

Stable bundles with small second Chern classes on
surfaces

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1 Introduction

Let X be a compact complex surface and E a topological complex vector bundle on X of rank r and Chern classes $c_1 \in H^2(X, \mathbb{Z})$, $c_2 \in H^4(X, \mathbb{Z}) \cong \mathbb{Z}$. When X is algebraic E admits a holomorphic structure if and only if c_1 lies in the Néron-Severi group $NS(X)$ of X (i.e. the image of c_1 in $H^2(X, \mathbb{R})$ is of type $(1,1)$). If moreover c_2 is large enough as compared to r and c_1^2 then E admits stable holomorphic structures with respect to any fixed polarization on X ([21]) and their moduli spaces have nice geometric properties. In particular they admit natural projective compactifications.

The situation changes if we let X be non-algebraic. For the existence of holomorphic structures in E , the condition $c_1 \in NS(X)$ is still necessary but no longer sufficient. In fact it was proved by Bănică and Le Potier that if E admits a holomorphic structure then

$$2rc_2 - (r-1)c_1^2 \geq 0$$

and that holomorphic structures exist when c_2 is large enough with respect to r and c_1 . For small values of c_2 however, the existence problem remains in general open. This situation is strikingly similar to that of the stable structures. Notice that the inequality above is exactly the Bogomolov inequality which is satisfied by the topological invariants of any stable vector bundle. A second problem arising in the non-algebraic case is finding a nice complex-analytic compactification of the moduli space of stable vector bundles.

In this paper we address these problems as follows.

We use deformations of the complex structure of a torus keeping a suitable Riemannian metric fixed in order to switch between stable structures in E over algebraic tori and so called irreducible structures in E over non-algebraic tori. This enables us to solve the existence problems for rank-two holomorphic vector bundles on non-algebraic tori and for stable rank-two vector bundles of degree zero on any two-dimensional torus. More precisely we prove:

Theorem 1.1 *A topological rank 2 complex vector bundle E on a non-algebraic two-dimensional complex torus X admits some holomorphic structure if and only if*

$$c_1(E) \in NS(X) \text{ and } 4c_2(E) - c_1(E)^2 \geq 0.$$

Theorem 1.2 *Let X be a complex 2-dimensional torus and ω a Kähler class on X . Let $c_1 \in NS(X)$ such that $c_1 \cdot \omega = 0$. Suppose that*

$$c_1^2 = \max\{(c_1 + 2b)^2 \mid b \in NS(X), b \cdot \omega = 0\}.$$

Then a topological rank 2 vector bundle with Chern classes c_1, c_2 admits a holomorphic structure stable with respect to ω if and only if

$$4c_2 - c_1^2 \geq 0,$$

except when

$$c_1 = 0 \text{ and } c_2 \in \{0, 1\}, \text{ or}$$

$$c_1^2 = -2 \text{ and } c_2 = 0.$$

In the excepted cases the holomorphic structures on E are unstable with respect to any polarization ω such that $c_1 \cdot \omega = 0$.

Remark that $c_1 \cdot \omega = 0$ implies $c_1^2 \leq 0$ by Hodge index and that the condition

$$c_1^2 = \max\{(c_1 + 2a) \mid a \in NS(X), a \cdot \omega = 0\}$$

can always be fulfilled by twisting E with a suitable line bundle. Neither the stability nor the invariant $4c_2(E) - c_1^2(E)$ are modified by such twists.

Remark also that if $c_1^2 < 0$ or $c_1 = 0$ there exist polarizations ω such that $c_1 \cdot \omega = 0$.

For "large c_2 " one can always construct locally free sheaves as extensions of coherent sheaves of smaller rank. This method cannot work for all c_2 . In fact, for c_2 below a certain bound all existing holomorphic structures are "irreducible" i.e. do not admit coherent subsheaves of lower rank (cf. [5], [3]). Some holomorphic structures for E with "small c_2 " have been constructed in [33]. We fill in the gaps left by [33] in the following way. We consider two suitable invariant metrics on our torus X and perform two "quaternionic deformations" (see 2.1.1 for the definition) such that the deformed torus has a convenient algebraic structure (see section 3). We construct a stable holomorphic structure in E with respect to this new complex structure of the base and use anti-self-dual connections to get a holomorphic structure for E over our original X .

To prove Theorem 1.2 we perform again a quaternionic deformation, this time changing the structure of X into a non-algebraic one. It is enough then to know which bundles admit here "irreducible" structures i.e. without coherent subsheaves of lower rank. So we reduce ourselves to

Theorem 1.3 *When X is a non-algebraic 2-dimensional torus and E a topological rank 2 vector bundle having $c_1(E) \in NS(X)$ such that $c_1(E)^2 = \max\{(c_1(E) + 2a)^2 \mid a \in NS(X)\}$, then E admits an irreducible holomorphic structure if and only if*

$$4c_2(E) - c_1(E)^2 \geq 0,$$

unless

$$c_1(E)^2 = 0 \text{ and } c_2(E) \in \{0, 1\} \text{ or,}$$

$$c_1(E)^2 = -2 \text{ and } c_2(E) = 0.$$

In the excepted cases all holomorphic structures are reducible.

Remark again that the intersection form on $NS(X)$ is negative semidefinite if X is a non-algebraic surface and that the condition

$$c_1(E)^2 = \max\{(c_1(E) + 2a)^2 \mid a \in NS(X)\}$$

is always fulfilled after a suitable twist of E .

In the last section we prove the existence of a natural compactification of the moduli space of stable structures in E (see Theorem 5.9). For this we have to impose some restrictions on X and E , the most important of which being to ask that there exist no semi-stable sheaves having the same topological invariants as E . Under this same condition Buchdahl constructed a compactification of the moduli space in [7]. But it is not clear whether his construction leads to a complex analytic space. The compactification we shall consider will be an open part of the corresponding moduli space of simple sheaves on X and thus will inherit a natural complex analytic structure. The idea of the proof is to show that the comparison map to the compactified space of anti-self-dual connections is proper. We have restricted ourselves to the situation of anti-self-dual connections, rather than considering the more general Hermite-Einstein connections, since our main objective was to construct compactifications for moduli spaces of stable vector bundles over non-Kählerian surfaces. (In this case one can always reduce oneself to this situation by a suitable twist). In particular, when X is a primary Kodaira surface our compactness theorem combined with the existence result of [33] gives rise to moduli spaces which are holomorphically symplectic compact manifolds.

A common feature of our existence and compactness results is the emergence of an exotic class of holomorphic vector bundles. These are such that all their deformations, including non locally free ones, are irreducible (cf. Definition 2.17). For the existence question this is precisely the class for which general construction methods are lacking. For tori we used direct images through unramified coverings in [31] and [33] and "quaternionic deformations" in this paper. (The first method proved itself useful also in the case of primary Kodaira surfaces whereas the second could be tried for K3 surfaces.) However when turning to the compactification problem, this same class of holomorphic vector bundles is the easiest to deal with. In fact over surfaces with odd first Betti number we could prove the compactness of the moduli space of stable sheaves only for this class.

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Note: The present text is up to some minor corrections that of my Habilitationsschrift submitted in September 1998 at the University of Osnabrück.

2 Preliminary Material

2.1 Self-duality and complex structures

We recall here some simple basic facts about 4-dimensional geometry. For a thorough discussion based on the representation theory of the involved symmetry groups see [26].

2.1.1 Self-duality

Let V be a 4-dimensional oriented real vector space. Further let $g : V \times V \rightarrow \mathbb{R}$ be a metric (scalar product) on V . g induces canonical metrics on the dual vector space V^* and on the exterior algebra ΛV^* by

$$g(x_1 \wedge \dots \wedge x_m, y_1 \wedge \dots \wedge y_m) := \det(g(x_i, y_k))_{i,k}.$$

Then one defines the Hodge operator

$$* : \Lambda^r V^* \longrightarrow \Lambda^{4-r} V^*$$

through the formula

$$\alpha \wedge * \beta = g(\alpha, \beta) \nu,$$

where $\alpha, \beta \in \Lambda^r V^*$ and $\nu \in \Lambda^4 V^*$ is the canonical volume form on V . For $r = 2$ we obtain an endomorphism of $\Lambda^2 V^*$ with $*^2 = 1$. We denote by Λ^+, Λ^- its eigenspaces belonging to the eigenvalues ± 1 . The elements of Λ^+ and Λ^- are called **self-dual**, respectively **anti-self-dual** forms.

Λ^+, Λ^- are maximal positive, respectively negative orthogonal subspaces of $\Lambda^2 V^*$ for the "intersection" form:

$$(\alpha, \beta) \longmapsto \alpha \wedge \beta / \nu = \alpha \cdot \beta.$$

Conversely, one can show that starting with the oriented 4-dimensional space V and with an orthogonal decomposition of $\Lambda^2 V^*$ into maximal positive and negative subspaces for the intersection form (it is of course enough to know one of them), there is a metric g on V , unique up to a constant, such that the given subspaces of $\Lambda^2 V^*$ coincide with the eigenspaces of the associated Hodge operator.

2.1.2 Complex structures and quaternionic deformations

We further consider a complex structure I on V which is compatible with the orientation and with the metric (i.e. $g(Iu, Iv) = g(u, v)$) and denote by V_I the complex vector space thus obtained. To g and I one can associate an element $\omega_I \in \Lambda^2 V^*$ by $\omega_I(u, v) := g(u, Iv)$. Then g and ω_I are the real and respectively the imaginary part of a hermitian metric h on V_I .

One sees easily that $\omega_I \cdot \omega_I = g(\omega_I, \omega_I) = 2$, so ω_I belongs to the sphere of radius $\sqrt{2}$ in Λ^+ .

One verifies that conversely, each element of this sphere is associated to a unique complex structure on V compatible with the orientation and with the metric. In fact these complex structures turn V into a module over the quaternions. (If ω_I and ω_J are orthogonal, the product $K := IJ$ is a new complex structure corresponding to $\omega_I \times \omega_J$ in Λ^+ .) We shall therefore say that two such complex structures are **quaternionic deformations** of each other (with respect to the fixed metric g).

2.1.3 Type decomposition and the positive cone

The complex structure I induces decompositions

$$V_{\mathbb{C}} := V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}, \quad V_{\mathbb{C}}^* := V^* \otimes \mathbb{C} = V^{*1,0} \oplus V^{*0,1}$$

into eigenspaces of the extension of I to $V_{\mathbb{C}}$ and $V_{\mathbb{C}}^*$, and further decompositions into type

$$\Lambda_{\mathbb{C}}^r V_{\mathbb{C}}^* = \bigoplus_{p+q=r} \Lambda^{p,q},$$

where

$$\Lambda^{p,q} := \Lambda_{\mathbb{C}}^p(V^{*1,0}) \otimes \Lambda_{\mathbb{C}}^q(V^{*0,1}).$$

In particular we get $\Lambda^2 V^* = (\Lambda^{2,0} \oplus \Lambda^{0,2})_{\mathbb{R}} \oplus \Lambda_{\mathbb{R}}^{1,1}$ where the \mathbb{R} -index denotes intersection of the corresponding space with $\Lambda^2 V^*$.

The property $\omega_I(Iu, Iv) = \omega_I(u, v)$ means that $\omega_I \in \Lambda_{\mathbb{R}}^{1,1}$. The two orthogonal decompositions of $\Lambda^2 V^*$,

$$\begin{aligned} \Lambda^2 V^* &= \Lambda^+ \oplus \Lambda^- \text{ and} \\ \Lambda^2 V^* &= (\Lambda^{2,0} \oplus \Lambda^{0,2})_{\mathbb{R}} \oplus \Lambda_{\mathbb{R}}^{1,1}, \end{aligned}$$

compare in the following way:

$$\begin{aligned} \Lambda^+ &= (\Lambda^{2,0} \oplus \Lambda^{0,2})_{\mathbb{R}} \oplus \langle \omega_I \rangle \\ \Lambda_{\mathbb{R}}^{1,1} &= \langle \omega_I \rangle \oplus \Lambda^-, \end{aligned}$$

where $\langle \omega_I \rangle$ is the line spanned by ω_I .

The intersection form on $\Lambda_{\mathbb{R}}^{1,1}$ has type (1,3) and thus the set $\{\eta \in \Lambda_{\mathbb{R}}^{1,1} \mid \eta \cdot \eta > 0\}$ has two components. The condition $\eta(Iu, u) > 0$ for one, or equivalently for all $u \neq 0$, $u \in V$, distinguishes one of these components which we call the **positive cone**, $\mathcal{C} := \{\eta \in \Lambda_{\mathbb{R}}^{1,1} \mid \eta \cdot \eta > 0, \eta(Iu, u) > 0 \text{ for some } u \neq 0, u \in V\}$.

The above facts now show that \mathcal{C} and the set of metrics on V compatible with a fixed complex structure I are in a natural bijective correspondence.

2.2 Line bundle cohomology on complex tori

Consider a $2g$ -dimensional real vector space V endowed with a complex structure I , a lattice $\Gamma \subset V$ and the complex torus $X = X_I := V_I/\Gamma$.

Using translation invariant differential forms on X we get the following natural isomorphisms for the de Rham and Dolbeault cohomology groups of X :

$$\begin{aligned} H^r(X, \mathbb{R}) &\cong \Lambda^r V^* \\ H^{p,q} &\cong \Lambda^{p,q} \\ H^r(X, \mathbb{Z}) &\cong \Lambda^r \Gamma^*, \end{aligned}$$

where $\Gamma^* := \text{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{Z}) \subset V^*$.

The first Chern class of a holomorphic line bundle on X is an element of

$$H^{1,1} \cap H^2(X, \mathbb{Z}) \cong \Lambda^{1,1} \cap \Lambda^2 \Gamma^*$$

and thus it is represented by a real skew-symmetric bilinear form E on V taking integer values on $\Gamma \times \Gamma$ and such that

$$E(Iu, Iv) = E(u, v), \text{ for all } u, v \in V.$$

To E one associates a hermitian form H on V_I such that $E = \text{Im } H$. The pairs (H, α) consisting of a hermitian form H on V_I such that $E := \text{Im } H$ is integer valued on $\Gamma \times \Gamma$ and a map $\alpha : \Gamma \times \Gamma \rightarrow U(1)$ satisfying

$$\alpha(\gamma_1 + \gamma_2) = \alpha(\gamma_1)\alpha(\gamma_2)(-1)^{E(\gamma_1, \gamma_2)}$$

are called **Appell-Humbert data**. Addition on the first component and multiplication on the second induce a group structure on the set of Appell-Humbert data. There is a natural way to construct a holomorphic line bundle $L(H, \alpha)$ on X out of the data (H, α) and this gives an isomorphism from the group of Appell-Humbert data to the Picard group of X . Moreover, through this isomorphism $c_1(L(H, \alpha))$ corresponds to $E := \text{Im } H$. Let

$$\begin{aligned} \text{Ker } H &:= \{u \in V \mid H(u, v) = 0, \quad \forall v \in V\} \\ &= \{u \in V \mid E(u, v) = 0, \quad \forall v \in V\}, \end{aligned}$$

$k = \dim \text{Ker } H$ and n the number of negative eigenvalues of H . We denote by $pf(E)$ the Pfaffian of E : for an oriented symplectic basis $(u_i)_{1 \leq i \leq 2g}$ of V with respect to E (i.e. such that $E(u_i, u_j) = 0$ for $|j - i| \neq g$)

$$pf(E) = \det(E(u_i, u_{j+g}))_{0 \leq i, j \leq g}.$$

Remark that there always exists such a basis which is also a basis for Γ over \mathbb{Z} . Moreover this can be chosen in such a way that $d_i | d_{i+1}$ where $d_i := E(u_i, u_{i+g})$. In this case we shall call the sequence (d_1, d_2, \dots, d_g) the **type** of $L(H, \alpha)$. We can now state the results we shall need on line bundle cohomology on tori (cf. [18]).

Theorem 2.1 (*Mumford-Kempf*)

- (a) $H^i(X, L(H, \alpha)) = 0$ for $i < n$ or $i > n + k$.
- (b) $H^{n+i}(X, L(H, \alpha)) \cong H^n(X, L(H, \alpha)) \otimes H^{0i}(\text{Ker } H/\Gamma \cap \text{Ker } H)$ for $0 \leq i \leq k$
- (c) $H^n(X, L(H, \alpha)) = 0$ if and only if $\alpha|_{\Gamma \cap \text{Ker } H} \neq 1$.

Theorem 2.2 (*Riemann-Roch*)

$$\begin{aligned} \chi(L(H, \alpha)) &:= \sum_{i=0}^g (-1)^i \dim H^i(X, L(H, \alpha)) \\ &= \frac{1}{g!} c_1(L(H, \alpha))^g = \frac{1}{g!} pf(E). \end{aligned}$$

When $g = 2$ we distinguish the following cases:

- $c_1(L(H, \alpha))^2 > 0$
implies H or $-H$ is positive definite (which is equivalent to saying that $L(H, \alpha)$ or $L(H, \alpha)^{-1}$ is ample) and according to this, the cohomology of $L(H, \alpha)$ is concentrated in degree 0 or 2.
- $c_1(L(H, \alpha))^2 < 0$
implies H is indefinite and the cohomology of $L(H, \alpha)$ is concentrated in degree one
- $c_1(L(H, \alpha))^2 = 0$ and $H \neq 0$
imply H or $-H$ is positive semi-definite and thus the cohomology of $L(H, \alpha)$ in degree 2, respectively 0, must vanish; for suitable $\alpha - s$ all cohomology groups will vanish in this case.

2.3 Stable bundles and anti-self-dual connections

In this section we recall some definitions and facts on stability and anti-self-dual connections. For a broader treatment we refer the reader to [20].

Let X be a compact (non-singular) complex surface.

Definition 2.3 A hermitian metric on X is called **Gauduchon metric** if its associated Kähler form is $\partial\bar{\partial}$ -closed.

By a result of Gauduchon any hermitian metric on X is conformally equivalent to a Gauduchon metric. We fix such a Gauduchon metric g on X and we denote by ω its Kähler form. We shall call the couple (X, g) or (X, ω) a **polarized surface** and ω the **polarization**. When ω is d -closed we replace (X, ω) by $(X, [\omega])$ where $[\omega]$ is the de Rham

cohomology class of ω . Let L be a holomorphic line bundle on X , h a hermitian metric in its fibers, A the associated Chern connection and F_A its curvature. Then the Chern class $c_1(L)$ of L is represented by the closed differential form

$$c_1(L, h) = -\frac{1}{2\pi i} F_A$$

in the de Rham cohomology group. Two such representatives (coming from different metrics on L) differ by a $\partial\bar{\partial}$ -exact form. Thus one can define the **degree of L with respect to g** by

$$\deg L = \deg_\omega L = \deg_g L := \int c_1(L, h) \wedge \omega.$$

If ω is closed and $[\omega]$ is its de Rham cohomology class, one gets

$$\deg L = c_1(L) \cdot [\omega] \in H^4(X, \mathbb{R}) \cong \mathbb{R}.$$

In general, however the degree is not a topological invariant:

Proposition 2.4 (a) *The degree map $\deg_g : \text{Pic}(X) \rightarrow \mathbb{R}$ is a Lie-group homomorphism.*

$$\deg_g : \text{Pic}^0(X) \longrightarrow \mathbb{R}$$

is surjective when $b_1(X)$ is odd and trivial when $b_1(X)$ is even.

(b) *When X is projective algebraic there exists to any Gauduchon metric g a Kähler metric g' such that $\deg_g = \deg_{g'}$ on $NS(X)$.*

Proof For (a) we refer the reader to [20].

(b) Let X be projective algebraic and ω the Kähler form of a Gauduchon metric g on X . Let W be the space of real $\partial\bar{\partial}$ -closed $(1,1)$ -forms modulo d - (or $\partial\bar{\partial}$ -) exact forms and $[\omega]$ the class of ω in W .

$NS(X) \otimes \mathbb{R}$ is a finite dimensional vector subspace of W on which the restriction of the intersection form from W is non-degenerated. Therefore we may consider the orthogonal projection $p([\omega]) \in NS(X) \otimes \mathbb{R}$. $p([\omega])$ gives the same "degree function" on $NS(X) \otimes \mathbb{R}$ as ω .

We want to show that $p([\omega])$ belongs to the ample cone of X . By Kleiman's criterion we need only to check that $p([\omega])$ is positive on the closed cone generated by the effective curves on X . But the elements of this cone are represented by closed positive $(1,1)$ -currents (cf. [8]) so the positivity condition is verified. \square

For a torsion-free sheaf \mathcal{F} of rank r on X we set

$$\deg \mathcal{F} := \deg(\Lambda^r \mathcal{F})^{\vee\vee}.$$

(For an \mathcal{O}_X -module \mathcal{G} we denote by \mathcal{G}^\vee the dual sheaf $\text{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{O}_X)$).

Definition 2.5 A coherent torsion-free sheaf \mathcal{F} on X will be called **stable** (respectively **semi-stable**) with respect to the polarization ω if for any coherent subsheaf \mathcal{F}' of \mathcal{F} with $0 < \text{rank } \mathcal{F}' < \text{rank } \mathcal{F}$ the following inequality holds

$$\frac{\deg \mathcal{F}'}{\text{rank } \mathcal{F}'} < \frac{\deg \mathcal{F}}{\text{rank } \mathcal{F}}$$

(respectively

$$\frac{\deg \mathcal{F}'}{\text{rank } \mathcal{F}'} \leq \frac{\deg \mathcal{F}}{\text{rank } \mathcal{F}}).$$

This notion has a differential-geometric counterpart through the "Kobayashi-Hitchin correspondence" which we describe below in the important special case of anti-self-dual connections.

For the space of complexified 2-forms \mathcal{A}^2 on X we have two decompositions induced fiberwise by the decompositions in 2.1:

$$\mathcal{A}^2 = \mathcal{A}^{2,0} \oplus \mathcal{A}^{1,1} \oplus \mathcal{A}^{0,2}$$

by type and

$$\mathcal{A}^2 = \mathcal{A}^+ \oplus \mathcal{A}^-$$

into self-dual and anti-self-dual parts.

Moreover

$$\begin{aligned} \mathcal{A}^+ &= \mathcal{A}^{2,0} \oplus \mathcal{A}^{0,2} \oplus \mathcal{A}^0 \cdot \omega, \\ \mathcal{A}^{1,1} &= \mathcal{A}^0 \cdot \omega \oplus \mathcal{A}^- \end{aligned}$$

and the sums are pointwise orthogonal. In particular the anti-self-dual forms are those (1,1)-forms which are pointwise orthogonal to ω . Let now E be a holomorphic vector bundle on X and h a hermitian metric in the fibers of E . (We shall denote also by E the associated locally free sheaf). There is then a unique unitary connection A on E compatible with the holomorphic structure of E , i.e. such that its (0,1)-part is exactly the $\bar{\partial}$ -operator on sections of E . The associated curvature F_A is of type (1,1), i.e. $F_A \in \mathcal{A}^{1,1}(X, \text{End } E)$.

The following converse holds:

Theorem 2.6 *If the curvature of a unitary connection in a hermitian vector bundle on X is of type (1,1) then the (0,1)-part of this connection defines a holomorphic structure in E .*

Definition 2.7 A unitary connection A in a hermitian differentiable vector bundle (E, h) on (X, ω) is called **anti-self-dual** if its curvature F_A belongs to $\mathcal{A}^-(\text{End}(E))$. (This definition depends on the polarization ω , of course). In particular it follows that $c_1(\det E, \det h) \wedge \omega = 0$.

Remark 2.8 *If A is an anti-self-dual connection of (E, h) then*

$$c_2(E) - \frac{1}{2}c_1^2(E) \geq 0.$$

One has indeed

$$\begin{aligned} c_2(E) - \frac{1}{2}c_1^2(E) &= \frac{1}{8\pi^2} \int \text{Tr}(F_A^2) \\ &= \frac{1}{8\pi^2} \int (|F_A^-|^2 - |F_A^+|^2) d\mu \\ &= \frac{1}{8\pi^2} \int |F_A^-|^2 d\mu = \frac{1}{8\pi^2} \|F_A^-\|^2 = \frac{1}{8\pi^2} \|F_A\|^2. \end{aligned}$$

Here we have denoted by F_A^+, F_A^- the self-dual and anti-self-dual parts of F_A , by $|\cdot|$ the punctual norms and by $\|\cdot\|$ the corresponding L^2 -norms over X .

We shall make use of the following special case of the "Kobayashi-Hitchin correspondence":

Theorem 2.9 *Let X be a compact complex surface, ω the associated $(1,1)$ -form of a Gauduchon metric on X and E a differential vector bundle on X .*

If E admits an anti-self-dual connection then the induced holomorphic structure in E is semi-stable (with respect to ω) and E splits holomorphically into a direct sum of stable vector bundles of degree zero which are invariant under the connection.

Conversely, any stable holomorphic bundle of degree zero on (X, ω) admits a hermitian metric (up to a positive constant unique) such that the associated connection is anti-self-dual.

We mention one more property of stable sheaves. First a

Definition 2.10 A coherent sheaf \mathcal{F} on X is called **simple** if its only (global) endomorphisms are the constant multiples of the identity.

Remark 2.11 (a) \mathcal{F} is simple if and only if every non-zero endomorphism of \mathcal{F} is an automorphism.

(b) If a torsion-free coherent sheaf is stable with respect to some polarization ω on X then it is simple.

Proof

(a) If every non-zero endomorphism of \mathcal{F} is an automorphism then such an endomorphism φ generates a finite-dimensional field extension of \mathbb{C} . But \mathbb{C} is algebraically closed so the extension is trivial.

- (b) Let φ be a non-zero endomorphism of a stable sheaf \mathcal{F} . Then its image sheaf $\text{Im } \varphi$ is non-trivial and torsion-free, thus $\text{rank}(\text{Im } \varphi) > 0$. If $\text{rank}(\text{Im } \varphi) < \text{rank}(\mathcal{F})$ then the slope inequalities for $\text{Im } \varphi$ as a subsheaf and as a quotient sheaf of \mathcal{F} give a contradiction. (One uses $\text{deg } \mathcal{F} = \text{deg}(\text{Im } \varphi) + \text{deg}(\text{Ker } \varphi)$). So $\text{rank}(\text{Im } \varphi) = \text{rank}(\mathcal{F})$. If $x \in X$ is a point where \mathcal{F} is non-singular and λ an eigen-value of $\varphi_x : \mathcal{F}/\mathfrak{m}_x\mathcal{F} \rightarrow \mathcal{F}/\mathfrak{m}_x\mathcal{F}$, then $\det(\varphi - \lambda \cdot \text{id}_{\mathcal{F}}) : \det \mathcal{F} \rightarrow \det \mathcal{F}$ will vanish identically. Thus $\text{rank } \text{Im}(\varphi - \lambda \cdot \text{id}_{\mathcal{F}}) < \text{rank } \mathcal{F}$ and $\varphi \equiv \lambda \cdot \text{id}_{\mathcal{F}}$. \square

2.4 Existence of holomorphic vector bundles on non-algebraic surfaces

Let X be a compact complex surface. In this section we recall some facts around the question: "Which topological (complex) vector bundles admit holomorphic structures?"

For line bundles the answer is easy. If \mathcal{C}^* , \mathcal{O}^* denote the sheaves of germs of continuous, resp. holomorphic nonvanishing functions on X , then the topological and the holomorphic line bundles are classified by $H^1(X, \mathcal{C}^*)$ and $H^1(X, \mathcal{O}^*)$ respectively. Out of the corresponding exponential sequences one gets a commutative square

$$\begin{array}{ccc} H^1(X, \mathcal{C}^*) & \xrightarrow{c_1} & H^2(X, \mathbb{Z}) \\ \uparrow & & \parallel \\ H^1(X, \mathcal{O}^*) & \xrightarrow{c_1} & H^2(X, \mathbb{Z}), \end{array}$$

where c_1 denotes taking the first Chern class. Thus a topological line bundle E admits a holomorphic structure if and only if its first Chern class, $c_1(E)$, lies in the Néron-Severi group of X ,

$$NS(X) := \text{Im}(H^1(X, \mathcal{O}^*) \longrightarrow H^2(X, \mathbb{Z})).$$

By Lefschetz' theorem on (1,1)-classes, $NS(X)$ is the pullback of $H^{1,1}(X)$ through the natural morphism $H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathbb{C})$, (cf. 2.3).

Consider now topological vector bundles of rank $r \geq 2$. By a result of Wu these are classified by their Chern classes

$$c_1 \in H^2(X, \mathbb{Z}), \quad c_2 \in H^4(X, \mathbb{Z}) \cong \mathbb{Z};$$

(we shall always identify $H^4(X, \mathbb{Z})$ to \mathbb{Z} using the orientation). More precisely: for every $(r, c_1, c_2) \in (\mathbb{N} \setminus \{0, 1\}) \times H^2(X, \mathbb{Z}) \times \mathbb{Z}$ there exists exactly one topological vector bundle (up to isomorphism) E having:

$$\text{rank } E = r, \quad c_1(E) = c_1, \quad c_2(E) = c_2.$$

Since E and $\det E := \Lambda^r E$ have the same first Chern class, the condition $c_1(E) \in NS(X)$ remains necessary for E to admit a holomorphic structure.

When X is algebraic this condition is also sufficient by a theorem of Scharzenberger, [29], i.e. for any $(r, c_1, c_2) \in (\mathbb{N} \setminus \{0, 1\}) \times NS(X) \times \mathbb{Z}$ there is a *holomorphic* vector bundle E on X with

$$\text{rank}(E) = r, \quad c_1(E) = c_1, \quad c_2(E) = c_2.$$

This result does not hold any more when X is non-algebraic. Bănică and Le Potier showed that in this case c_2 cannot be arbitrarily small. In fact if one defines the discriminant of a vector bundle E by

$$\Delta = \Delta(E) := \frac{1}{r}(c_2(E) - \frac{(r-1)}{2r}c_1(E)^2),$$

one has

Theorem 2.12 ([3]) *The discriminant of a holomorphic vector bundle E (of rank $r \geq 2$) on a nonalgebraic-surface is always non-negative:*

$$\Delta(E) \geq 0.$$

The fact which lies behind this result is a theorem of Kodaira asserting that a compact complex surface is non-algebraic precisely when the intersection form on $NS(X)$ is negative semi-definite.

We recall at this place that Chern classes for arbitrary coherent sheaves may be easily introduced over surfaces by using global locally free resolutions. These exist also in the non-projective case in dimension two as proved by Schuster, [28]. For higher dimensions more sophisticated constructions are needed; cf. [30], [14]. The above theorem remains true for torsion-free coherent sheaves E as well.

In order to state the available existence results we need some more definitions.

Definition 2.13 A torsion-free coherent sheaf \mathcal{F} on X is called **reducible** if it admits a coherent subsheaf \mathcal{F}' with $0 < \text{rank } \mathcal{F}' < \text{rank } \mathcal{F}$, (and **irreducible** otherwise). \mathcal{F} is called **filtrable** if it has a filtration $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_{r-1} \subset \mathcal{F}_r = \mathcal{F}$ by coherent subsheaves \mathcal{F}_i with $\text{rank } \mathcal{F}_i = i$, for $0 \leq i \leq r = \text{rank } \mathcal{F}$.

We shall use the same terminology for holomorphic vector bundles thinking of their sheaves of germs of holomorphic sections.

Remark that if X is algebraic (and thus projective), every torsion-free coherent sheaf \mathcal{F} on X is filtrable. (Consider a non-zero section in a twist of \mathcal{F} by some sufficiently ample line bundle). The situation is different in the non-algebraic case:

Theorem 2.14 *There exist irreducible rank-two holomorphic vector bundles on any non-algebraic surface.*

This theorem has been proved in [3] and [32] by showing that not all deformations of holomorphic structures of a suitably chosen reducible vector bundle can be reducible.

For filtrable holomorphic structures one has the following existence result due to Bănică and Le Potier.

Theorem 2.15 [3] *If X is non-algebraic then a topological vector bundle E on X admits a filtrable holomorphic structure if and only if $c_1(E) \in NS(X)$ and*

$$c_2(E) \geq \inf \left\{ c_2 \left(\bigoplus_{i=1}^{\text{rank } E} L_i \right) \mid L_i \in \text{Pic}(X) \right\},$$

with one exception:

When X is a K3 surface without non-constant meromorphic functions,

$$c_2(E) = \inf \left\{ c_2 \left(\bigoplus_{i=1}^{\text{rank } E} L_i \right) \mid L_i \in \text{Pic}(X) \right\} + 1$$

and $c_1(E) \in \text{rank}(E)NS(X)$ then there are no holomorphic structures on E .

For the proof one constructs extensions of non locally free sheaves having locally free middle terms. These middle-terms provide the looked for holomorphic vector bundles. A different construction strategy would be to start with a known locally free sheaf and then construct others with larger second Chern class by surgery and deformation. This is suggested by the following.

Proposition 2.16 *Let X be a compact complex surface with Kodaira dimension $\text{kod}(X) = -\infty$ or with $\text{kod}(X) = 0$ and $p_g(X) = 1$. Let E be a holomorphic vector bundle on X whose rank exceeds 1 and n a positive integer. Then there exists a holomorphic vector bundle F on X with*

$$\text{rank}(F) = \text{rank}(E), \quad c_1(F) = c_1(E), \quad c_2(F) = c_2(E) + n.$$

excepting the case when X is K3 without non-constant meromorphic functions, E is a twist of the trivial line bundle by some line bundle and $n = 1$.

Proof

Take E as in the Proposition. We may assume that for given rank, $\text{rank}(E) = r$, and first Chern class, $c_1(E) = a \in NS(X)$, the second Chern class of E is minimal among all $c_2(F)$ with F holomorphic vector bundle of rank r on X with $c_1(F) = a$.

We consider a filtration $0 = E_0 \subset E_1 \subset \dots \subset E_m = E$ by coherent subsheaves such that E_i/E_{i-1} are irreducible of positive ranks r_i for $1 \leq i \leq m$. Since by taking double duals the second Chern class decreases, the minimality assumption on $c_2(E)$ implies that the quotient sheaves E_i/E_{i-1} are locally free.

When all r_i equal 1, E is filtrable and the assertion follows from Theorem 2.15.

Let then $r_{i_0} > 1$, take n distinct points x_1, \dots, x_n on X and let \mathcal{F}_0 be the kernel of a surjective sheaf morphism from E_{i_0}/E_{i_0-1} to the skyscraper sheaf $\mathcal{O}_{x_1} \oplus \dots \oplus \mathcal{O}_{x_n}$.

Then $c_1(\mathcal{F}_0) = c_1(E_{i_0}/E_{i_0-1})$, $c_2(\mathcal{F}_0) = c_2(E_{i_0}/E_{i_0-1}) + n$ and \mathcal{F}_0 is singular at x_1, \dots, x_n .

We look for a locally free sheaf \mathcal{F} in the versal deformation of \mathcal{F}_0 . If \mathcal{F} exists we just take $F = \bigoplus_{i \neq i_0} (E_i/E_{i-1}) \oplus \mathcal{F}$. In order to prove that a locally free deformation of \mathcal{F}_0 exists it is enough to show that the versal deformation of \mathcal{F}_0 is non-obstructed and thus is smooth of the expected dimension. Then count dimensions as in Claim 3 in the proof of our Theorem 5.9. The non-obstructedness follows if we show that $\text{Ext}^2(X; \mathcal{F}_0, \mathcal{F}_0) \cong H^2(X, \mathcal{O})$ (cf. § 5.2). But $\text{Ext}^2(X; \mathcal{F}_0, \mathcal{F}_0) \cong \text{Hom}(X; \mathcal{F}_0, \mathcal{F}_0 \otimes K_X)^*$. By taking double duals $\text{Hom}(X; \mathcal{F}_0, \mathcal{F}_0 \otimes K_X)$ injects into $\text{Hom}(X; \mathcal{F}_0^{\vee\vee}, \mathcal{F}_0^{\vee\vee} \otimes K_X)$.

Let φ be a non-zero homomorphism $\varphi : \mathcal{F}_0^{\vee\vee} \rightarrow \mathcal{F}_0^{\vee\vee} \otimes K_X$. Then $\det \varphi : \det \mathcal{F}_0^{\vee\vee} \rightarrow (\det \mathcal{F}_0^{\vee\vee}) \otimes K_X^{\otimes r_{i_0}}$ cannot vanish identically since $\mathcal{F}_0^{\vee\vee} = E_{i_0}/E_{i_0-1}$ was irreducible. Thus it induces a non-zero section of $K_X^{\otimes r_{i_0}}$. This forces $\text{kod}(X) = 0$ and $P_g(X) = 1$ by assumption.

Let s be a non-zero section of K_X and consider $\lambda \cdot \text{id}_{\mathcal{F}_0^{\vee\vee}} \otimes s - \varphi \in \text{Hom}(X; \mathcal{F}_0^{\vee\vee}, \mathcal{F}_0^{\vee\vee} \otimes K_X)$ for $\lambda \in \mathbb{C}$. For a point $x \in X$ away from the vanishing locus of $\det \varphi$ and from the vanishing locus of s the polynomial $\det(\lambda \cdot \text{id}_{\mathcal{F}_0^{\vee\vee}} \otimes s - \varphi)(x)$ vanishes for some $\lambda := \lambda_x \in \mathbb{C} \setminus \{0\}$. Since $\text{kod}(X) = 0$, $\det(\lambda_x \cdot \text{id}_{\mathcal{F}_0^{\vee\vee}} \otimes s - \varphi)$ will vanish identically on X . Again by the irreducibility of $\mathcal{F}_0^{\vee\vee}$ this implies that $\varphi = \lambda_x \cdot \text{id}_{\mathcal{F}_0^{\vee\vee}} \otimes s = \text{id}_{\mathcal{F}_0^{\vee\vee}} \otimes \lambda_x \cdot s$. Thus the natural map $H^0(X, K_X) \rightarrow \text{Hom}(X; \mathcal{F}_0^{\vee\vee}, \mathcal{F}_0^{\vee\vee} \otimes K_X)$ is an isomorphism, and this goes over to $H^0(X, K_X) \rightarrow \text{Hom}(X; \mathcal{F}_0, \mathcal{F}_0 \otimes K_X)$ proving our claim. \square

The existence problem remains in general open when $c_2(E)$ is smaller than the bound given in the last Theorem. For such a value of $c_2(E)$ a holomorphic structure is automatically non-filtrable. One can consider an even lower bound under which holomorphic structures will be automatically irreducible.

The problem now is to construct such irreducible bundles (which cannot be deformed to reducible ones). Then one may use them as building blocks for the construction of new bundles. We introduce a definition which is actually on the topological invariants of a vector bundle more than on the bundle itself, but we leave it in this form to preserve analogy.

Definition 2.17 A torsion-free sheaf \mathcal{F} on X is called **stably irreducible** if every torsion-free sheaf \mathcal{F}' with

$$\text{rank}(\mathcal{F}') = \text{rank}(\mathcal{F}), \quad c_1(\mathcal{F}') = c_1(\mathcal{F}), \quad c_2(\mathcal{F}') \leq c_2(\mathcal{F})$$

is irreducible.

Stably irreducible bundles have been constructed on 2-dimensional tori and on primary

Kodaira surfaces in [31] and [33]. In both cases this was done by considering direct image sheaves through unramified coverings of the base surface X .

In the next paragraph we use "quaternionic deformations" of tori to construct new examples. This idea could also work for $K3$ surfaces but in general construction methods for stably irreducible bundles are lacking.

Finally, since stably irreducible bundles as well as their deformations are stable with respect to any polarization on X , we shall be able to construct some natural compactifications for their moduli spaces in § 5. We shall further discuss the relation to stability there.

3 Vector bundles on non-algebraic 2-tori

Let X be a 2 dimensional non-algebraic torus, and E a differential complex vector bundle of rank 2 on X having $c_1(E) \in NS(X)$. As recalled in 2.4 a necessary condition for E to admit some holomorphic structure is that

$$\Delta(E) := \frac{1}{2}(c_2(E) - \frac{1}{4}c_1(E)^2) \geq 0.$$

We want to prove that this condition is also sufficient. If 4 divides $c_1(E)^2$ this has been proved in [31], [33]. The statement holds also when $c_1(E)^2 = -2$ since we can construct a filtrable vector bundle E as a direct sum of holomorphic line bundles having

$$\Delta(E) = \frac{1}{2}\left(0 - \frac{1}{4}(-2)\right) = \frac{1}{4},$$

and then apply the Proposition 2.16 to increase c_2 . Here we shall deal with the case

$$c_1(E)^2 = -2(4k \pm 1),$$

k a positive integer. (Recall that the self intersection of an element of $NS(X)$ is non positive since X is non-algebraic). If L is a holomorphic line bundle on X then

$$\begin{aligned} c_1(E \otimes L) &= c_1(E) + 2c_1(L) \\ \Delta(E \otimes L) &= \Delta(E), \end{aligned}$$

so it will be enough to solve the existence problem for some vector bundle E' of rank 2 with $c_1(E') \in c_1(E) + 2NS(X)$ and

$$\Delta(E') = \Delta(E).$$

In particular we may always suppose that $c_1(E)$ is a primitive element in $NS(X)$. X will be considered as the quotient $X_I = V_I/\Gamma$ of a fixed real 4-dimensional vector space V endowed with a complex structure I through the fixed lattice Γ . Let a be a primitive element in

$$NS(X_I) \cong H^2(X, \mathbb{Z}) \cap H^{1,1} \cong \Lambda^2 \Gamma^* \cap \Lambda_I^{1,1}.$$

We first connect the complex structure I to a new complex structure K by two quaternionic deformations such that $NS(X_K)$ is generated by a and an ample class of a special type.

Lemma 3.1 *Let X_I be a complex 2-dimensional torus and $a \in NS(X)$ a primitive element such that $a^2 < 0$.*

- (a) *There exists an invariant hermitian metric h on X_I which makes a (seen as an invariant 2-form) anti-self-dual. Moreover this metric may be chosen such that all integer elements in Λ^- are multiples of a .*

- (b) If h is chosen as above and g is its real part, then there exists a dense open set of quaternionic deformations J of I with respect to g , such that $NS(X_J)$ is cyclic generated by a .
- (c) If $a^2 = -2(4k \pm 1)$ there exists $\eta \in H^2(X, \mathbb{Z})$ of type $(1, k)$ with $a \cdot \eta = 0$.

For a suitably chosen complex structure J as above there exists a quaternionic deformation K of it, with respect to a possibly new metric on X_J , such that η is proportional to the imaginary part ω_K of the associated hermitian metric and $NS(X_K)$ is generated over \mathbb{Q} by a and η .

Proof

- (a) By 2.1.3 it is enough to find an element $\omega_I \in \mathcal{C}$ with $\omega_I \cdot a = 0$ but $\omega_I \cdot b \neq 0$ if $b \in NS(X)$ is not a multiple of a . For the existence of such an ω_I we just remark that the intersection form on the orthogonal of a in $\Lambda_{\mathbb{R}}^{1,1}$ has type $(1,2)$, since $a^2 < 0$.
- (b) Using 2.1.2 one sees that it suffices to take ω_J on the sphere of radius $\sqrt{2}$ in Λ^+ and not belonging to a line of the form $(\langle b \rangle \oplus \Lambda^-) \cap \Lambda^+$, where $b \in H^2(X, \mathbb{Z}) \setminus \langle a \rangle$.
- (c) We choose a symplectic integer basis for a . Since a was primitive the associated matrix will have the form:

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -(4k \pm 1) \\ -1 & 0 & 0 & 0 \\ 0 & (4k \pm 1) & 0 & 0 \end{pmatrix}.$$

Any element $\eta \in H^2(X, \mathbb{Z})$ is represented in this basis by an integer-valued skew-symmetric matrix

$$S = \begin{pmatrix} 0 & \theta & \alpha & \beta \\ -\theta & 0 & \gamma & \delta \\ -\alpha & -\gamma & 0 & \tau \\ -\beta & -\delta & -\tau & 0 \end{pmatrix}.$$

The condition $a \cdot \eta = 0$ becomes:

$$\delta = \alpha(4k \pm 1).$$

η is of type $(1, k)$ if there exists some basis of Γ with respect to which its associated matrix is:

$$\Xi = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & k \\ -1 & 0 & 0 & 0 \\ 0 & -k & 0 & 0 \end{pmatrix}.$$

We thus need a transformation matrix $M \in SL(4, \mathbb{Z})$ changing Ξ into S by

$$S = M^t \Xi M$$

such that for S , $\delta = \alpha(4k \pm 1)$. It is easy to see that such an M exists. (In fact one can show that the $\eta - s$ satisfying the given conditions span $\langle a \rangle^\perp$ over \mathbb{R} .)

So let now η be of type $(1, k)$ with $\eta \cdot a = 0$. Let Λ^+ be the space of invariant self-dual forms with respect to the metric which was fixed in (a).

If $\eta \in \Lambda^+$ we just take ω_K proportional to η on the sphere of radius $\sqrt{2}$ in Λ^+ . For the new complex structure K , $NS(X_K)$ will be generated over \mathbb{Q} by a and η . Let indeed $b = c \cdot \eta + \mu \in NS(X_K)$ with $c \in \mathbb{R}$ and $\mu \in \Lambda^-$. Since $\eta \cdot b = c\eta^2$ is an integer, c must be rational and μ lies in $\Lambda^- \cap H^2(X, \mathbb{Q}) = \mathbb{Q} \cdot a$.

When $\eta \notin \Lambda^+$ we have to change the metric on X . We do this as follows. Since $\eta \notin \Lambda^+$, $\eta^\perp \cap \Lambda^+$ is a 2-dimensional subspace of Λ^+ . Let ω be an element of the sphere of radius $\sqrt{2}$ in Λ^+ around 0, which is orthogonal to $\eta^\perp \cap \Lambda^+$. In fact ω is proportional to the projection of η on Λ^+ . We choose ω_J as in (b) but close to ω . We claim that the intersection form is positive definite on $(\Lambda_J^{2,0} \oplus \Lambda_J^{0,2})_{\mathbb{R}} \oplus \langle \eta \rangle$.

Indeed, let $\omega_K := \frac{\sqrt{2} \cdot \eta}{\eta^2}$, $\nu \in (\Lambda_J^{2,0} \oplus \Lambda_J^{0,2})_{\mathbb{R}} = \omega_J^\perp \cap \Lambda^+$ of norm $\sqrt{2}$ and consider an element $s\omega_K + t\nu$ in $(\Lambda_J^{2,0} \oplus \Lambda_J^{0,2})_{\mathbb{R}} \oplus \langle \eta \rangle$; $s, t \in \mathbb{R}$. We have $(s\omega_K + t\nu)^2 = 2s^2 + 2t^2 + 2st\omega_K \cdot \nu = 2s^2 + 2t^2 + 2st\omega_K^+ \cdot \nu$, where $\omega_K = \omega_K^+ + \omega_K^-$ is the decomposition in self-dual and anti-self-dual parts of ω_K . There exists a real number c depending on η and Λ^+ such that $\omega_K^+ = c\omega$. When ω_J is sufficiently close to ω we have $|\omega_K^+ \cdot \nu| = |c\omega \cdot \nu| < \frac{1}{2}$ for all $\nu - s$, hence our claim. Thus we may view $(\Lambda_J^{2,0} \oplus \Lambda_J^{0,2})_{\mathbb{R}} \oplus \langle \eta \rangle$ as the space of self-dual forms with respect to a new Riemannian metric g' which is compatible with J .

Since $NS(X_J)$ is generated by a , a will span $\Lambda_{g'}^- \cap H^2(X, \mathbb{Z})$ too. The complex structure K corresponding to ω_K is the quaternionic deformation of J (with respect to g') we have been looking for. \square

The next move is to construct a stable vector bundle E on X_K with $c_1(E) = a$ and smallest possible $c_2(E)$, i.e. such that $\Delta(E) = \frac{1}{4}$.

Lemma 3.2 *Let X_K be a complex 2-dimensional torus whose Néron-Severi group is generated over \mathbb{Q} by a and η where a is primitive with $a^2 = -2(4k \pm 1)$, k a positive integer and η is an ample class of type $(1, k)$ orthogonal to a . Then there exists a rank 2 vector bundle E on X_K , stable with respect to η and having $c_1(E) = a$, $c_2(E) = \frac{1}{4}(a^2 + 2)$.*

Proof

We first prove the existence of a holomorphic rank 2 vector bundle E on X_K with the given invariants which is simple. We begin with the case $a^2 = -2(4k + 1)$. Let A, L be line bundles on X_K having $c_1(A) = a$, $c_1(L) = \eta$. We have $\chi(A^{-1} \otimes L^{-2}) = \frac{1}{2}(a + 2\eta)^2 = -1$ by Riemann-Roch hence $\text{Ext}^1(X; L \otimes A, L^{-1}) \cong H^1(X_K, A^{-1} \otimes L^{-2}) \neq 0$ and there exists a nontrivial extension

$$0 \longrightarrow L^{-1} \longrightarrow E \longrightarrow A \otimes L \longrightarrow 0.$$

Notice that E has the required Chern classes. The fact that E is simple is implied by the vanishing of $\text{Hom}(L^{-1}, A \otimes L)$ and of $\text{Hom}(A \otimes L, L^{-1})$ as one can easily check.

Let now $a^2 = -2(4k - 1)$, $k \geq 1$. As before we consider two line bundles A and L on X_K having $c_1(A) = a$ and $c_1(L) = \eta$. Since L is ample and $(2\eta + a)^2 = 2 > 0$, the bundle $L^2 \otimes A$ will have a nontrivial section vanishing on a divisor, say D . For numerical reasons D must have a reduced component. We may then choose a point p on the regular part of D , seen as a subvariety of X_K . Let Z be the reduced subspace of X_K consisting of the point p .

We want to construct E as the middle term of an extension

$$0 \longrightarrow L^{-1} \longrightarrow E \longrightarrow \mathcal{I}_Z \otimes A \otimes L \longrightarrow 0.$$

Such an extension is given by an element $\theta \in \text{Ext}^1(X_K; \mathcal{I}_Z \otimes A \otimes L, L^{-1})$. By a criterion of Serre E is locally free if and only if the image of θ through the canonical mapping

$$\text{Ext}^1(X_K; \mathcal{I}_Z \otimes A \otimes L, L^{-1}) \longrightarrow H^0(X_K, \mathcal{E}xt^1(\mathcal{I}_Z \otimes A \otimes L, L^{-1}))$$

generates the sheaf $\mathcal{E}xt^1(\mathcal{I}_Z \otimes A \otimes L, L^{-1})$; cf. [25] I.5.

From the exact sequence of the first terms of the Ext spectral sequence.

$$\begin{aligned} 0 &\longrightarrow H^1(X_K; \mathcal{H}om(\mathcal{I}_Z \otimes A \otimes L, L^{-1})) \\ &\longrightarrow \text{Ext}^1(X_K; \mathcal{I}_Z \otimes A \otimes L, L^{-1}) \longrightarrow H^0(X_K, \mathcal{E}xt^1(\mathcal{I}_Z \otimes A \otimes L, L^{-1})) \\ &\longrightarrow H^2(X_K, \mathcal{H}om(\mathcal{I}_Z \otimes A \otimes L, L^{-1})) \longrightarrow \text{Ext}^2(X_K; \mathcal{I}_Z \otimes A \otimes L, L^{-1}), \end{aligned}$$

we see that in our situation

$$\text{Ext}^1(X_K, \mathcal{I}_Z \otimes A \otimes L, L^{-1}) \longrightarrow H^0(X_K, \mathcal{E}xt^1(\mathcal{I}_Z \otimes A \otimes L, L^{-1}))$$

is an isomorphism, and since

$$\mathcal{E}xt^1(\mathcal{I}_Z \otimes A \otimes L, L^{-1}) \cong \mathcal{O}_Z,$$

a non-zero θ will give a locally free middle term E .

Let then E be such a locally free sheaf. We shall prove that E is simple. The first term of the exact sequence

$$0 \longrightarrow \text{Hom}(E, L^{-1}) \longrightarrow \text{End}(E) \longrightarrow \text{Hom}(E, \mathcal{I}_Z \otimes A \otimes L)$$

vanishes, so it will be enough to prove that

$$\text{Hom}(E, \mathcal{I}_Z \otimes A \otimes L) \cong H^0(X_K, E^\vee \otimes \mathcal{I}_Z \otimes A \otimes L) \cong H^0(X_K, E \otimes \mathcal{I}_Z \otimes L)$$

is one-dimensional. But this holds since

$$H^0(X_K, \mathcal{I}_Z \otimes \mathcal{I}_Z \otimes L^2 \otimes A) \cong H^0(X_K, \mathcal{I}_Z \otimes \mathcal{I}_Z(D))$$

is one-dimensional; (use the fact that p is a simple point on the regular part of D and that D does not move, so $\mathcal{I}_Z^2(D)$ can have no global section).

We now turn to the proof of the stability of E (in both cases).

Suppose E were not stable. There would exist then a locally free subsheaf B of rank 1 of E having $\deg_\eta B \geq 0$. Let

$$b := c_1(B) = s\eta + ta; \quad s, t \in \mathbb{Q}.$$

Since η and a are primitive, s and t must be either both in \mathbb{Z} or both in $\frac{1}{2} \cdot \mathbb{Z}$. Moreover, $\deg B \geq 0$ is equivalent to $s \geq 0$. We may also suppose that E/B is torsion-free.

Thus there exists a 2-codimensional subspace Y of X_K such that E sits in an exact sequence

$$0 \longrightarrow B \longrightarrow E \longrightarrow \mathcal{I}_Y \otimes A \otimes B^{-1} \longrightarrow 0.$$

Hence

$$\frac{1}{4}(a^2 + 2) = c_2(E) = b(a - b) + c_2(\mathcal{I}_Y) = b(a - b) + h^0(\mathcal{O}_Y).$$

On the other side, since E is simple

$$0 = \text{Hom}(\mathcal{I}_Y \otimes A \otimes B^{-1}, B) = H^0(A^{-1} \otimes B^2)$$

and thus $(2b - a)^2 \leq 0$. This further gives $4h^0(\mathcal{O}_Y) \leq 2$, hence $Y = \emptyset$ and $(2b - a)^2 = -2$.

Using the two exact sequences we have for E and again the fact that E is simple we get

$$H^0(L \otimes A \otimes B^{-1}) \neq 0 \text{ and } H^0(L^{-1} \otimes A^{-1} \otimes B) = 0,$$

hence $s \in \{0, \frac{1}{2}\}$. But then b cannot fulfill the condition $(2b - a)^2 = -2$. \square

Proof of Theorem 1.1

We start with a non-algebraic torus X_I , a primitive element $a \in NS(X_I)$ with $a^2 = -2(4k \pm 1)$, $k \geq 1$, and a differentiable vector bundle E on X_I with $c_1(E) = a$ and

$$\Delta(E) := \frac{1}{2} \left(c_2(E) - \frac{c_1(E)^2}{4} \right) = \frac{1}{4}.$$

We shall show that E admits a holomorphic structure.

Consider deformations X_J, X_K of X_I as in Lemma 3.1. Now Lemma 3.2 shows the existence over (X_K, η) of a stable holomorphic structure on E . By the Kobayashi-Hitchin correspondence we get an anti-self-dual connection on E with respect to the Riemannian metric which corresponds to η . The curvature of this connection remains of type (1,1) when considered on X_J , and thus gives a holomorphic structure E_J on E over X_J . Now

$$\frac{1}{4} = \Delta(E) < \frac{c_1(E)^2}{8} = k \pm \frac{1}{4}.$$

By Theorem 2.15 E_J must be irreducible. Thus E_J is stable with respect to any polarization on X_J . In particular E_J has an anti-self-dual connection with respect to ω_J too.

By deforming back to X_I we obtain a holomorphic structure E_I on E over X_I in the same way as on X_J .

By Proposition 2.16 any rank 2 topological vector bundle with $c_1(E) = a$ and $\Delta(E) \geq 0$ will admit some holomorphic structure. \square

4 Stability versus irreducibility

Irreducible vector bundles are stable with respect to any polarization on the base surface. In particular they admit anti-self-dual connections if their associated determinant bundles do.

In this paragraph we construct stable vector bundles of rank two on any polarized 2-dimensional torus X using the anti-self-dual connections which exist on irreducible vector bundles over some non-algebraic quaternionic deformation of X . In order to do this we first prove Theorem 1.3 which tells us which rank 2 vector bundles on a non-algebraic torus admit irreducible structures.

We start with a non-existence result.

Lemma 4.1 *Let X be a 2-dimensional complex torus, ω a Kähler class on it and $a \in NS(X)$. When $a = 0$ and $c \in \{0, 1\}$ or when $a^2 = -2$, $a \cdot \omega = 0$ and $c = 0$, there exists no stable (with respect to ω) rank 2 vector bundle E on X with $c_1(E) = a$ and $c_2(E) = c$.*

Proof

Suppose that E is a stable rank 2 vector bundle on X of degree zero with respect to ω . We consider its Fourier-Mukai transform:

Let \mathcal{P} be the Poincaré line-bundle on $X \times \text{Pic}^0(X)$ where $\text{Pic}^0(X)$ denotes the variety of topologically trivial holomorphic line bundles on X . Let $p_1 : X \times \text{Pic}^0(X) \rightarrow X$, $p_2 : X \times \text{Pic}^0(X) \rightarrow \text{Pic}^0(X)$ be the projections and $E^\wedge := R^1 p_{2*}(p_1^*(E) \otimes \mathcal{P})$. Since $H^0(X; E \otimes L)$ and $H^2(X; E \otimes L)$ vanish for all $L \in \text{Pic}^0(X)$ it follows that E^\wedge is locally free of rank $-\chi(E) = c_2(E) - \frac{1}{2}c_1(E)^2$.

When one computes the Chern classes of E^\wedge with Grothendieck-Riemann-Roch, one gets:

$$\begin{aligned} \text{rank}(E^\wedge) &= c_2(E) - \frac{1}{2}c_1(E)^2, \\ \text{rank } E &= c_2(E^\wedge) - \frac{1}{2}c_1(E^\wedge)^2 \\ &= c_2(E^\wedge) - \frac{1}{2}c_1(E)^2, \end{aligned}$$

hence $c_2(E^\wedge) = 2 + \frac{1}{2}c_1(E)^2$.

Thus in the considered cases we get $\text{rank}(E^\wedge) \leq 1$ and $c_2(E^\wedge) \neq 0$ which contradicts the locally freeness of E^\wedge . \square

Next we need a reformulation of part (b) of Lemma 3.1 which will ensure the existence of a convenient quaternionic deformation of I .

Lemma 4.2 *Let (X_I, ω_I) be a polarized 2-dimensional torus. There exists then a quaternionic deformation J of I such that $NS(X_J) \subset \Lambda^-$.*

The proof is the same as for part (b) of Lemma 1: just take ω_J on the sphere of radius $\sqrt{2}$ in Λ^+ and not on lines of the type $(\langle b \rangle \oplus \Lambda^-) \cap \Lambda^+$ with $b \in H^2(X, \mathbb{Z}) \setminus \Lambda^-$.

With these preparations the proofs of theorems 1.3 and 1.2 will follow from our Theorem 1.1 and the argument in [3] § 5 where irreducible vector bundles are found in the versal deformation of reducible ones provided that the discriminant is sufficiently large.

More precisely, if X is a non-algebraic torus and E is a holomorphic rank 2 vector bundle on X with $c_1(E)$ primitive in $NS(X)$, then it is proved in [3] 5.10 that there exist irreducible vector bundles in the versal deformation of E provided

$$\Delta(E) \geq 1 + \frac{1}{8}c_1(E)^2.$$

Now let E be a topological vector bundle of rank 2 on a non-algebraic torus X with $c_1(E) \in NS(X)$ and $\Delta(E) \geq 0$. We investigate when E admits an irreducible holomorphic structure. We may suppose that $c_1(E)$ is a primitive element in $NS(X)$, otherwise we twist by a line bundle, etc. By Theorem 1.1 E admits a holomorphic structure which we denote also by E . By Theorem 2.15 this holomorphic structure is necessarily irreducible if

$$\Delta(E) < -\frac{1}{8}c_1(E)^2.$$

By the above mentioned result from [3] we find an irreducible structure in the versal deformation of E as soon as

$$\Delta(E) \geq 1 + \frac{1}{8}c_1(E)^2.$$

It remains to consider the case when

$$-\frac{1}{8}c_1(E)^2 \leq \Delta(E) < 1 + \frac{1}{8}c_1(E)^2.$$

But now

$$c_1(E)^2 = 0 \text{ and } c_2(E) \in \{0, 1\} \text{ or } c_1(E)^2 = -2 \text{ and } c_2(E) = 0.$$

By Lemma 3.1 there is a Kähler class ω on X such that $c_1(E) \cdot \omega = 0$ and by Lemma 4.1 E cannot be stable with respect to ω . Thus this is exactly the case when E cannot admit irreducible structures. The proof of theorem 1.3 is completed.

We can now prove Theorem 1.2. Let X_I be a complex torus and ω_I a Kähler class on it. Let $a \in NS(X_I)$ such that $a \cdot \omega_I = 0$ and

$$a^2 = \max\{(a + 2b)^2 \mid b \in NS(X), b \cdot \omega_I = 0\}.$$

(Recall that the intersection form is negative definite on the orthogonal of ω_I in $H^{1,1}$).

Let $c \in \mathbb{Z}$ be such that

$$\begin{aligned} c &\geq 2 \text{ if } a = 0, \\ c &\geq 1 \text{ if } a^2 = -2 \text{ and} \\ c &\geq \frac{a^2}{4} \text{ if } a^2 \leq -4. \end{aligned}$$

We have to show that stable rank 2 vector bundles E with $c_1(E) = a$ and $c_2(E) = c$ exist on (X_I, ω_I) . (The other implication follows from Lemma 4.1 already.)

For this we consider a quaternionic deformation J as in Lemma 4.2. X_J is non-algebraic and $\Lambda_I^- = \Lambda_J^-$. Hence

$$\begin{aligned} a^2 &= \max\{(a + 2b)^2 \mid b \in NS(X_I), b \cdot \omega_I = 0\} \\ &= \max\{(a + 2b)^2 \mid b \in NS(X_J)\} \end{aligned}$$

and by Theorem 1.3 there exists an irreducible rank 2 vector bundle E_J on X_J with $c_1(E_J) = a$, $c_2(E_J) = c$. In particular E_J is stable with respect to ω_J and thus admits an anti-self-dual connection. This connection remains anti-self-dual on (X_I, ω_I) , as the Riemannian structure does not change, and induces a holomorphic structure E_I over X_I on the underlying differentiable bundle. By the Kobayashi-Hitchin correspondence E_I is stable or splits into a sum of line bundles of degree zero which are invariant under the connection. In the last case we should get a splitting of E_J as a direct sum of holomorphic line bundles on X_J , which is not the case. Thus E_I is stable and Theorem 1.2 is proved.

5 Compactifying moduli spaces of stable bundles

5.1 The stable irreducibility condition

In this paragraph we show that certain components of the moduli space of simple sheaves over a compact complex surface X are compact. This will always work over the stable irreducible range. However we should like to point out here that when X is Kählerian one can relax the stable irreducibility condition in the following way.

Let g be a Gauduchon metric on X with Kähler form ω and let $\mathcal{M}^{st}(E, L)$ denote the moduli space of stable holomorphic structures in a vector bundle E of rank $r > 1$, determinant $L \in \text{Pic}(X)$ and second Chern class $c \in H^4(X, \mathbb{Z}) \cong \mathbb{Z}$. We consider the following condition on $(r, c_1(L), c)$:

$$(*) \quad \text{every semi-stable vector bundle } \mathcal{E} \text{ with } \text{rank}(\mathcal{E}) = r, \\ c_1(\mathcal{E}) = c_1(L) \text{ and } c_2(\mathcal{E}) \leq c \text{ is stable.}$$

Under this condition Buchdahl constructed a compactification of $\mathcal{M}^{st}(E, L)$ in [7]. We shall show that under this same condition one can compactify $\mathcal{M}^{st}(E, L)$ allowing simple coherent sheaves in the border. For simplicity we shall restrict ourselves to the case $\deg_\omega L = 0$. When $b_1(X)$ is odd we can always reduce ourselves to this case by a suitable twist with a topologically trivial line bundle; (see the following Remark).

The condition $(*)$ takes a different aspect according to the parity of the first Betti number of X or equivalently, according to the existence or non-existence of a Kähler metric on X .

Remark 5.1 (a) *When $b_1(X)$ is odd $(*)$ is equivalent to: "every torsion free sheaf \mathcal{F} on X with $\text{rank}(\mathcal{F}) = r$, $c_1(\mathcal{F}) = c_1(L)$ and $c_2(\mathcal{F}) \leq c$ is irreducible", i.e. $(r, c_1(L), c)$ describes the topological invariants of a stably irreducible vector bundle.*

(b) *When $b_1(X)$ is even and $c_1(L)$ is not a torsion class in $H^2(X, \mathbb{Z}_r)$ one can find a Kähler metric g such that $(r, c_1(L), c)$ satisfies $(*)$ for all c .*

(c) *When $b_1(X)$ is odd or when $\deg L = 0$, $(*)$ implies $c < 0$.*

(d) *If $b_2(X) = 0$ then there is no coherent sheaf on X whose invariants satisfy $(*)$.*

Proof

It is clear that the stable irreducibility condition is stronger than $(*)$. Now if a sheaf \mathcal{F} is not irreducible it admits some subsheaf \mathcal{F}' with $0 < \text{rank } \mathcal{F}' < \text{rank } \mathcal{F}$. When $b_1(X)$ is odd the degree function $\deg_\omega : \text{Pic}^0(X) \rightarrow \mathbb{R}$ is surjective, so twisting by suitable invertible sheaves $L_1, L_2 \in \text{Pic}^0(X)$ gives a semi-stable but not stable sheaf $(L_1 \otimes \mathcal{F}') \oplus (L_2 \otimes (\mathcal{F}/\mathcal{F}'))$

with the same Chern classes as \mathcal{F} . Since by taking double-duals the second Chern class decreases, we get a locally free sheaf

$$(L_1 \otimes (\mathcal{F}')^{\vee\vee} \oplus (L_2 \otimes (\mathcal{F}/\mathcal{F}')^{\vee\vee})$$

which contradicts (*) for $(\text{rank}(\mathcal{F}), c_1(\mathcal{F}), c_2(\mathcal{F}))$. This proves (a).

For (b) it is enough to take a Kähler class ω such that

$$\omega(r' \cdot c_1(L) - r \cdot \alpha) \neq 0 \text{ for all } \alpha \in NS(X)/\text{Tors}(NS(X))$$

and integers r' with $0 < r' < r$. This is possible since the Kähler cone is open in $H^{1,1}(X)$.

For (c) just consider $(L \otimes L_1) \oplus \mathcal{O}_X^{\otimes(r-1)}$ for a suitable $L_1 \in \text{Pic}^0(X)$ in case $b_1(X)$ odd.

Finally, suppose $b_2(X) = 0$. Then X admits no Kähler structure hence $b_1(X)$ is odd. If \mathcal{F} were a coherent sheaf on X whose invariants satisfy (*) we should have

$$\Delta(\mathcal{F}) = \frac{1}{r} \left(c_2 - \frac{(r-1)}{2r} c_1(L)^2 \right) = \frac{1}{r} c_2 < 0$$

contradicting Theorem 2.12. \square

5.2 The moduli space of simple sheaves

We fix as usual a complex surface X , although most constructions here should work over a general base space.

The existence of a coarse moduli space Spl_X for simple (torsion-free) sheaves over a compact complex space has been proved in [17] (also in the relative context). The idea is to consider all versal deformations of simple sheaves over X , to show that they are universal and to consider the union of these parameter spaces modulo the natural equivalence relation; cf. [27]. The resulting complex space will be possibly non-Hausdorff. Each of its points will have a neighborhood isomorphic to the base of the versal deformation of the corresponding simple sheaf.

The following is a well-known separation criterion for the moduli space of simple sheaves.

Remark 5.2 *If $\mathcal{F}_1, \mathcal{F}_2$ represent distinct non-separated classes of isomorphism of simple sheaves in Spl_X then there exist non-trivial morphisms $\epsilon_1 : \mathcal{F}_1 \rightarrow \mathcal{F}_2$, $\epsilon_2 : \mathcal{F}_2 \rightarrow \mathcal{F}_1$ such that $\epsilon_1 \circ \epsilon_2 = 0$ and $\epsilon_2 \circ \epsilon_1 = 0$.*

In particular when \mathcal{F}_1 and \mathcal{F}_2 are stable with respect to some polarization on X , $[\mathcal{F}_1]$ and $[\mathcal{F}_2]$ are separated in Spl_X .

Proof

If $[\mathcal{F}_1], [\mathcal{F}_2]$ are non-separated in Spl_X one gets sequences of points $s_{1,n}$ and $s_{2,n}$ converging to the centers of the bases of versal deformations of \mathcal{F}_1 and \mathcal{F}_2 and such that $\mathcal{F}_{s_{1,n}} \cong \mathcal{F}_{s_{2,n}}$. By semi-continuity one deduces $\text{Hom}(X; \mathcal{F}_1, \mathcal{F}_2) \neq 0$ and $\text{Hom}(X; \mathcal{F}_2, \mathcal{F}_1) \neq 0$, (cf. [4]).

Hence non-trivial morphism $\epsilon_1 : \mathcal{F}_1 \rightarrow \mathcal{F}_2$, $\epsilon_2 : \mathcal{F}_2 \rightarrow \mathcal{F}_1$ exist. Since $\mathcal{F}_1, \mathcal{F}_2$ are simple but non-isomorphic we have

$$\epsilon_1 \circ \epsilon_2 = 0 \text{ and } \epsilon_2 \circ \epsilon_1 = 0.$$

Suppose now that $\mathcal{F}_1, \mathcal{F}_2$ are stable with respect to some polarization on X and let ϵ_1, ϵ_2 be morphisms as above.

Then

$$\frac{\deg \mathcal{F}_1}{\text{rank } \mathcal{F}_1} < \frac{\deg \text{Im}(\epsilon_1)}{\text{rank } \text{Im}(\epsilon_1)} < \frac{\deg \mathcal{F}_2}{\text{rank } \mathcal{F}_2} < \frac{\deg \text{Im}(\epsilon_2)}{\text{rank } \text{Im}(\epsilon_2)} < \frac{\deg \mathcal{F}_1}{\text{rank } \mathcal{F}_1}$$

gives a contradiction. (One could have relaxed the stability assumption on one of the sheaves to semi-stability.) \square

In order to give a better description of the base of the versal deformation of a coherent sheaf \mathcal{F} we need to compare it to the deformation of its determinant line bundle $\det \mathcal{F}$.

For surfaces the determinant line bundle may be constructed using a global locally free resolution for \mathcal{F} . (A more general construction is done in [16]). In fact, one can show that locally free resolutions exist also in the relative case.

Proposition 5.3 *Let X be a nonsingular compact complex surface, $(S, 0)$ a complex space germ, \mathcal{F} a coherent sheaf on $X \times S$ flat over S and $q : X \times S \rightarrow X$ the projection. If the central fiber $\mathcal{F}_0 := \mathcal{F}|_{X \times \{0\}}$ is torsion-free then there exists a locally free resolution of \mathcal{F} over $X \times S$ of the form*

$$0 \longrightarrow q^*G \longrightarrow E \longrightarrow \mathcal{F} \longrightarrow 0$$

where G is a locally free sheaf on X .

Proof

In [28] it is proven that a resolution of \mathcal{F}_0 of the form

$$0 \longrightarrow G \longrightarrow E_0 \longrightarrow \mathcal{F}_0 \longrightarrow 0$$

exists on X with G and E_0 locally free on X as soon as the rank of G is large enough and

$$H^2(X, \mathcal{H}om(\mathcal{F}_0, G)) = 0.$$

We only have to notice that when \mathcal{F}_0 and G vary in some flat families over S then one can extend the above exact sequence over $X \times S$. For this we use the spectral sequence

$$H^p(S, \mathcal{E}xt^q(p; \mathcal{F}, q^*G)) \implies \text{Ext}^{p+q}(X \times S; \mathcal{F}, q^*G)$$

relating the relative and global Ext-s. Here $p : X \times S \rightarrow S$ denotes the projection. There is an exact sequence

$$\begin{aligned} 0 &\longrightarrow H^1(S, p_* \mathcal{H}om(\mathcal{F}, q^*G)) \longrightarrow \text{Ext}^1(X \times S, \mathcal{F}, q^*G) \\ &\longrightarrow H^0(S, \mathcal{E}xt^1(p; \mathcal{F}, q^*G)) \longrightarrow H^2(S, p_* \mathcal{H}om(\mathcal{F}, q^*G)). \end{aligned}$$

If we choose S to be Stein we thus get surjectivity for the natural map

$$\text{Ext}^1(X \times S; \mathcal{F}, q^*G) \longrightarrow H^0(S, \mathcal{E}xt^1(p; \mathcal{F}, q^*G)).$$

We can apply the base change theorem for the relative Ext^1 sheaf if we know that $\text{Ext}^2(X; \mathcal{F}_0, G) = 0$ (cf. [4] Korollar 1). But in the spectral sequence

$$H^p(X, \mathcal{E}xt^q(\mathcal{F}_0, G)) \implies \text{Ext}^{p+q}(X; \mathcal{F}_0, G)$$

relating the local Ext -s to the global ones, all degree two terms vanish since $H^2(X; \mathcal{H}om(\mathcal{F}_0, G)) = 0$ by assumption.

Thus by base change

$$\text{Ext}^1(X; \mathcal{F}_0, G) \cong \mathcal{E}xt^1(p; \mathcal{F}, q^*G)_0 / \mathfrak{m}_{S,0} \cdot \mathcal{E}xt^1(p; \mathcal{F}, q^*G)$$

and the natural map

$$\text{Ext}^1(X \times S; \mathcal{F}, q^*G) \longrightarrow \text{Ext}^1(X; \mathcal{F}_0, G)$$

given by restriction is surjective. \square

Let X, S and \mathcal{F} be as above. Proposition 5.3 allows us to define a morphism

$$\det : (S, 0) \longrightarrow (\text{Pic}(X), \det \mathcal{F}_0)$$

by associating to \mathcal{F} its **determinant line bundle** $\det \mathcal{F}$. More generally if \mathcal{F} has a locally free resolution

$$0 \longrightarrow F^{-m} \longrightarrow \dots \longrightarrow F^{-1} \longrightarrow F^0 \longrightarrow \mathcal{F} \longrightarrow 0$$

then $\det \mathcal{F} := \bigotimes_i \det(F^i)^{(-1)^i}$ where

$$\det F^i := \Lambda^{\text{rank}(F^i)} F^i.$$

We get a line bundle $\det \mathcal{F}$ on $X \times S$ such that $(\det \mathcal{F})_0 = (\det \mathcal{F}_0)$ since \mathcal{F} is flat. The wanted map

$$\det : S \longrightarrow \text{Pic}(X)$$

follows now from the universal property of $\text{Pic}(X)$.

The tangent space at the isomorphism class $[\mathcal{F}] \in \text{Spl}_X$ of a simple sheaf \mathcal{F} is $\text{Ext}^1(X; \mathcal{F}, \mathcal{F})$ since Spl_X is locally around $[\mathcal{F}]$ isomorphic to the base of the versal deformation of \mathcal{F} . The space of obstructions to the extension of a deformation of \mathcal{F} is $\text{Ext}^2(X; \mathcal{F}, \mathcal{F})$.

In order to state the next theorem which compares the deformations of \mathcal{F} and $\det \mathcal{F}$, we have to recall the definition of the **trace** maps

$$tr^q : \text{Ext}^q(X; \mathcal{F}, \mathcal{F}) \longrightarrow H^q(X, \mathcal{O}_X).$$

When \mathcal{F} is locally free one defines $tr_{\mathcal{F}} : \mathcal{E}nd(\mathcal{F}) \longrightarrow \mathcal{O}_X$ in the usual way by taking local trivializations of \mathcal{F} . Suppose now that \mathcal{F} has a locally free resolution F^\bullet . (See [30] and [14] for more general situations.) Then one defines

$$tr_{F^\bullet} : \mathcal{H}om^\bullet(F^\bullet, F^\bullet) \longrightarrow \mathcal{O}_X$$

by

$$tr_{F^\bullet} |_{\mathcal{H}om(F^i, F^j)} = \begin{cases} (-1)^i tr_{F^i}, & \text{for } i = j \\ 0, & \text{for } i \neq j. \end{cases}$$

Here we denoted by $\mathcal{H}om^\bullet(F^\bullet, F^\bullet)$ the complex having $\mathcal{H}om^n(F^\bullet, F^\bullet) = \bigoplus_i \mathcal{H}om(F^i, F^{i+n})$ and differential

$$d(\varphi) = d_{F^\bullet} \circ \varphi - (-1)^{\deg \varphi} \cdot \varphi \circ d_{F^\bullet}$$

for local sections $\varphi \in \mathcal{H}om^n(F^\bullet, F^\bullet)$. tr_{F^\bullet} becomes a morphism of complexes if we see \mathcal{O}_X as a complex concentrated in degree zero.

Thus tr_{F^\bullet} induces morphisms at hypercohomology level. Since the hypercohomology groups of $\mathcal{H}om^\bullet(F^\bullet, F^\bullet)$ and of \mathcal{O}_X are $\text{Ext}^q(X; \mathcal{F}, \mathcal{F})$ and $H^q(X, \mathcal{O}_X)$ respectively, we get our desired maps

$$tr^q : \text{Ext}^q(X; \mathcal{F}, \mathcal{F}) \longrightarrow H^q(X, \mathcal{O}_X).$$

Using tr^0 over open sets of X we get a sheaf homomorphism $tr : \mathcal{E}nd(\mathcal{F}) \longrightarrow \mathcal{O}_X$. Let $\mathcal{E}nd_0(\mathcal{F})$ be its kernel. Then we have a naturally split exact sequence:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E}nd(\mathcal{F}) \longrightarrow \mathcal{E}nd_0(\mathcal{F}) \longrightarrow 0$$

inducing a commutative diagram:

$$\begin{array}{ccc} H^q(X, \mathcal{O}_X) \oplus H^q(X, \mathcal{E}nd_0(\mathcal{F})) & \xlongequal{\quad} & H^q(X, \mathcal{E}nd(\mathcal{F})) \longrightarrow \text{Ext}^q(X; \mathcal{F}, \mathcal{F}) \\ & & \downarrow \begin{array}{l} H^q(tr) \\ \swarrow tr^q \end{array} \\ & & H^q(X, \mathcal{O}_X). \end{array}$$

In particular tr^q are always surjective. If one denotes the kernel of $tr^q : \text{Ext}^q(X; \mathcal{F}, \mathcal{F}) \longrightarrow H^q(X, \mathcal{O}_X)$ by $\text{Ext}^q(X, \mathcal{F}, \mathcal{F})_0$ one gets natural maps $H^q(X, \mathcal{E}nd_0(\mathcal{F})) \longrightarrow \text{Ext}^q(X, \mathcal{F}, \mathcal{F})_0$, which are isomorphisms for \mathcal{F} locally free.

This construction generalizes immediately to give trace maps

$$tr^q : \text{Ext}^q(X; \mathcal{F}, \mathcal{F} \otimes N) \longrightarrow H^q(X, N)$$

for locally free sheaves N on X or for sheaves N such that $\mathcal{T}or_i^{\mathcal{O}_X}(N, \mathcal{F})$ vanish for $i > 0$.

The following easy Lemma says that the trace map is graded symmetric with respect to the Yoneda pairing

$$\text{Ext}^p(X; \mathcal{F}, \mathcal{G}) \times \text{Ext}^q(X; \mathcal{G}, \mathcal{E}) \longrightarrow \text{Ext}^{p+q}(X, \mathcal{F}, \mathcal{E}).$$

Lemma 5.4 *If \mathcal{F} and \mathcal{G} are sheaves on X allowing finite locally free resolutions and $u \in \text{Ext}^p(X; \mathcal{F}, \mathcal{G})$, $v \in \text{Ext}^q(X; \mathcal{G}, \mathcal{F})$ then*

$$\text{tr}^{p+q}(u \cdot v) = (-1)^{p \cdot q} \text{tr}^{p+q}(v \cdot u).$$

Theorem 5.5 *Let X be a compact complex surface, $(S, 0)$ be a germ of a complex space and \mathcal{F} a coherent sheaf on $X \times S$ flat over S such that $\mathcal{F}_0 := \mathcal{F}|_{X \times \{0\}}$ is torsion-free. The following holds.*

(a) *The tangent map of $\det : S \rightarrow \text{Pic}(X)$ in 0 factorizes as*

$$T_0 S \xrightarrow{KS} \text{Ext}^1(X; \mathcal{F}, \mathcal{F}) \xrightarrow{\text{tr}^1} H^1(X, \mathcal{O}_X) = T_{[\det \mathcal{F}_0]}(\text{Pic}(X)).$$

(b) *If T is a zero-dimensional complex space such that $\mathcal{O}_{S,0} = \mathcal{O}_{T,0}/I$ for an ideal I of $\mathcal{O}_{T,0}$ with $I \cdot \mathfrak{m}_{T,0} = 0$, then the obstruction $\text{ob}(\mathcal{F}, T)$ to the extension of \mathcal{F} to $X \times T$ is mapped by*

$$\begin{aligned} \text{tr}^2 \otimes_{\mathbb{C}} \text{id}_I : \text{Ext}^2(X; \mathcal{F}_0, \mathcal{F}_0 \otimes_{\mathbb{C}} I) &\cong \text{Ext}^2(X; \mathcal{F}_0, \mathcal{F}_0) \otimes_{\mathbb{C}} I \longrightarrow \\ &\longrightarrow H^2(X, \mathcal{O}_X) \otimes_{\mathbb{C}} I \cong \text{Ext}^2(X; \det \mathcal{F}_0, (\det \mathcal{F}_0) \otimes_{\mathbb{C}} I) \end{aligned}$$

to the obstruction to the extension of $\det \mathcal{F}$ to $X \times T$ which is zero.

Proof

(a) We may suppose that S is the double point $(0, \mathbb{C}[\epsilon])$. We define the Kodaira-Spencer map by means of the Atiyah class (cf. [13]).

For a complex space Y let $p_1, p_2 : Y \times Y \rightarrow Y$ be the projections and $\Delta \subset Y \times Y$ the diagonal. Tensoring the exact sequence

$$0 \longrightarrow \mathcal{I}_{\Delta}/\mathcal{I}_{\Delta}^2 \longrightarrow \mathcal{O}_{Y \times Y}/\mathcal{I}_{\Delta}^2 \longrightarrow \mathcal{O}_{\Delta} \longrightarrow 0$$

by $p_2^* \mathcal{F}$ for \mathcal{F} locally free on Y and applying $p_{1,*}$ gives an exact sequence on Y

$$0 \longrightarrow \mathcal{F} \otimes \Omega_Y \longrightarrow p_{1,*}(p_2^* \mathcal{F} \otimes (\mathcal{O}_{Y \times Y}/\mathcal{I}_{\Delta}^2)) \longrightarrow \mathcal{F} \longrightarrow 0.$$

The class $A(\mathcal{F}) \in \text{Ext}^1(Y; \mathcal{F}, \mathcal{F} \otimes \Omega_Y)$ of this extension is called the **Atiyah class** of \mathcal{F} . When \mathcal{F} is not locally free but admits a finite locally free resolution F^\bullet one gets again a class $A(\mathcal{F})$ in $\text{Ext}^1(Y; \mathcal{F}, \mathcal{F} \otimes \Omega_Y)$ seen as first cohomology group of $\mathcal{H}om^\bullet(F^\bullet, F^\bullet \otimes \Omega_Y)$.

Consider now $Y = X \times S$ with X and S as before, $p : Y \rightarrow S$, $q : Y \rightarrow X$ the projections and \mathcal{F} as in the statement of the theorem.

The decomposition $\Omega_{X \times S} = q^* \Omega_X \oplus p^* \Omega_S$ induces

$$\begin{aligned} \text{Ext}^1(X \times S; \mathcal{F}, \mathcal{F} \otimes \Omega_{S \times X}) &\cong \\ \text{Ext}^1(X \times S; \mathcal{F}, \mathcal{F} \otimes q^* \Omega_X) \oplus \text{Ext}^1(X \times S; \mathcal{F}, \mathcal{F} \otimes p^* \Omega_S). \end{aligned}$$

The component $A_S(\mathcal{F})$ of $A(\mathcal{F})$ lying in $\text{Ext}^1(X \times S; \mathcal{F}, \mathcal{F} \otimes p^*\Omega_S)$ induces the "tangent vector" at 0 to the deformation \mathcal{F} through the isomorphisms

$$\begin{aligned} \text{Ext}^1(X \times S; \mathcal{F}, \mathcal{F} \otimes p^*\Omega_S) &\cong \text{Ext}^1(X \times S; \mathcal{F}, \mathcal{F} \otimes p^*\mathfrak{m}_{S,0}) \cong \\ \text{Ext}^1(X \times S; \mathcal{F}, \mathcal{F}_0) &\cong \text{Ext}^1(X; \mathcal{F}_0, \mathcal{F}_0). \end{aligned}$$

Applying now $tr^1 : \text{Ext}^1(Y; \mathcal{F}, \mathcal{F} \otimes \Omega_Y) \rightarrow H^1(Y; \Omega_Y)$ to the Atiyah class $A(\mathcal{F})$ gives the first Chern class of \mathcal{F} , $c_1(\mathcal{F}) := tr^1(A(\mathcal{F}))$, (cf. [14], [30]).

It is known that

$$c_1(\mathcal{F}) = c_1(\det \mathcal{F}), \text{ i.e. } tr^1(A(\mathcal{F})) = tr^1(A(\det \mathcal{F})).$$

Now $\det \mathcal{F}$ is invertible so

$$tr^1 : \text{Ext}^1(Y, \det \mathcal{F}, (\det \mathcal{F}) \otimes \Omega_Y) \longrightarrow H^1(Y, \Omega_Y)$$

is just the canonical isomorphism. Since tr^1 is compatible with the decomposition $\Omega_{X \times S} = q^*\Omega_X \oplus p^*\Omega_S$ we get $tr^1(A_S(\mathcal{F})) = A_S(\det \mathcal{F})$ which proves (a).

- (b) In order to simplify notation we drop the index 0 from $\mathcal{O}_{S,0}$, $\mathfrak{m}_{S,0}$, $\mathcal{O}_{T,0}$, $\mathfrak{m}_{T,0}$ and we use the same symbols \mathcal{O}_S , \mathfrak{m}_S , \mathcal{O}_T , \mathfrak{m}_T for the respective pulled-back sheaves through the projections $X \times S \rightarrow S$, $X \times T \rightarrow T$.

There are two exact sequences of \mathcal{O}_S -modules:

$$0 \longrightarrow \mathfrak{m}_S \longrightarrow \mathcal{O}_S \longrightarrow \mathbb{C} \longrightarrow 0, \quad (1)$$

$$0 \longrightarrow I \longrightarrow \mathfrak{m}_T \longrightarrow \mathfrak{m}_S \longrightarrow 0. \quad (2)$$

(Use $I \cdot \mathfrak{m}_T = 0$ in order to make \mathfrak{m}_T an \mathcal{O}_S -module.)

Let $j : \mathbb{C} \rightarrow \mathcal{O}_S$ be the \mathbb{C} -vector space injection given by the \mathbb{C} -algebra structure of \mathcal{O}_S . j induces a splitting of (1). Since \mathcal{F} is flat over S we get exact sequences over $X \times S$

$$\begin{aligned} 0 \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_S} \mathfrak{m}_S \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_0 \longrightarrow 0 \\ 0 \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_S} I \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_S} \mathfrak{m}_T \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_S} \mathfrak{m}_S \longrightarrow 0 \end{aligned}$$

which remain exact as sequences over \mathcal{O}_X . Thus we get elements in $\text{Ext}^1(X; \mathcal{F}_0, \mathcal{F} \otimes_{\mathcal{O}_S} \mathfrak{m}_S)$ and $\text{Ext}^1(X; \mathcal{F} \otimes_{\mathcal{O}_S} \mathfrak{m}_S, \mathcal{F} \otimes_{\mathbb{C}} I)$ whose Yoneda composite $ob(\mathcal{F}, T)$ in $\text{Ext}^2(X; \mathcal{F}_0, \mathcal{F} \otimes_{\mathbb{C}} I)$ is represented by the 2-fold exact sequence

$$0 \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_S} I \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_S} \mathfrak{m}_T \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_0 \longrightarrow 0$$

and is the obstruction to extending \mathcal{F} from $X \times S$ to $X \times T$, as is well-known.

Consider now a resolution

$$0 \longrightarrow q^*G \longrightarrow E \longrightarrow \mathcal{F} \longrightarrow 0$$

of \mathcal{F} as provided by Proposition 5.3, i.e. with G locally free on X and E locally free on $X \times S$. Our point is to compare $ob(\mathcal{F}, T)$ to $ob(E, T)$.

Since \mathcal{F} is flat over S we get the following commutative diagrams with exact rows and columns by tensoring this resolution with the exact sequences (1) and (2):

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & q^*G \otimes_{\mathbb{C}} \mathfrak{m}_S & \longrightarrow & q^*G & \longrightarrow & G_0 \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & E \otimes_{\mathcal{O}_S} \mathfrak{m}_S & \longrightarrow & E & \longrightarrow & E_0 \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{F} \otimes_{\mathcal{O}_S} \mathfrak{m}_S & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}_0 \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array} \quad (1')$$

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & q^*G \otimes_{\mathbb{C}} I & \longrightarrow & q^*G \otimes_{\mathbb{C}} \mathfrak{m}_T & \longrightarrow & q^*G \otimes_{\mathbb{C}} \mathfrak{m}_S \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & E \otimes_{\mathcal{O}_S} I & \longrightarrow & E \otimes_{\mathcal{O}_S} \mathfrak{m}_T & \longrightarrow & E \otimes_{\mathcal{O}_S} \mathfrak{m}_S \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{F} \otimes_{\mathcal{O}_S} I & \longrightarrow & \mathcal{F} \otimes_{\mathcal{O}_S} \mathfrak{m}_T & \longrightarrow & \mathcal{F} \otimes_{\mathcal{O}_S} \mathfrak{m}_S \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array} \quad (2')$$

Using the section $j : \mathbb{C} \rightarrow \mathcal{O}_S$ we get an injective morphism of \mathcal{O}_X sheaves

$$G_0 \xrightarrow{id_{q^*G} \otimes j} q^*G \otimes_{\mathbb{C}} \mathfrak{m}_T \longrightarrow E \otimes_{\mathcal{O}_S} \mathfrak{m}_T$$

which we call j_G .

From (1') we get a short exact sequence over X in the obvious way

$$0 \longrightarrow (E \otimes_{\mathcal{O}_S} \mathfrak{m}_S) \oplus j_G(G_0) \longrightarrow E \longrightarrow \mathcal{F}_0 \longrightarrow 0$$

Combining this with the middle row of (2') we get a 2-fold extension

$$0 \longrightarrow (E \otimes_{\mathcal{O}_S} I) \oplus G_0 \longrightarrow (E \otimes_{\mathcal{O}_S} \mathfrak{m}_T) \oplus G_0 \longrightarrow E \longrightarrow \mathcal{F}_0 \longrightarrow 0$$

whose class in $\text{Ext}^2(X; \mathcal{F}_0, (E \otimes_{\mathcal{O}_S} I) \otimes G_0)$ we denote by u .

Let v be the surjection $E \rightarrow \mathcal{F}$ and

$$v' := \begin{pmatrix} v \otimes id_I \\ 0 \end{pmatrix} : (E \otimes_{\mathcal{O}_S} I) \oplus G_0 \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_S} I,$$

$$v'' = \begin{pmatrix} v_0 \\ 0 \end{pmatrix} : E_0 \oplus G_0 \longrightarrow \mathcal{F}_0,$$

the \mathcal{O}_X -morphisms induced by v .

The commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (E \otimes_{\mathcal{O}_S} I) \oplus G_0 & \longrightarrow & (E \otimes_{\mathcal{O}_S} \mathfrak{m}_T) \oplus G_0 & \longrightarrow & E & \longrightarrow & \mathcal{F}_0 & \longrightarrow & 0 \\ & & \downarrow v' & & \downarrow (v \otimes id_{\mathfrak{m}_T}) & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F} \otimes_{\mathcal{O}_S} I & \longrightarrow & \mathcal{F} \otimes_{\mathcal{O}_S} \mathfrak{m}_T & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}_0 & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (E \otimes_{\mathcal{O}_S} I) \oplus G_0 & \longrightarrow & (E \otimes_{\mathcal{O}_S} \mathfrak{m}_T) \oplus G_0 & \longrightarrow & E \oplus G_0 & \longrightarrow & E_0 \oplus G_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow id & & \downarrow (id_{j_G}) & & \downarrow v'' & & \\ 0 & \longrightarrow & (E \otimes_{\mathcal{O}_S} I) \oplus G_0 & \longrightarrow & (E \otimes_{\mathcal{O}_S} \mathfrak{m}_T) \oplus G_0 & \longrightarrow & E & \longrightarrow & \mathcal{F}_0 & \longrightarrow & 0 \end{array}$$

show that $ob(\mathcal{F}, T) = v' \cdot u$ and

$$(ob(E, T), 0) = u \cdot v'' \in \text{Ext}^2(X; E_0 \oplus G_0, (E \otimes_{\mathcal{O}_S} I) \oplus G_0).$$

We may restrict ourselves to the situation when I is generated by one element. Then we have canonical isomorphisms of \mathcal{O}_X -modules $E_0 \cong E \otimes_{\mathcal{O}_S} I$ and $\mathcal{F}_0 \cong \mathcal{F} \otimes_{\mathcal{O}_S} I$. By these one may identify v' and v'' . Now the Lemma 5.4 on the graded symmetry of the trace map with respect to the Yoneda pairing gives $tr^2(ob(\mathcal{F}, T)) = tr^2(ob(E, T))$.

But E is locally free and the assertion (b) of the theorem may be proved for it as in the projective case by a cocycle computation.

Thus $tr^2(ob(E, T)) = ob(\det E)$ and since $\det(E) = (\det \mathcal{F}) \otimes q^*(\det G)$ and $q^*(\det G)$ is trivially extendable, the assertion (b) is true for \mathcal{F} as well. \square

The theorem should be true in a more general context. In fact the proof of (a) is valid for any compact complex manifold X and flat sheaf \mathcal{F} over $X \times S$. Our proof of (b) is in a way symmetric to the proof of Mukai in [24] who uses a resolution for \mathcal{F} of a special form in the projective case (see also [2]). The middle term of his resolution is a trivial bundle over $X \times S$, whereas in our case we use Proposition 5.3 to get a resolution whose last term is trivial in the S -direction. Therefore we need to require that the assumptions of Proposition 5.3 are fulfilled.

Notation For a compact complex surface X and an element L in $\text{Pic}(X)$ we denote by $Spl_X(L)$ the fiber of the morphism $\det : Spl_X \rightarrow \text{Pic}(X)$ over L .

Corollary 5.6 *For a compact complex surface X and $L \in \text{Pic}(X)$ the tangent space to $\text{Spl}_X(L)$ at an isomorphism class $[F]$ of a simple torsion-free sheaf F with $[\det F] = L$ is $\text{Ext}^1(X; F, F)_0$. When $\text{Ext}^2(X; F, F)_0 = 0$, $\text{Spl}_X(L)$ and Spl_X are smooth of dimensions*

$$\dim \text{Ext}^1(X; F, F)_0 = 2 \text{rank}(F)^2 \Delta(F) - (\text{rank}(F)^2 - 1) \chi(\mathcal{O}_X)$$

and

$$\dim \text{Ext}^1(X; F, F) = \dim \text{Ext}^1(X; F, F)_0 + h^1(\mathcal{O}_X)$$

respectively.

Proof It is enough to notice that tensoring with an element B of $\text{Pic}(X)$ gives an isomorphism between $\text{Spl}_X(L)$ and $\text{Spl}_X(L \otimes B^{\text{rank}(F)})$ making $\det : \text{Spl}_X \rightarrow \text{Pic}(X)$ into a fibre bundle over each connected component of $\text{Pic}(X)$. One applies then the Theorem and computes the dimensions with Riemann-Roch. \square

We end this paragraph by a remark on the symplectic structure of the moduli space Spl_X when X is symplectic.

Recall that a complex manifold M is called **holomorphically symplectic** if it admits a global nondegenerate closed holomorphic two-form ω . For a surface X , being holomorphically symplectic thus means that the canonical line bundle K_X is trivial. The Enriques-Kodaira classification of surfaces gives us exactly three classes of holomorphically symplectic surfaces:

- 2-dimensional complex tori
- $K3$ surfaces (i.e. surfaces X with $h^1(\mathcal{O}_X) = 0$ and K_X trivial) and
- primary Kodaira surfaces (i.e. topologically non-trivial elliptic principal bundles over elliptic curves).

For such an X , Spl_X is smooth and holomorphically symplectic as well. The smoothness follows immediately from the above Corollary since for a simple sheaf F on X one has by Serre duality:

$$\begin{aligned} \text{Ext}^2(X; F, F) &\cong \text{Ext}^0(X; F, F \otimes K_X)^* \\ &= \text{Hom}(X; F, F \otimes K_X)^* = \text{Hom}(X; F, F)^* \cong \mathbb{C}. \end{aligned}$$

A two-form ω is defined at $[F]$ on Spl_X as the composition:

$$\begin{aligned} T_{[F]} \text{Spl}_X \times T_{[F]} \text{Spl}_X &\cong \text{Ext}^1(X; F, F) \times \text{Ext}^1(X; F, F) \longrightarrow \\ &\longrightarrow \text{Ext}^2(X; F, F) \xrightarrow{\text{tr}^2} H^2(X, \mathcal{O}_X) \cong H^2(X, K_X) \cong \mathbb{C}. \end{aligned}$$

It can be shown exactly as in the algebraic case that ω is closed and nondegenerate on Spl_X (cf. [24], [13]).

Moreover, it is easy to see that the restriction of ω to the fibers $\text{Spl}_X(L)$ of $\det : \text{Spl}_X \rightarrow \text{Pic}(X)$ remains nondegenerate, in other words that $\text{Spl}_X(L)$ are holomorphically symplectic subvarieties of Spl_X .

5.3 The moduli space of ASD connections and the comparison map

5.3.1 The moduli space of anti-self-dual connections

In this subsection we recall some results about the moduli spaces of anti-self-dual connections in the context we shall need. The reader is referred to [9], [12] and [20] for a thorough treatment of these questions.

We start with a compact complex surface X equipped with a Gauduchon metric g and a differential (complex) vector bundle E with a hermitian metric h in its fibers. The space of all C^∞ unitary connections on E is an affine space modelled on $\mathcal{A}^1(X, \text{End}(E, h))$ and the C^∞ unitary automorphism group \mathcal{G} , also called gauge-group, operates on it. Here $\text{End}(E, h)$ is the bundle of skew-hermitian endomorphisms of (E, h) . The subset of anti-self-dual connections is invariant under the action of the gauge-group and we denote the corresponding quotient by

$$\mathcal{M}^{ASD} = \mathcal{M}^{ASD}(E).$$

A unitary connection A on E is called **reducible** if E admits a splitting in two parallel sub-bundles. This happens exactly then when the holonomy group H_A of A is reducible, i.e. when the linear representation of H_A on \mathbb{C}^r induced by the canonical representation of $U(r)$ is reducible, where $r := \text{rank } E$.

It is easy to see that the isotropy group of A , $\Gamma_A := \{u \in \mathcal{G} \mid u(A) = A\}$, coincides with the centralizer of H_A in $U(r)$. On the other side, by Schur's Lemma a subgroup of $U(r)$ is irreducible if and only if its centralizer is the center of $U(r)$.

When A is anti-self-dual we consider the following elliptic complex:

$$0 \longrightarrow \mathcal{A}^0(\text{End}(E, h)) \xrightarrow{d_A} \mathcal{A}^1(\text{End}(E, h)) \xrightarrow{d_A^+} \mathcal{A}^+(\text{End}(E, h)) \longrightarrow 0$$

where d_A is the derivation associated to A and d_A^+ is d_A followed by projection on the anti-self dual component of $\mathcal{A}^2(\text{End}(E, h))$. Let H_A^i be the cohomology groups of this complex. The cohomology in degree zero H_A^0 measures the irreducibility of A . Indeed, by the above remarks, A is irreducible if and only if $H_A^0 \cong i\mathbb{R} \cdot \text{id}_E$. The two other cohomology spaces describe the tangent space of \mathcal{M}^{ASD} at $[A]$ and the obstruction space respectively: locally around $[A]$, $\mathcal{M}^{ASD}(E)$ looks like $f^{-1}(0)/\Gamma_A$, where $f : H_A^1 \longrightarrow H_A^2$ is a Γ_A -invariant map. In particular, if A is irreducible and $H_A^2 = 0$, then $\mathcal{M}^{ASD}(E)$ is smooth at A .

In order to improve this smoothness criterion we use as in the previous section the determinant map

$$\det : \mathcal{M}^{ASD}(E) \longrightarrow \mathcal{M}^{ASD}(\det E)$$

which associates to A the connection $\det A$ in $\det E$. (On $\det E$ we consider the hermitian metric $\det h$ induced by h). This is a fiber bundle over $\mathcal{M}^{ASD}(\det E)$ and the fibers

$\mathcal{M}^{ASD}(E, [a])$ are described locally by the deformation complex

$$0 \longrightarrow \mathcal{A}^0(\text{End}_0(E, h)) \xrightarrow{d_A} \mathcal{A}^1(\text{End}_0(E, h)) \xrightarrow{d_A^+} \mathcal{A}^+(\text{End}_0(E, h)) \longrightarrow 0.$$

Here $[a]$ denotes the gauge equivalence class of the unitary connection a in $\det E$ and $\text{End}_0(E, h)$ the bundle of trace-free skew-symmetric endomorphisms of E . If now $H_{A,0}^i$ are the cohomology groups of this complex, one sees that the vanishing of $H_{A,0}^0$ and $H_{A,0}^2$ entails smoothness for $\mathcal{M}^{ASD}(E, [a])$ and $\mathcal{M}^{ASD}(E)$ at $[A]$.

We recalled in 2.3 that the $(0,1)$ part $\bar{\partial}_A$ of an anti-self-dual connection A induces a semi-stable holomorphic structure on E . We can compare the deformation complex for $\mathcal{M}^{ASD}(E, [a])$ to the Dolbeault complex of $\mathcal{E}nd_0(E) := \mathcal{E}nd_0(E, \bar{\partial}_A)$ using the natural projections

$$\mathcal{A}^i(\text{End}_0(E, h)) \longrightarrow \mathcal{A}^{0,i}(\mathcal{E}nd_0(E)).$$

The morphism of complexes thus obtained induces natural isomorphisms on cohomology:

$$\begin{aligned} H^0(X, \mathcal{E}nd_0(E)) &\cong H_{A,0}^0 \otimes \mathbb{C}, \\ H^1(X, \mathcal{E}nd_0(E)) &\cong H_{A,0}^1, \\ H^2(X, \mathcal{E}nd_0(E)) &\cong H_{A,0}^2. \end{aligned}$$

In fact, there is a more precise formulation of the Kobayashi-Hitchin correspondence as in the following Theorem. We denote by $\mathcal{M}^{st}(E) = \mathcal{M}_g^{st}(E)$ the moduli space of stable holomorphic structures in E and by $\mathcal{M}^{st}(E, L)$ the fiber of the determinant map $\det : \mathcal{M}^{st}(E) \longrightarrow \text{Pic}(X)$ over an element L of $\text{Pic}(X)$.

Theorem 5.7 *Let X be a compact complex surface, g a Gauduchon metric on X , E a differentiable vector bundle over X , a an anti-self-dual connection on $\det E$ (with respect to g) and L the element in $\text{Pic}(X)$ given by $\bar{\partial}_a$ on $\det E$. Then $\mathcal{M}^{st}(E, L)$ is an open part of $\text{Spl}_X(L)$ and the mapping $A \mapsto \bar{\partial}_A$ gives rise to a real-analytic isomorphism between the moduli space $\mathcal{M}^{ASD,*}(E, [a])$ of irreducible anti-self-dual connections which induce $[a]$ on $\det E$ and $\mathcal{M}^{st}(E, L)$.*

We may also look at $\mathcal{M}^{ASD}(E, [a])$ in the following way. We consider all anti-self-dual connections inducing a fixed connection a on $\det E$ and factor by those gauge transformations in \mathcal{G} which preserve a . This is the same as taking gauge transformations of (E, h) which induce a constant multiple of the identity on $\det E$. Since constant multiples of the identity leave each connection invariant, whether on $\det E$ or on E , we may as well consider the action of the subgroup of \mathcal{G} inducing the identity on $\det E$. We denote this group by $S\mathcal{G}$, the quotient space by $\mathcal{M}^{ASD}(E, a)$ and by $\mathcal{M}^{ASD,*}(E, a)$ the part consisting of irreducible connections. There is a natural injective map

$$\mathcal{M}^{ASD}(E, a) \longrightarrow \mathcal{M}^{ASD}(E, [a])$$

which associates to an $S\mathcal{G}$ -equivalence class of a connection A its \mathcal{G} -equivalence class. The surjectivity of this map depends on the possibility to lift any unitary gauge transformation

of $\det E$ to a gauge transformation of E . This possibility exists if E has a rank-one differential sub-bundle, in particular when $r := \text{rank } E > 2$, since then E has a trivial sub-bundle of rank $r - 2$. In this case one constructs a lifting by putting in this rank-one component the given automorphism of $\det E$ and the identity on the orthogonal complement.

Another case when the lifting exists is when $\text{rank } E = 2$ and $\det E$ is topologically trivial, for now a choice of a non-trivial section in $\det E$ leads to an $SU(2)$ -structure in E . The automorphism of $\det E$ leads to a change of $SU(2)$ -structures and it is known that over a (complex) surface all $SU(2)$ -structures in a given vector bundle are equivalent, being classified by the second Chern class (cf. [11] Theorem E5).

Finally, a lifting also exists for all gauge transformations of $(\det E, \det h)$ admitting an r -th root. More precisely, denoting the gauge group of $(\det E, \det h)$ by $\mathcal{U}(1)$, it is easy to see that the elements of the subgroup $\mathcal{U}(1)^r := \{u^r \mid u \in \mathcal{U}(1)\}$ can be lifted to elements of \mathcal{G} . Since the obstruction to taking r -th roots in $\mathcal{U}(1)$ lies in $H^1(X, \mathbb{Z}_r)$, as one deduces from the corresponding short exact sequence, we see that $\mathcal{U}(1)^r$ has finite index in $\mathcal{U}(1)$. From this it is not difficult to infer that $\mathcal{M}^{ASD}(E, [a])$ is isomorphic to a topologically disjoint union of finitely many parts of the form $\mathcal{M}^{ASD}(E, a_k)$ with $[a_k] = [a]$ for all k .

5.3.2 The Uhlenbeck compactification

We continue by stating some results we need on the Uhlenbeck compactification of the moduli space of anti-self-dual connections. References for this material are [9] and [12].

Let (X, g) and (E, h) be as in 5.3.1. For each non-negative integer k we consider hermitian bundles (E_{-k}, h_{-k}) on X with $\text{rank } E_{-k} = \text{rank } E =: r$, $(\det E_{-k}, \det h_{-k}) \cong (\det E, \det h)$, $c_2(E_{-k}) = c_2(E) - k$. Set

$$\begin{aligned}\bar{\mathcal{M}}^U(E) &:= \bigcup_{k \in \mathbb{N}} (\mathcal{M}^{ASD}(E_{-k}) \times S^k X) \\ \bar{\mathcal{M}}^U(E, [a]) &:= \bigcup_{k \in \mathbb{N}} (\mathcal{M}^{ASD}(E_{-k}, [a]) \times S^k X) \\ \bar{\mathcal{M}}^U(E, a) &:= \bigcup_{k \in \mathbb{N}} (\mathcal{M}^{ASD}(E_{-k}, a) \times S^k X)\end{aligned}$$

where $S^k X$ is the k -th symmetric power of X . The elements of these spaces are called **ideal connections**. The unions are finite since by Remark 2.8 the second Chern class of a hermitian vector bundle admitting an anti-self-dual connection is bounded below (by $\frac{1}{2}c_1^2$).

To an element $([A], Z) \in \bar{\mathcal{M}}^U(E)$ one associates a Borel measure

$$\mu([A], Z) := |F_A|^2 + 8\pi^2 \delta_Z$$

where δ_Z is the Dirac measure whose mass at a point x of X equals the multiplicity $m_x(Z)$ of x in Z . We denote by $m(Z)$ the total multiplicity of Z .

A topology for $\bar{\mathcal{M}}^U(E)$ is determined by the following neighborhood basis for $([A], Z)$:

$$V_{U,N,\epsilon}([A], Z) = \{([A'], Z') \in \bar{\mathcal{M}}^U(E) \mid \mu([A'], Z') \in U \text{ and there is an } L_3^2 \text{-isomorphism } \psi : E_{-m(Z)}|_{X \setminus N} \longrightarrow E_{-m(Z')}|_{X \setminus N} \text{ such that } \|A - \psi^*(A')\|_{L_2^2(X \setminus N)} < \epsilon\}$$

where $\epsilon > 0$ and U and N are neighborhoods of $\mu([A], Z)$ and $\text{supp}(\delta_Z)$ respectively.

This topology is first-countable and Hausdorff and induces the usual topology on each $\mathcal{M}^{ASD}(E_{-k}) \times S^k X$. Most importantly, by the weak compactness theorem of Uhlenbeck $\bar{\mathcal{M}}^U(E)$ is compact when endowed with this topology. $\mathcal{M}^{ASD}(E)$ is an open part of $\bar{\mathcal{M}}^U(E)$ and its closure $\bar{\mathcal{M}}^{ASD}(E)$ inside $\bar{\mathcal{M}}^U(E)$ is called the **Uhlenbeck compactification** of $\mathcal{M}^{ASD}(E)$. Analogous statements are valid for $\mathcal{M}^{ASD}(E, [a])$ and $\mathcal{M}^{ASD}(E, a)$.

Using a technique due to Taubes, one can obtain a neighborhood of an irreducible ideal connection $([A], Z)$ in the border of $\mathcal{M}^{ASD}(E, a)$ by gluing to A "concentrated" $SU(r)$ anti-self-dual connections over S^4 . One obtains "cone bundle neighborhoods" for each such ideal connection $([A], Z)$ when $H_{A,0}^2 = 0$. For the precise statements and the proofs we refer the reader to [9] chapters 7 and 8 and to [12] 3.4. As a consequence of this description and of the connectivity of the moduli spaces of $SU(r)$ anti-self-dual connections over S^4 (see [22]) we have the following weaker property which will suffice to our needs.

Proposition 5.8 *Around an irreducible ideal connection $([A], Z)$ with $H_{A,0}^2 = 0$ the border of the Uhlenbeck compactification $\bar{\mathcal{M}}^{ASD}(E, a)$ is **locally non-disconnecting** in $\bar{\mathcal{M}}^{ASD}(E, a)$, i.e. there exist arbitrarily small neighborhoods V of $([A], Z)$ in $\bar{\mathcal{M}}^{ASD}(E, a)$ with $V \cap \mathcal{M}^{ASD}(E, a)$ connected.*

5.3.3 The comparison map

We fix (X, g) a compact complex surface together with a Gauduchon metric on it, (E, h) a hermitian vector bundle over X , a an unitary anti-self-dual connection on $(\det E, \det h)$ and denote by L the (isomorphism class of the) holomorphic line bundle induced by $\bar{\partial}_a$ on $\det E$. Let $c_2 := c_2(E)$ and $r := \text{rank } E$. We denote by $\mathcal{M}^{st}(r, L, c_2)$ the subset of Spl_X consisting of isomorphism classes of non-necessarily locally free sheaves F (with respect to g) with $\text{rank } F = r, \det F = L, c_2(F) = c_2$.

In 5.3.1 we have mentioned the existence of a real-analytic isomorphism between $\mathcal{M}^{st}(E, L)$ and $\mathcal{M}^{ASD,*}(E, [a])$. When X is algebraic, $\text{rank } E = 2$ and a is the trivial connection this isomorphism has been extended to a continuous map from the Gieseker compactification of $\mathcal{M}^{st}(E, \mathcal{O})$ to the Uhlenbeck compactification of $\mathcal{M}^{ASD}(E, 0)$ in [23] and [19]. The proof given in [23] adapts without difficulty to our case to show the continuity of the natural extension

$$\Phi : \mathcal{M}^{st}(r, L, c_2) \longrightarrow \bar{\mathcal{M}}^U(E, [a]).$$

Φ is defined by $\Phi([F]) = ([A], Z)$, where A is the unique unitary anti-self-dual connection inducing the holomorphic structure on $F^{\vee\vee}$ (which is locally free and which we pre-endow

with a hermitian metric as in 5.3.2) and Z describes the singularity set of F with multiplicities $m_x(Z) := \dim_{\mathbb{C}}(F_x^{\vee\vee}/F_x)$ for $x \in X$. The main result of this paragraph asserts that under certain conditions for X and E this map is proper as well.

Theorem 5.9 *Let X be a non-algebraic compact complex surface which has either Kodaira dimension $\text{kod}(X) = -\infty$ or has trivial canonical bundle and let g be a Gauduchon metric on X . Let (E, h) be a hermitian vector bundle over X , $r := \text{rank } E$, $c_2 := c_2(E)$, a unitary anti-self-dual connection on $(\det E, \det h)$ and L the holomorphic line bundle induced by $\bar{\partial}_a$ on $\det E$. If $(r, c_1(L), c_2)$ satisfies condition (*) from 5.1 then the following hold:*

- (a) *the natural map $\Phi : \mathcal{M}^{\text{st}}(r, L, c_2) \longrightarrow \bar{\mathcal{M}}^U(E, [a])$ is continuous and proper,*
- (b) *any unitary automorphism of $(\det E, \det h)$ lifts to an automorphism of (E, h) and*
- (c) *$\mathcal{M}^{\text{st}}(r, L, c_2)$ is a compact complex (Hausdorff) manifold.*

Proof

Under the Theorem's assumptions we prove the following claims.

Claim 1. Spl_X is smooth and of the expected dimension at points $[F]$ of $\mathcal{M}^{\text{st}}(r, L, c_2)$.

By Corollary 5.6 for such a stable sheaf F we have to check that $\text{Ext}^2(X; F, F)_0 = 0$. When K_X is trivial this is equivalent to $\dim(\text{Ext}^2(X; F, F)) = 1$ and by Serre duality further to $\dim(\text{Hom}(X; F, F)) = 1$ which holds since stable sheaves are simple. (See Remark 2.11).

So let now X be non-algebraic and $\text{kod}(X) = -\infty$. By surface classification $b_1(X)$ must be odd and Remark 5.1 shows that F is irreducible. In this case $\text{Ext}^2(X; F, F) = 0$ as in the proof of Proposition 2.16.

Claim 2. $\mathcal{M}^{\text{st}}(r, L, c_2)$ is open in Spl_X .

This claim is known to be true over the open part of Spl_X parameterizing simple locally free sheaves and holds possibly in all generality. Here we give an ad-hoc proof.

If b_1 is odd or if the degree function $\deg_g : \text{Pic}(X) \longrightarrow \mathbb{R}$ vanishes identically the assertion follows from the condition (*). Suppose now that X is non-algebraic with b_1 even and trivial canonical bundle. Let F be a torsion-free sheaf on X with $\text{rank } F = r$, $\det F = L$ and $c_2(F) = c_2$. If F is not stable then F sits in a short exact sequence

$$0 \longrightarrow F_1 \longrightarrow F \longrightarrow F_2 \longrightarrow 0$$

with F_1, F_2 torsion-free coherent sheaves on X . Let $r_1 := \text{rank } F_1, r_2 := \text{rank } F_2$.

We first show that the possible values for $\deg F_1$ lie in a discrete subset of \mathbb{R} .

An easy computation gives

$$-\frac{c_1(F_1)^2}{r_1} - \frac{c_1(F_2)^2}{r_2} = -\frac{c_1(F)^2}{r} + 2r\Delta(F) - 2r_1\Delta(F_1) - 2r_2\Delta(F_2).$$

Since by Theorem 2.12 all discriminants are non-negative we get

$$-\frac{c_1(F_1)^2}{r_1} - \frac{c_1(F_2)^2}{r_2} \leq -\frac{c_1(F)^2}{r} + 2r\Delta(F).$$

In particular $c_1(F_1)^2$ is bounded by a constant depending only on $(r, c_1(L), c_2)$. Since X is non-algebraic the intersection form on $NS(X)$ is negative semi-definite. In fact, by [6] $NS(X)/Tors(NS(X))$ can be written as a direct sum $N \oplus I$ where the intersection form is negative definite on N , I is the isotropy subgroup for the intersection form and I is cyclic. We denote by c a generator of I . It follows the existence of a finite number of classes b in N for which one can have $c_1(F_1) = b + \alpha c$ modulo torsion, with $\alpha \in \mathbb{N}$. Thus $\deg F_1 = \deg b + \alpha \deg c$ lies in a discrete subset of \mathbb{R} .

Let now $b \in NS(X)$ be such that $0 < \deg b \leq |\deg F_1|$ for all possible subsheaves F_1 as above with $\deg F_1 \neq 0$. We consider the torsion-free stable central fiber \mathcal{F}_0 of a family of sheaves \mathcal{F} on $X \times S$ flat over S . Suppose that $\text{rank}(\mathcal{F}_0) = r$, $\det \mathcal{F}_0 = L$, $c_2(\mathcal{F}_0) = c_2$. We choose an irreducible vector bundle G on X with $c_1(G) = -b$. Then $H^2(X, \text{Hom}(\mathcal{F}_0, G)) = 0$, so if $\text{rank } G$ is large enough we can apply Proposition 5.3 to get an extension

$$0 \longrightarrow q^*G \longrightarrow E \longrightarrow \mathcal{F} \longrightarrow 0$$

with E locally free on $X \times S$, for a possibly smaller S . (As in Proposition 5.3 we have denoted by q the projection $X \times S \longrightarrow S$.) It is easy to check that E_0 doesn't have any subsheaf of degree larger than $-\deg b$. Thus E_0 is stable. Hence small deformations of E_0 are stable as well. As a consequence we get that small deformations of \mathcal{F}_0 will be stable. Indeed, it is enough to consider for a destabilizing subsheaf F_1 of \mathcal{F}_s , for $s \in S$, the induced extension

$$0 \longrightarrow G \longrightarrow E_1 \longrightarrow F_1 \longrightarrow 0.$$

Then E_1 is a subsheaf of E_s with $\deg E_1 = \deg G + \deg F_1 \geq 0$. This contradicts the stability of E_s .

Claim 3. Any neighborhood in Spl_X of a point $[F]$ of $\mathcal{M}^{\text{st}}(r, L, c_2)$ contains isomorphism classes of locally free sheaves.

The proof goes as in the algebraic case by considering the "double-dual stratification" and making a dimension estimate. Here is a sketch of it.

If one takes a flat family \mathcal{F} of torsion free sheaves on X over a reduced base S , one may consider for each fiber \mathcal{F}_s , $s \in S$, the injection into the double-dual $\mathcal{F}_s^{\vee\vee} := \text{Hom}(\text{Hom}(\mathcal{F}_s, \mathcal{O}_{X \times \{s\}}), \mathcal{O}_{X \times \{s\}})$. The double-duals form a flat family over some Zariski-open subset of S . To see this consider first $\mathcal{F}^\vee := \text{Hom}(\mathcal{F}, \mathcal{O}_{X \times S})$. Since \mathcal{F} is flat over S , one gets $(\mathcal{F}_s)^\vee = \mathcal{F}_s^\vee$. \mathcal{F}^\vee is flat over the complement of a proper analytic subset of S and one repeats the procedure to

obtain $\mathcal{F}^{\vee\vee}$ and $\mathcal{F}^{\vee\vee}/\mathcal{F}$ flat over some Zariski open subset S' of S . Over $X \times S'$, $\mathcal{F}^{\vee\vee}$ is locally free and $(\mathcal{F}^{\vee\vee}/\mathcal{F})_s = \mathcal{F}_s^{\vee\vee}/\mathcal{F}_s$ for $s \in S'$.

Take now S a neighborhood of $[F]$ in $\mathcal{M}^{st}(r, L, c_2)$. Suppose that

$$\text{length}(\mathcal{F}_{s_0}^{\vee\vee}/\mathcal{F}_{s_0}) = k > 0$$

for some $s_0 \in S'$. Taking S' smaller around s_0 if necessary, we find a morphism ϕ from S' to a neighborhood T of $[\mathcal{F}_{s_0}^{\vee\vee}]$ in $\mathcal{M}^{st}(r, L, c_2 - k)$ such that there exists a locally free universal family \mathcal{E} on $X \times T$ with $\mathcal{E}_{t_0} \cong \mathcal{F}_{s_0}^{\vee\vee}$ for some $t_0 \in T$ and $(\text{id}_X \times \phi)^*\mathcal{E} = \mathcal{F}^{\vee\vee}$. Let D be the relative Douady space of quotients of length k of the fibers of \mathcal{E} and let $\pi : D \rightarrow T$ be the projection. There exists an universal quotient \mathcal{Q} of $(\text{id}_X \times \pi)^*\mathcal{E}$ on $X \times D$. Since $\mathcal{F}^{\vee\vee}/\mathcal{F}$ is flat over S' , ϕ lifts to a morphism $\tilde{\phi} : S' \rightarrow D$ with $(\text{id}_X \times \tilde{\phi})^*\mathcal{Q} = \mathcal{F}^{\vee\vee}/\mathcal{F}$. By the universality of S' there exists also a morphism (of germs) $\psi : D \rightarrow S'$ with $(\text{id}_X \times \psi)^*\mathcal{F} = \text{Ker}((\text{id}_X \times \pi)^*\mathcal{E} \rightarrow \mathcal{Q})$. One sees now that $\psi \circ \tilde{\phi}$ must be an isomorphism, in particular $\dim S' \leq \dim D$. Since S' and T have the expected dimensions, it is enough to compute now the relative dimension of D over T . This is $k(r+1)$. On the other side by Corollary 5.6 $\dim S' - \dim T = 2kr$. This forces $r = 1$ which is excluded by hypothesis.

After these preparations of a relatively general nature we get to the actual proof of the Theorem. We start with (b).

If $b_2^-(X)$ denotes the number of negative eigenvalues of the intersection form on $H^2(X, \mathbb{R})$, then for our surface X we have $b_2^-(X) > 0$. This is clear when K_X is trivial by classification and follows from the index theorem and Remark 5.1 (d) when $b_1(X)$ is odd. In particular, taking $p \in H^2(X, \mathbb{Z})$ with $p^2 < 0$ one constructs topologically split rank two vector bundles F with given first Chern class l and arbitrarily large second Chern class: just consider $(L \otimes P^{\otimes n}) \oplus (P^*)^{\otimes n}$ where L and P are line bundles with $c_1(L) = l$, $c_1(P) = p$ and $n \in \mathbb{N}$. If E has rank two we take F with $\det F \cong \det E$ and $c_2(F) \geq c_2(E) = c_2$. (When $r > 2$ assertion (b) is trivial; cf. section 5.3.1). We consider an anti-self-dual connection A in E inducing a on $\det E$ and $Z \subset X$ consisting of $c_2(F) - c_2(E)$ distinct points. By the computations from the proof of Claim 1 we see that A is irreducible and $H_{A,0}^2 = 0$. Using the gluing procedure mentioned in section 5.3.2, one sees that a neighborhood of $([A], Z)$ in $\bar{\mathcal{M}}^U(F, [a])$ contains classes of irreducible anti-self-dual connections in F . We have seen in section 5.3.1 that any unitary automorphism of $\det F$ lifts to an unitary automorphism u of F . If we take a sequence of anti-self-dual connections (A_n) in F with $\det A_n = a$ and $([A_n])$ converging to $([A], Z)$, we get by applying u a limit connection B for subsequence of $(u(A_n))$. Since $\bar{\mathcal{M}}^U(F, [a])$ is Hausdorff, there exists an unitary automorphism \tilde{u} of E with $\tilde{u}^*(B) = A$. It is clear that \tilde{u} induces the original automorphism u on $\det F \cong \det E$.

In order to prove (a) we need the following elementary topological lemma.

Lemma 5.10 *Let $\pi : Z \rightarrow Y$ be a continuous surjective map between Hausdorff topological spaces. Suppose Z locally compact, Y locally connected and that there is a locally non-disconnecting closed subset Y_1 of Y with $Z_1 := \pi^{-1}(Y_1)$ compact and $\overset{\circ}{Z}_1 = \emptyset$. Suppose*

further that π restricts to a homeomorphism

$$\pi|_{Z \setminus Z_1, Y \setminus Y_1}: Z \setminus Z_1 \longrightarrow Y \setminus Y_1.$$

Then for any neighborhood V of Z_1 in Z , $\pi(V)$ is a neighborhood of Y_1 in Y . If in addition Y is compact, then Z is compact as well.

Proof Under the above assumptions let V be a neighborhood of Z_1 in Z . We shall show that $\pi(V)$ is a neighborhood of Y_1 in Y . When $Z_1 = \emptyset$ the conclusion is trivial, so suppose the contrary holds. Since Z is locally compact, there is a compact neighborhood K of Z_1 with $K \subset V$. So $Z_1 \subset \overset{\circ}{K}$ but $Z_1 \neq \overset{\circ}{K}$ since $\overset{\circ}{Z}_1 = \emptyset$. The partition

$$Z = Z_1 \cup \partial K \cup (\overset{\circ}{K} \setminus Z_1) \cup (Z \setminus K)$$

induces by taking images a partition of Y . $\pi(\partial K) \cap Y_1 = \emptyset$ so $W := Y_1 \cup \pi(\overset{\circ}{K} \setminus Z_1) \cup \pi(Z \setminus K)$ is an open neighborhood of Y_1 in Y .

Let y be a point in Y_1 and U_y a connected neighborhood of y with $U_y \subset W$. Y_1 is locally non-disconnecting, hence $U_y \setminus Y_1$ is contained either in $\pi(\overset{\circ}{K} \setminus Z_1)$ or in $\pi(Z \setminus K)$. Let now $z \in \pi^{-1}(y)$ and V_z be a neighborhood of z such that $V_z \subset \overset{\circ}{K}$ and $\pi(V_z) \subset U_y$. Since Z_1 has no interior points, $\emptyset \neq \pi(V_z \setminus Z_1) \subset \pi(\overset{\circ}{K} \setminus Z_1) \cap U_y$ and thus $U_y \subset \pi(\overset{\circ}{K})$. This shows that $\pi(\overset{\circ}{K})$ is an open neighborhood of Y_1 .

If Y is compact, then $Y \setminus \pi(\overset{\circ}{K})$ is also compact and Z is the union of the two compact sets K and $\pi^{-1}(Y \setminus \pi(\overset{\circ}{K})) = Z \setminus \overset{\circ}{K}$. Thus Z is compact too and the Lemma is proved.

We complete now the proof of the Theorem by induction on c_2 . For fixed r and $c_1(E)$, $c_2(E)$ is bounded below if E is to admit an anti-self-dual connection; cf. Remark 2.8. If we take c_2 minimal, then $\mathcal{M}^{st}(r, L, c_2) = \mathcal{M}^{st}(E, L)$ and $\mathcal{M}^{ASD,*}(E, [a]) = \mathcal{M}^{ASD}(E, [a])$ is compact. From Theorem 5.7 we obtain that Φ is a homeomorphism in this case.

Take now c_2 arbitrary but such that the hypotheses of the Theorem hold and assume that the assertions of the Theorem are true for any smaller c_2 . We apply Lemma 5.10 to the following situation:

$$Z := \mathcal{M}^{st}(r, L, c_2), \quad Y := \bar{\mathcal{M}}^U(E, [a]) = \bar{\mathcal{M}}^{ASD}(E, [a]) \cong \bar{\mathcal{M}}^{ASD}(E, a).$$

The last equalities hold according to Claim 3 and Claim 4. Let further Y_1 be the border $\bar{\mathcal{M}}^{ASD}(E, a) \setminus \mathcal{M}^{ASD}(E, a)$ of the Uhlenbeck compactification and Z_1 be the locus $\mathcal{M}^{st}(r, L, c_2) \setminus \mathcal{M}^{st}(E, L)$ of singular stable sheaves in Spl_X . Z is smooth by Claim 1 and Hausdorff by Remark 5.2, Y_1 is locally non-disconnecting by Proposition 5.8, $\overset{\circ}{Z}_1 = \emptyset$ by Claim 3 and $\pi|_{Z \setminus Z_1, Y \setminus Y_1}$ is a homeomorphism by Theorem 5.7. In order to be able to apply Lemma 5.10 and thus close the proof we only need to check that Z_1 is compact.

We want to reduce this to the compactness of $\mathcal{M}^{st}(r, L, c_2 - 1)$ which is ensured by the induction hypothesis.

We consider a finite open covering (T_i) of $\mathcal{M}^{st}(r, L, c_2 - 1)$ such that over each $X \times T_i$ an universal family \mathcal{E}_i exists. The relative Douady space D_i parameterizing quotients of length one in the fibers of \mathcal{E}_i is proper over (T_i) . In fact it was shown in [15] that $D_i \cong \mathbb{P}(\mathcal{E}_i)$. If $\pi_i : D_i \rightarrow T_i$ are the projections, we have universal quotients \mathcal{Q}_i of $\pi_i^*\mathcal{E}_i$ and $\mathcal{F}_i := \text{Ker}(\pi_i^*\mathcal{E}_i \rightarrow \mathcal{Q}_i)$ are flat over D_i . This induces canonical morphisms $D_i \rightarrow Z_1$. It is enough to notice that their images cover Z_1 , or equivalently, that any singular stable sheaf F over X sits in an exact sequence of coherent sheaves

$$0 \longrightarrow F \longrightarrow E \longrightarrow Q \longrightarrow 0$$

with length $Q = 1$ and E torsion-free. Such an extension is induced from

$$0 \longrightarrow F \longrightarrow F^{\vee\vee} \longrightarrow F^{\vee\vee}/F \longrightarrow 0$$

by any submodule Q of length one of $F^{\vee\vee}/F$. (To see that such Q exist recall that $(F^{\vee\vee}/F)_x$ is artinian over $\mathcal{O}_{X,x}$ and use Nakayama's Lemma). The Theorem is proved. \square

Remark 5.11 *As a consequence of this theorem we get that when X is a 2-dimensional complex torus or a primary Kodaira surface and (r, L, c_2) is chosen in the stable irreducible range as in section 3 or in [1], then $\mathcal{M}^{st}(r, L, c_2)$ is a holomorphically symplectic compact complex manifold.*

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