial\Delta)_{-\epsilon} \times \text{Ch}(\partial\Delta)_\epsilon)$). Let A be an apartment containing $\mathfrak{F}_1^\infty, \mathfrak{F}_2^\infty$, which exists by (MA iii). Let $\phi : A \rightarrow \mathbb{A}$ be an isomorphism of apartments. Let F_1^v, F_2^v be vectorial chambers of \mathbb{A} whose directions are $\phi(\mathfrak{F}_1^\infty)$ and $\phi(\mathfrak{F}_2^\infty)$. Let $w \in W^v$ such that $F_1^v = w.F_2^v$ (resp. $F_1^v = -w.F_2^v$). One sets $\delta(\mathfrak{F}_1^\infty, \mathfrak{F}_2^\infty) = w$ (resp. $\delta^*(\mathfrak{F}_1^\infty, \mathfrak{F}_2^\infty) = w$). These definitions do not depend on the choices we made. Then $\delta : \text{Ch}(\partial\Delta)_\epsilon \times \text{Ch}(\partial\Delta)_\epsilon \rightarrow W^v$ is a W^v -distance on $\text{Ch}(\partial\Delta)_\epsilon$ for the definition of 2.3.1 of [Rém02].

By Théorème 3.7 of [Rou11], the **codistance** $\delta^* : (\text{Ch}(\partial\Delta)_\epsilon \times \text{Ch}(\partial\Delta)_{-\epsilon}) \cup (\text{Ch}(\partial\Delta)_{-\epsilon} \times \text{Ch}(\partial\Delta)_\epsilon) \rightarrow W^v$ defines a twinning of $\text{Ch}(\partial\Delta)_+$ and $\text{Ch}(\partial\Delta)_-$ (see 2.5.1 of [Rém02] for the definition of a twinning).

2.4.3 Cone topology on $\text{Ch}(\partial\Delta)$

Let $+\infty$ be the germ at infinity of C_f^v . One chooses a ray $\Xi_{+\infty} \subset C_f^v$ based at 0 and one denotes by $\xi_{+\infty}$ its germ. Let $c \in \text{Ch}(\partial\Delta)$. Then there exists $\epsilon \in \{-, +\}$ and a isomorphism of apartments such that $c = \phi(\epsilon\infty)$. One sets $\xi_c = \phi(\epsilon\xi_{+\infty})$. This definition does not depend on the choices we made.

Let $x \in \Delta$ and $\mathfrak{F}^\infty \in \partial\Delta$ (resp. ξ be the direction of a generic ray). Then there exists a unique sector-face $Q_{x, \mathfrak{F}^\infty}$ (resp. a unique generic ray $[x, \xi)$) based at x and whose direction is \mathfrak{F}^∞ (resp. whose direction is ξ). Indeed, by (MA iii), there exists an apartment A containing x and \mathfrak{F}^∞ . Let F be a sector face of A whose direction is \mathfrak{F}^∞ . Then $Q_{x, \mathfrak{F}^\infty}$ is the translate of F in A based at x . This does not depend on the choices we made by Proposition 4.7 1) of [Rou11]. One obtains the result for rays similarly.

Let $x \in \Delta$, $c \in \text{Ch}(\partial\Delta)$ and $r \in [x, \xi_c)$. One sets

$$U_{x,r,c} = \{c' \in \text{Ch}(\partial\Delta) \mid [x, r] \subset [x, \xi_{c'}]\}.$$

The **cone topology** on $\text{Ch}(\partial\Delta)$ is the topology generated by the $U_{x,r,c}$ such that $c \in \text{Ch}(\partial\Delta)$ and $r \in [x, \xi_c)$. By Lemma 3.3 and Proposition 3.5 of [CMR17], this topology depend neither on the choice of $\xi_{+\infty}$ nor on the choice of $x \in \Delta$.

The cone topology is Hausdorff (Lemma 3.13 of [CMR17]). Indeed, if $c \neq c' \in \text{Ch}(\partial\Delta)$, one chooses an apartment A containing c, c' . One can suppose that $x \in A$. Then for all $r \in (x, \xi_c)$ and $r' \in (x, \xi_{c'})$, one has $c \in U_{x,r,c}$, $c' \in U_{x,r',c'}$ and $U_{x,r,c} \cap U_{x,r',c'} = \emptyset$.

From now on, we assume that $x = 0$.

3 Non-satisfaction of the axiom (TTB3)

If $c \in \text{Ch}(\partial\Delta)$ and $w \in W^v$, one sets $E_{\leq w}(c) = \{d \in \text{Ch}(\partial\Delta)_\epsilon \mid \delta(c, d) \leq w\}$, where ϵ is the sign of c and \leq is the Bruhat order.

One says that $\text{Ch}(\partial\Delta)_\epsilon = \lim_{\rightarrow} E_{\leq w}(c)$ topologically if it satisfies:

“ subset $U \subset \text{Ch}(\partial\Delta)$ is open if and only if $U \cap E_{\leq w}(c)$ is open in $E_{\leq w}(c)$ (for the topology induced by the cone topology) for all $w \in W^v$ ”.

The aim of this note is to prove the following proposition:

Proposition 3.1. *Let Δ be a thick measure such that W^v is infinite. We equip $\text{Ch}(\partial\Delta)$ with the cone topology. Then the axiom (TTB3) of [HKM13] is not satisfied, which means that for all $c \in \text{Ch}(\partial\Delta)$, one has topologically $\text{Ch}(\partial\Delta)_\epsilon \neq \lim_{\rightarrow} E_{\leq w}(c)$, where ϵ is the sign of c . In particular, $\text{Ch}(\partial\Delta)$ is not a topological twin building in the definition of [HKM13].*

In order to prove this, we construct for each chamber c in $\text{Ch}(\partial\Delta)$ a set $U \subset \text{Ch}(\partial\Delta)$ such that:

- $U \cap E_{\leq w}(c)$ is open for all $w \in W^v$
- U is not open.

Let us fix some notation. Let Δ be a thick measure with W^v infinite. Let $c \in \text{Ch}(\partial\Delta)$ and ϵ be its sign. We will consider only chambers of sign ϵ . Let A be an apartment containing c . One identifies A and \mathbb{A} . If $d \in \text{Ch}(\partial\Delta)$ and $r \in [0, \xi_d)$, one writes $U_{r,d}$ instead of $U_{0,r,d}$.

If $w \in W^v$, one sets $E_{\leq w} = E_{\leq w}(c)$ and $F_w = \bigcup_{v \in W^v \mid \ell(v) \leq \ell(w)} E_{\leq v}$.

Let $\alpha \in \Phi$ such that $D(\alpha, k) \not\supseteq c$ for all $k \in \mathbb{R}$.

If $w \in W^v$, one chooses an apartment A_w such that $\mathbb{A} \cap A_w = D(\alpha, \ell(w))$, which is possible by Proposition 2.9 of [Rou11]. By (MA ii), there exists a unique isomorphism $\phi_w : \mathbb{A} \rightarrow A_w$ fixing $\mathbb{A} \cap A_w$ pointwise. One identifies \mathbb{R}_+ and $[0, \xi_c)$. Let $r_w \in \mathbb{R}_+$ such that $D(\alpha, \ell(w)) \cap [0, \xi_c) = [0, r_w]$.

If A is an apartment such that $A \cap \mathbb{A}$ contains 0 in its interior, there exists a unique chamber d of ∂A such that $[0, \xi_d) \cap [0, \xi_c) \neq \{0\}$ and we denote it by c_A . Indeed, the enclosure d of the ray of A based at 0 and containing the germ of $[0, \xi_c)$ in 0 satisfies $[0, \xi_d) \cap [0, \xi_c) \neq \emptyset$. The uniqueness is a consequence of the fact that two vectorial faces of \mathbb{A} are equal or disjoint. In particular if $\phi : \mathbb{A} \rightarrow A$ is an isomorphism fixing a neighborhood of 0, then $c_A = \phi(c)$.

Let $d \in \text{Ch}(\partial\Delta)$, A be an apartment containing $Q_{0,d}$ and $\phi : \mathbb{A} \rightarrow A$ be an isomorphism sending $Q_{0,c}$ on $Q_{0,d}$. If $r \in \mathbb{R}_+$, one sets $[0, r]_d = \phi([0, r])$ and $U_{r,d} = U_{\phi(r),d}$.

3.1 Construction of a sequence $(d_w)_{w \in W^v}$

The aim of this subsection is to construct a sequence $(d_w)_{w \in W^v}$ of chambers such that for all $w \in W^v$, $d_w \notin F_w$ and $[0, \xi_{d_w}) \cap [0, \xi_c) = [0, r_w]$.

An element $s \in W^v$ is called a **reflection** of \mathbb{A} if it is of the shape $w.r_i.w^{-1}$ for some $w \in W^v$ and $i \in I$. Using isomorphisms of apartments, we extend this notion to each apartment of Δ . Let M be a wall of \mathbb{A} . One writes $M = (w.\alpha_i)^{-1}(\{0\})$ for some $i \in I$ and $w \in W^v$. Then $w.r_i.w^{-1}$ is a reflection fixing M and thus the number of reflections of \mathbb{A} fixing 0 is infinite.

Lemma 3.2. *Let A be an apartment such that $A \cap \mathbb{A}$ contains 0 in its interior. Let A_1 be an apartment such that $A_1 \cap \mathbb{A}$ contains 0 in its interior and such that $A \cap A_1$ is a half-apartment which does not contain c_A . Let M be the wall of $A \cap A_1$ and $A_2 = M \cup A_1 \setminus A \cup A \setminus A_1$. Let s be the reflection of A_2 fixing M . Then:*

1. One has $c_{A_1} = s(c_A)$.
2. Let $s' : A \rightarrow A$ be a reflection fixing M and $f : A \rightarrow \mathbb{A}$ be a isomorphism. Let $s'' \in W^v$ be the vectorial part of $f \circ s' \circ f^{-1} \in W^v$. Then $\delta(c_A, c_{A_1}) = s''$.

Proof. Let $\phi_0 : A_1 \xrightarrow{A_1 \cap A_2} A_2$, $\phi_1 : A \xrightarrow{A \cap A_2} A_2$ and $\phi_2 : A \xrightarrow{A \cap A_1} A_1$. Then by Lemma 3.4 of [Héb16], the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{\phi_2} & A_1 \\ \downarrow \phi_1 & & \downarrow \phi_0 \\ A_2 & \xrightarrow{s} & A_2 \end{array}$$

One has $c_{A_1} = \phi_2(c_A) = \phi_0 \circ \phi_2(c_A)$ and $c_A = \phi_1(c_A)$, which proves 1, from which we deduce 2. \square

Lemma 3.3. *The set $\{\delta(c_{A_w}, c) \mid w \in W^v\}$ is finite.*

Proof. Let c' be a chamber of \mathbb{A} such that $c' \subset D(\alpha, k)$ for all $k \in \mathbb{Z}$. Let $w \in W^v$. One has $c_{A_w} = \phi_w(c)$. One has $\delta(c', c) = \delta(\phi_w(c'), \phi_w(c)) = \delta(c', c_{A_w})$ and the lemma follows. \square

For the next lemma, one uses the Tits preorder on Δ . Let us define it. As the Tits preorder \leq on \mathbb{A} is W -invariant, one can define a Tits preorder \leq_A on each apartment. Let $x, y \in \Delta$. By Proposition 5.4 of [Rou11], if there exists an apartment A containing $\{x, y\}$ for which $x \leq_A y$, then $x \leq_{A'} y$ for each apartment A' containing $\{x, y\}$. Thus this defines a relation \leq on Δ . By Théorème 5.9 of [Rou11], this relation is a preorder and we call it the **Tits preorder**.

Lemma 3.4. *Let $w \in W^v$. Then there exists $d_w \in \text{Ch}(\partial\Delta)$ such that $d_w \notin F_w$ and $[0, \xi_{d_w}) \cap [0, \xi_c) = [0, r_w]$.*

Proof. Let $s'' \in W^v$ be a reflection such that $\ell(s'') > \ell(w) + 1 + \max\{\ell(\delta(c_{A_w}, c)) \mid w \in W^v\}$. Let M'' be the fixed wall of s'' and $M' = \phi_w(M'')$. Let A_1 be an apartment satisfying the following conditions:

- $A_w \cap A_1$ is a half-apartment containing $[0, r_w]$ in its interior,
- the wall of $A_w \cap A_1$ is parallel to M' ,
- $A_w \cap A_1$ does not contain c_{A_w} .

Let $d_w = c_{A_1}$. Then $\ell(\delta(d_w, c)) \geq \ell(\delta(d_w, c_{A_w})) - \ell(\delta(c_{A_w}, c)) = \ell(s'') - \ell(\delta(c_{A_w}, c)) \geq \ell(w) + 1$.

By construction, $[0, \xi_{d_w}) \cap [0, \xi_c)$ contains $[0, r_w]$. Suppose that there exists $z \in [0, \xi_{d_w}) \cap [0, \xi_c) \setminus [0, r_w]$. Then $z \geq r_w$. By Proposition 5.4 of [Rou11], $[r_w, z]_{A_1} = [r_w, z]_{\mathbb{A}}$. Moreover for $z' \in (r_w, z)_{A_1}$ near enough from r_w , $z' \in A_w$ and consequently $z' \in A_w \cap \mathbb{A}$. This is absurd because $A_w \cap \mathbb{A} \cap [0, \xi_c) = [0, r_w]$. Therefore $[0, \xi_{d_w}) \cap [0, \xi_c) = [0, r_w]$ and the lemma is proved. \square

3.2 Construction of U

Let $(d_w)_{w \in W^v}$ such that for all $w \in W^v$, $d_w \notin F_w$ and $[0, \xi_{d_w}) \cap [0, \xi_c) = [0, r_w]$, where $[0, r_w] = [0, \xi_c) \cap D(\alpha, \text{ell}(w))$. Let $\mathcal{D} = \{d_w \mid w \in W^v\}$ and $\overline{\mathcal{D}} = \mathcal{D} \cup \{c\}$. We now construct a set U containing c , such that $U \cap \mathcal{D} = \emptyset$ and such that $U \cap E_{\leq w}$ is open for all $w \in W^v$.

Lemma 3.5. *Let V be an nonempty open set and $d \in V$. Then there exists $r \in \mathbb{R}_+^*$ such that $V \supset U_{r,d}$.*

Proof. By definition, there exist $J \subset \text{Ch}(\partial\Delta)$ and $(r_{d'}) \in (\mathbb{R}_+^*)^J$ such that $V = \bigcup_{d' \in J} U_{r_{d'}, d'}$. Let $d' \in J$ such that $d \in U_{r_{d'}, d'}$. Then $U_{r_{d'}, d} = U_{r_{d'}, d'}$, thus $U_{r_{d'}, d} \subset V$, which proves the lemma. \square

Lemma 3.6. *Let $d \in \text{Ch}(\partial\Delta)$. Then $\bigcap_{r \in \mathbb{R}_+^*} U_{r,d} = \{d\}$.*

Lemma 3.7. *Let $d \in \text{Ch}(\partial\Delta) \setminus \overline{\mathcal{D}}$. Then there exists $a_d \in \mathbb{R}_+$ such that $U_{a_d, d} \cap \overline{\mathcal{D}} = \emptyset$.*

Proof. As $\text{Ch}(\partial\Delta)$ is Hausdorff, there exist open sets $V_d \ni d$ and $V_c \ni c$ such that $V_c \cap V_d = \emptyset$. One has $\lim_{\ell(w) \rightarrow +\infty} d_w = c$ and thus for $\ell(w)$ large enough, $d_w \in V_c$. Therefore $V_d \cap \overline{\mathcal{D}}$ is finite. By Lemma 3.5, one can suppose, reducing V_d if necessary that $V_d = U_{r,d}$ for some $r \in \mathbb{R}_+$. We conclude with Lemma 3.6. \square

Lemma 3.8. *Let $w \in W^v$. Then $(U_{r_w,c} \cap E_{\leq w}) \cap \mathcal{D}$ is empty.*

Proof. Let $v \in W^v$. If $\ell(v) < \ell(w)$, then $[0, \xi_{d_v}] \cap [0, \xi_c] = [0, r_v] \subsetneq [0, r_w]$ and thus $d_v \notin U_{r_w,c}$. If $\ell(w) \leq \ell(v)$, then $d_v \notin F_v$ by construction. As $F_v \supset E_{\leq w}$, the lemma follows. \square

If $w \in W^v$, one sets

$$U_w = \bigcup_{d \in (U_{r_w,c} \cap E_{\leq w}) \setminus \{c\}} U_{r_w+a_d,d},$$

which is well-defined by the lemma above. One sets $U = \bigcup_{w \in W^v} U_w \cup \{c\}$. By construction, $U \cap \mathcal{D} = \emptyset$.

Lemma 3.9. *Let $w \in W^v$. Then $U_w \subset U_{r_w,c}$.*

Proof. Let $d \in U_{r_w,c}$ and $d' \in U_{r_w+a_d,d}$. Then

$$[0, \xi_{d'}] \cap [0, \xi_d] \supset [0, r_w + a_d]_d \supset [0, r_w]_d = [0, r_w]_c,$$

thus $d' \in U_{r_w,c}$ and the lemma follows. \square

Lemma 3.10. *Let $w \in W^v$. Then $(U_w \cup \{c\}) \cap E_{\leq w} = U_{r_w,c} \cap E_{\leq w}$.*

Proof. Let $d \in (U_{r_w,c} \cap E_{\leq w}) \setminus \{c\}$. Then $d \in U_{r_w+a_d,d}$ and thus

$$d \in \bigcup_{d' \in (U_{r_w,c} \cap E_{\leq w}) \setminus \{c\}} U_{r_w+a_{d'},d'} = U_w.$$

Therefore $U_{r_w,c} \cap E_{\leq w} \subset (U_w \cup \{c\}) \cap E_{\leq w}$.

By Lemma 3.9, $(U_w \cup \{c\}) \cap E_{\leq w} \subset U_{r_w,c} \cap E_{\leq w}$ and the lemma follows. \square

The following lemma implies Proposition 3.1.

Lemma 3.11. *The set U is not open but for all $w \in W^v$, $U \cap E_{\leq w}$ is open in $E_{\leq w}$.*

Proof. One has $d_w \rightarrow c$, $U \ni c$ but $U \cap \mathcal{D} = \emptyset$ and thus U is not open.

Let $w \in W^v$. Then

$$U \cap E_{\leq w} = ((U_w \cup \{c\}) \cap E_{\leq w}) \cup \left(\bigcup_{v \in W^v \setminus \{w\}} (U_v \cap E_{\leq w}) \right) = (U_{r_w,c} \cup \bigcup_{v \in W^v \setminus \{w\}} U_v) \cap E_{\leq w}$$

is open in $E_{\leq w}$. \square

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