Propagation Estimates for the Boson Star Equation

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Abstract

We consider the boson star equation with a general two-body interaction potential w and initial data ψ_0 in a Sobolev space. Under general assumptions on w, namely that w decomposes as a sum of a finite, signed measure and an essentially bounded function, we prove that the (local in time) solution cannot propagate faster than the speed of light, up to a sharp exponentially small remainder term. If w is short-range and ψ_0 is regular and small enough, we prove in addition asymptotic phase-space propagation estimates and minimal velocity estimates for the (global in time) solution, depending on the momentum of the scattering state associated to ψ_0.

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1 Introduction

In this paper we study the pseudo-relativistic (or semi-relativistic) Hartree equation

$$\begin{cases} i\partial_t\psi = (\langle\nabla\rangle + w * |\psi|^2)\psi, \quad x \in \mathbb{R}^d, \quad t \geq 0, \\ \psi|_{t=0} = \psi_0, \end{cases} \tag{1.1}$$

in any dimension $d \geq 3$. Here and throughout the paper, we use the notation $\langle\nabla\rangle$ for the Fourier multiplier

$$\langle\nabla\rangle := \sqrt{1-\Delta} = \mathcal{F}^{-1}\sqrt{1+|\xi|^2}\mathcal{F},$$

where $\mathcal{F}$ is the Fourier transform normalized such that $\mathcal{F}$ is unitary on L^2.

Eq. (1.1) describes the effective dynamics, in the mean-field limit, of an N-body quantum system of pseudo-relativistic bosons of mass $m = 1$, with two-body gravitational interaction given by the convolution potential w; see [18]. It is thus used as a model for a pseudo-relativistic boson star. Here we work in units such that Planck's constant divided by 2π and the velocity of light are equal to 1. We also assume that the mass of the bosons is 1 for simplicity, but we could consider the kinetic energy $\sqrt{m^2-\Delta}$ (or $\sqrt{m^2-\Delta}-m$), for any $m > 0$, without changing our results.

A physically relevant choice of the interaction potential w is the attractive Newtonian potential, $w(x) = -\kappa|x|^{-1}$ for some $\kappa > 0$, but we will consider in this paper general potentials in suitable spaces, imposing different conditions depending on the results. The assumption $w \geq 0$ corresponds to repulsive, or defocusing interactions, while $w \leq 0$ corresponds to attractive, or focusing ones.

We use the shorthand ψ_t for a solution $(t,x) \mapsto \psi(t,x)$ to (1.1). The boson star equation (1.1) exhibits (formally at this stage) three important conserved quantities: the mass,

$$M(\psi_t) := \int_{\mathbb{R}^d} |\psi_t|^2; \tag{1.2}$$

the energy, defined by

$$E(\psi_t) := \frac{1}{2}\int_{\mathbb{R}^d} \left|\langle\nabla\rangle^{\frac{1}{2}}\psi_t\right|^2 + \frac{1}{4}\int_{\mathbb{R}^d}(w * |\psi_t|^2)|\psi_t|^2 \tag{1.3}$$

and the momentum,

$$P(\psi_t) := -\frac{i}{2}\int_{\mathbb{R}^d} \bar{\psi}_t \nabla\psi_t. \tag{1.4}$$

We consider in this paper a translation invariant boson star equation, but several of our results might be extended to a boson star placed in an external potential $V(x)$, provided that suitable regularity and decay assumptions on V are imposed. In this case, of course, the momentum is not a conserved quantity anymore.

For the attractive Newtonian potential $w(x) = -\kappa|x|^{-1}$ with $\kappa > 0$, the convergence, in the mean-field limit, of the ground state energy of the pseudo-relativistic N-body Schrödinger Hamiltonian to the ground state energy of the static, pseudo-relativistic Hartree equation corresponding to (1.1), was established by Lieb and Yau in their seminal paper [47]. The fermionic case is also considered in [47], in relation with the Chandrasekhar theory of stellar collapse. As mentioned before, the convergence in the mean-field limit of the time-dependent pseudo-relativistic Schrödinger equation to the pseudo-relativistic Hartree equation (1.1) was proven in [18]. See also [40, 41] for explicit rates of convergence in the mean-field limit.

In [45], local and global well-posedness are studied in dimension $d = 3$ for (1.1) with a general external potential V and a Yukawa-type interaction potential, $w(x) = \kappa|x|^{-1}e^{-\mu|x|}$ with $\kappa \in \mathbb{R}$, $\mu \geq 0$, and for initial data ψ_0 in the Sobolev space H^s, with $s \geq \frac{1}{2}$ (for attractive interaction potentials, a smallness condition must be imposed, either on ψ_0 or on w). The local well-posedness is extended in [35] to low-regularity initial states, namely $\psi_0 \in H^s$ with $s > \frac{1}{4}$. Here it should be noticed that the energy space corresponds to the Sobolev regularity $H^{1/2}$. In [11–13], global existence – and small initial data scattering – are proven for (sums of) short-range interaction potentials of the form $w_i(x) = \kappa_i|x|^{-\alpha_i}$, for some suitable values of α_i; see also [36] for related results in the case of Yukawa-type potentials $w(x) = \kappa|x|^{-1}e^{-\mu|x|}$ with $\mu > 0$. We will compare some of these results to ours below, after the statement of our main contributions. For long-range potentials $w(x) = \kappa|x|^{-1}$, "modified scattering states" must be introduced; see [53] for global existence and small initial data scattering in this case.

In relation with gravitational collapse, finite time blow-up for (1.1) is proven in [27] for attractive Newtonian potentials $w(x) = -\kappa|x|^{-1}$ with $\kappa > 0$ and the mass $M(\psi_0) = \|\psi_0\|^2_{L^2}$ of the initial state larger than some critical value. The blow-up phenomenon is analysed in the mean-field limit in [51]. For the existence of solitary waves and stability results around them, under suitable assumptions, we refer to [25, 26]. We will not study finite time blow-up nor the existence of solitary waves here.

Our main concern is the speed of propagation of boson stars whose dynamics is given by (1.1). We generally consider a large class of interaction potentials,

$$w \in \mathcal{M} + L^\infty, \tag{1.5}$$

where $\mathcal{M}$ stands for the Banach space of finite, signed Radon measures on $\mathbb{R}^d$. In particular, we can take w as a Dirac delta measure and hence consider a pure-power cubic non-linearity. Depending on the results, we will restrict the class of admissible potentials.

We will begin with establishing local and global existence of solutions to (1.1). Our proof follows the usual strategy of applying a fixed point argument to solve Duhamel's equation associated to (1.1) in a suitable function space. For the global existence, we will distinguish two distinct regimes: the long-range regime, $w \in L^{d/2+1,\infty} + L^\infty$, and the short-range regime, $w \in \mathcal{M} + L^q$, $1 \leq q < \frac{2d}{3}$, with possibly a smallness condition involving the initial state ψ_0. Compared to the extensively studied non-linear Schrödinger or Hartree equations (see e.g. [9, 10, 29, 61] and references therein), a difficulty, as in previous works, comes from the loss of derivatives in the dispersive estimates associated to the semi-relativistic kinetic energy, which, for the $L^1 \to L^\infty$ estimate, may be written as

$$\big\|e^{-it\langle\nabla\rangle}f\big\|_{L^\infty} \lesssim |t|^{-\frac{d}{2}}\|\langle\nabla\rangle^{\frac{d}{2}+1}f\|_{L^1}, \tag{1.6}$$

see Appendix B.

Given a solution ψ_t to (1.1), we will aim at estimating the speed of propagation of the boson star, in the sense of proving time-decay estimates for the probability that the velocity $\frac{x}{t}$ of the state ψ_t belongs to a certain domain, possibly depending on the position and momentum of the initial state ψ_0. We will establish both a maximal and a minimal velocity estimate, expressed under different forms. Our maximal velocity bound is of the form

$$\|\mathbf{1}_Y \psi_t\|_{L^2} \leq e^{t-\mathrm{dist}(X,Y)} \|\mathbf{1}_X \psi_0\|_{L^2}, \tag{1.7}$$

for any convex subsets $X, Y \subset \mathbb{R}^d$, where $\mathrm{dist}(X,Y)$ stands for the distance from X to Y. It holds for any local in time solution ψ_t to (1.1), for any interaction potential $w \in \mathcal{M} + L^\infty$, provided that the initial state ψ_0 is regular enough, depending on w. For times $t \ll \mathrm{dist}(X,Y)$, (1.7) shows that the probability for the star to travel from X to Y is exponentially small, hence justifying that the maximal velocity of propagation for (1.1) is equal to the velocity of light (1 in our units). A feature of (1.7) is that it gives a sharp exponentially small error (in a sense that will be made precise below). Our proof is adapted from [58]. The maximal velocity estimate (1.7) can be extended to general disjoint subsets X, Y, dropping the convexity assumption, at the cost of losing optimality. See below for more details.

Our (asymptotic) minimal velocity estimate takes the form

$$\Big\| \mathbf{1}_{[0,\alpha)}\Big(\frac{x^2}{t^2}\Big)\psi_t \Big\|_{L^2} \to 0, \qquad t \to \infty, \tag{1.8}$$

for any initial state ψ_0 associated to a scattering state with an "asymptotic instantaneous velocity" larger than α. The precise definition of the instantaneous velocity operator will be given below. In order to construct initial states with localized asymptotic instantaneous velocity, we will need first to establish "phase-space" propagation estimates and to prove the existence, and right-invertibility, of wave operators on a suitable set of small and regular initial data. For this we will have to restrict the admissible class of potentials to short-range interaction potentials $w \in \mathcal{M} + L^q$ with $q < \frac{d}{2}$.

Propagation estimates, including maximal velocity, minimal velocity and phase-space propagation estimates, played a crucial role in the eighties and nineties in the scattering theory of N-body quantum systems, especially in the proof of asymptotic completeness of the wave operators, see, among others, [15, 19, 28, 31, 38, 55, 56, 59]. Propagation estimates were later extended to the framework of non-relativistic QED in e.g. [4, 17, 23, 24]. In the recent years, proving bounds on the maximal speed of propagation for quantum information in various physical contexts has been the subject of many works. We refer to [2] and [58] for the development of general methods for proving maximal velocity estimates for quantum systems, the former by the means of adiabatic space-time localization observables (ASTLO), while the latter is based on analyticity properties leading to exponential bounds. In relation with Lieb-Robinson bounds, maximal velocity of quantum transport for Bose-Hubbard type Hamiltonians has been established in [20–22, 43, 44]. For related works in the case of open quantum systems described by a Lindblad master equation, we refer to [7, 8, 57]. Finally a maximal velocity estimate for the (non-relativistic) Hartree equation, using the ASTLO method, has been derived in [1].

Our paper is organized as follows. In the next section, we describe our main results in precise terms. Section 3 contains preliminary technical estimates that are subsequently used in the proof of our main results. In Section 4, we prove local existence for (1.1), for the general class of potentials w satisfying (1.5). Sections 5 and 6 are devoted to the proof of global existence, in the long-range and short-range regimes, respectively. In Section 7, we prove the maximal velocity estimate (1.7) and in Section 8 we study the scattering theory for short-range potentials and establish the minimal velocity bound (1.8). Appendix A recalls estimates on commutators between weights and fractional derivatives in L^p spaces. In Appendix B we derive various time-decay estimates for the linear flow $e^{-it\langle\nabla\rangle}$ that are important ingredients in the proofs of our main results.

2 Main results

In this section we state and comment our main results on the existence and properties of solutions to (1.1). In what follows L^p and $L^{p,\infty}$ stand for the usual Lebesgue and weak Lebesgue spaces over $\mathbb{R}^d$, respectively; we also denote by $\mathcal{M}$ the space of signed, finite Radon measures on $\mathbb{R}^d$, equipped with the total variation norm $\|\mu\|_{\mathcal{M}} = |\mu|(\mathbb{R}^d)$. We introduce the following class of interaction potentials

$$\mathcal{W}_{d,s} = \begin{cases} L^{\frac{d}{2s},\infty} + L^\infty & \text{if } s < \frac{d}{2}, \\ \bigcup_{q>1}(L^{q,\infty} + L^\infty) & \text{if } s = \frac{d}{2}, \\ \mathcal{M} + L^\infty & \text{if } s > \frac{d}{2}, \end{cases} \tag{2.1}$$

where, in the sequel, s will correspond to the regularity of the initial data ψ_0 in (1.1).

Remark 2.1.

1. *As mentioned in the introduction, for $s > \frac{d}{2}$, w can be chosen as a sum of a Dirac delta measure and a function in any L^p space, $1 \le p \le \infty$, leading in (1.1) to a potential which is a sum of a cubic non-linearity and a convolution non-linearity by a function in a broad class.*

2. *The class $L^{\frac{d}{2s},\infty}+L^\infty$ includes the interactions given by (sums of) convolution potentials of the form $w(x) = \frac{1}{|x|^\alpha}$ for $\alpha \in [0, 2s]$. Indeed, $\frac{1}{|x|^{2s}} \in L^{\frac{d}{2s},\infty}(\mathbb{R}^d)$, while for $\alpha \in [0, 2s)$ we have*

$$\frac{1}{|x|^\alpha} = \frac{\mathbb{1}_{|x|\le 1}}{|x|^\alpha} + \frac{\mathbb{1}_{|x|>1}}{|x|^\alpha} \le \frac{1}{|x|^{2s}} + \frac{\mathbb{1}_{|x|>1}}{|x|^\alpha} \in L^{\frac{d}{2s},\infty}(\mathbb{R}^d) + L^\infty(\mathbb{R}^d).$$

3. *For $s = \frac{d}{2}$, as usual, we cannot reach the endpoint case $L^{1,\infty}$ because of the absence of embedding of H^s into L^∞. Note the equality $\bigcup_{q>1}(L^{q,\infty} + L^\infty) = \bigcup_{q>1}(L^q + L^\infty)$. Here we do not try to refine further the functional space considered in this critical case.*

To simplify the exposition, we only consider non-negative times to state (and prove) our results, even if they clearly also hold for negative times. Similarly, all our results hold without change if one replaces $\langle\nabla\rangle$ by $-\langle\nabla\rangle$ in (1.1).

2.1 Local existence

We begin with the local existence of solutions to (1.1) for initial data in H^s, $s \ge 0$.

Theorem 2.2 (Local existence I). *Let $s \geq 0$, ψ_0 in H^s and w in $\mathcal{W}_{d,s}$. Then there exists $T_{\max}$ in $(0,\infty]$ such that (1.1) admits a unique solution*

$$\psi \in \mathcal{C}^0([0,T_{\max}), H^s) \cap \mathcal{C}^1([0,T_{\max}), H^{s-1}),$$

where either $T_{\max} = \infty$ or $\lim_{t\to T_{\max}} \|\psi_t\|_{H^s} = \infty$. Moreover, for any T in $[0, T_{\max})$, the map

$$H^s \ni \psi_0 \mapsto \psi|_{[0,T]} \in \mathcal{C}^0([0,T], H^s) \cap \mathcal{C}^1([0,T], H^{s-1})$$

is continuous.

The proof of Theorem 2.2 relies on a standard fixed point argument in H^s to solve Duhamel's equation associated to (1.1), using multilinear estimates proven in Section 3. We also remark that the mass $M(\psi_t) = \|\psi_t\|_{L^2}^2$ is a conserved quantity for (1.1) (see Lemma 5.1 for a precise statement), therefore the previous theorem with $s = 0$ implies global existence for $w \in L^\infty$ and $\psi_0 \in L^2$.

For initial states with limited regularity, the class of admissible potentials w for local existence can be extended thanks to the Strichartz estimates given in Appendix B. In the next statement $L^{b,2}$ stands for the usual Lorentz space, see Subsection 3.2 for the definition.

Theorem 2.3 (Local existence II). *Let $0 \leq s < (d+1)/(2d-2)$, w in $L^{(d+1)/(4s),\infty} + L^\infty$ and ψ_0 in H^s. There exists $T_{\max}$ in $(0,\infty]$ such that (1.1) admits a unique solution*

$$\psi \in \mathcal{C}^0([0,T_{\max}), H^s) \cap \mathcal{C}^1([0,T_{\max}), H^{s-1}) \cap L^a_{\mathrm{loc}}([0,T_{\max}), L^{b,2}), \tag{2.2}$$

where $\frac{1}{a} = \frac{(d-1)s}{d+1}$ and $\frac{1}{b} = \frac{1}{2} - \frac{2s}{d+1}$.

If $d \geq 4$, $s \geq (d+1)/(2d-2)$, $w \in L^{(d-1)/2,\infty} + L^\infty$ and $\psi_0 \in H^s$, then (1.1) admits a unique solution satisfying (2.2), with $\frac{1}{a} = \frac{1}{2}$ and $\frac{1}{b} = \frac{1}{2} - \frac{1}{d-1}$.

If $d = 3$, $s \geq 1$, $w \in L^{q,\infty} + L^\infty$ with $q > 1$ and $\psi_0 \in H^s$, then (1.1) admits a unique solution satisfying (2.2) with $\frac{1}{a} = \frac{1}{2q}$ and $\frac{1}{b} = \frac{1}{2} - \frac{1}{2q}$.

Similarly as in Theorem 2.2, both a blow-up alternative and a continuity property of the solution with respect to the initial data hold under the conditions of Theorem 2.3. See Theorem 4.2 below for a more precise and more general statement involving any admissible pair (a,b) for the Strichartz estimates; as observed in [30], a useful property of (1.1) is that Strichartz estimates hold for both wave and Schrödinger admissible pairs. In Appendix B, we justify that Strichartz estimates hold in Lorentz spaces, in order to be able to consider w in the weak space $L^{(d-1)/2,\infty}$ for $d \geq 4$.

As shown in [35], the regularity condition on the initial states can be improved, at least for the Coulomb potential. Indeed, local existence for initial data $\psi_0 \in H^s$ for $s > \frac{1}{4}$ in dimension 3 is proven in [35], in the case where $w(x) = -|x|^{-1}$ is the attractive Coulomb potential (which belongs to $L^{3,\infty}$), while Theorem 2.3 requires $s \geq \frac{1}{3}$. In this paper we do not try to optimize the regularity condition of initial states to ensure local existence for a given potential.

2.2 Maximal velocity estimates

Our first main result provides a maximal velocity estimate for the boson star between two convex subsets of $\mathbb{R}^d$. It holds for any local in time solutions to (1.1).

Recall that $\mathrm{dist}(X,Y)$ stands for the distance between two subsets of $\mathbb{R}^d$.

Theorem 2.4 (Sharp maximal velocity estimate for convex subsets). *Let $X, Y \subset \mathbb{R}^d$ be two convex subsets, $s \geq 0$, w in $\mathcal{W}_{d,s}$ and ψ_0 in H^s such that $\mathbf{1}_X \psi_0 = \psi_0$.*

If ψ is the local solution to (1.1) on $[0, T_{\max})$ given by Theorem 2.2 or Theorem 2.3, then

$$\forall t \in [0, T_{\max}), \qquad \|\mathbf{1}_Y \psi_t\|_{L^2} \leq e^{t - \mathrm{dist}(X,Y)} \|\psi_0\|_{L^2}. \tag{2.3}$$

Remark 2.5.

1. *The maximal velocity bound (2.3) also holds if one replaces the conditions on w and ψ_0 ensuring the local existence of solutions to (1.1) given by Theorem 2.3, by the slightly more general conditions ensuring local existence given by Theorem 4.2.*

2. *We recall that the speed of light is equal to 1 in our units. For the relativistic dispersion relation $\sqrt{-c^2\Delta + m^2c^4} - mc^2$, a simple scaling argument gives, instead of (2.3), the maximal velocity bound*

$$\|\mathbf{1}_Y \psi_t\|_{L^2} \leq e^{mc^2(ct - \mathrm{dist}(X,Y))} \|\psi_0\|_{L^2}.$$

3. *The exponentially small error term in the maximal velocity estimate (2.3) is "sharp" in the sense that:*

   (a) *If $C < 1$, there exist convex subsets X and Y such that the estimate*

   $$\forall t \in [0, T_{\max}), \quad \|\mathbf{1}_Y \psi_t\|_{L^2} \leq C e^{t - \mathrm{dist}(X,Y)} \|\psi_0\|_{L^2},$$

   does not *hold. This is obvious since, if $X = Y$ then $\|\mathbf{1}_Y \psi_t\|_{L^2} = \|\psi_0\|_{L^2}$ at $t = 0$ for any ψ_0 as in the statement of Theorem 2.4.*

   (b) *If $c < 1$, there exist convex subsets X, Y and w, ψ_0 as in the statement of Theorem 2.4 such that the estimate*

   $$\forall t \in [0, T_{\max}), \quad \|\mathbf{1}_Y \psi_t\|_{L^2} \leq e^{ct - \mathrm{dist}(X,Y)} \|\psi_0\|_{L^2},$$

   does not *hold. This statement is proven in the case of the free evolution, $w = 0$, in our companion paper* [6, Corollary B.2].

Our proof of Theorem 2.4 is based on an elegant and powerful argument recently introduced in [58] for linear Schrödinger-type equations of the form $i\partial_t \psi = \omega(-i\nabla)\psi_t + V(x)\psi_t$, with $\omega(\xi)$ admitting an analytic continuation to a bounded region of $\mathbb{C}^d$. Substantial modifications are however necessary to accommodate the non-linear dynamics considered here, and to reach the sharp bound stated in (2.3) (note that the bound obtained in [58], for more general dispersion relations and general open sets $X, Y \in \mathbb{R}^d$, is of the form $\|\mathbf{1}_Y \psi_t\|_{L^2} \leq C_{\mu,c} e^{\mu(ct - \mathrm{dist}(X,Y))} \|\psi_0\|_{L^2}$ for some $C_{\mu,c} > 0$ and any $\mu < 1$ and $c > 1$, without explicit control on $C_{\mu,c}$).

The idea of the proof of Theorem 2.4 is to estimate, for a suitably chosen function ℓ:

$$\|\mathbf{1}_Y \psi_t\|_{L^2} \leq \underbrace{\|\mathbf{1}_Y e^{\ell(x)}\|_{\mathcal{B}(L^2)}}_{\leq \exp\left(-\frac{\mathrm{dist}(X,Y)}{2}\right)} \underbrace{\|e^{-\ell(x)}\psi_t\|_{L^2}}_{\leq \exp(t)\exp\left(-\frac{\mathrm{dist}(X,Y)}{2}\right)\|\psi_0\|_{L^2}},$$

The function ℓ is constructed thanks to the convexity of X and Y, through a separation argument; see [22] for a similar construction and see Section 7 for the precise expression of ℓ

we use here. The main technical issue to justify this estimate lies in the proof of the fact that $e^{-\ell(x)}\psi_t$ is well-defined in L^2 and can be estimated by Gronwall's Lemma. Instead of using an analyticity argument as in [58], we introduce a bounded approximation ℓ_ε and establish estimates that are uniform in ε. A careful analysis allows us to obtain the sharp bound stated in (2.3). Some technical results entering the proof of Theorem 2.4 are deferred to our companion paper [6] where we prove a maximal velocity estimate for the non-autonomous pseudo-relativistic Schrödinger equation.

Using results from [6] together with a covering argument in the spirit of that used in [58], we can extend the maximal velocity bound to non-convex subsets X, Y, up to a polynomial growth in $\mathrm{dist}(X,Y)$:

Proposition 2.6 (Maximal velocity estimate for general subsets). *There exists $C_d > 0$ such that, if $X, Y \subseteq \mathbb{R}^d$ are Borel subsets, $s \geq 1$, w is in $\mathcal{W}_{d,s}$ and ψ_0 is in H^s with $\mathbf{1}_X\psi_0 = \psi_0$, then the local solution ψ to (1.1) on $[0, T_{\max})$ given by Theorem 2.2 satisfies*

$$\forall t \in [0, T_{\max}), \qquad \|\mathbf{1}_Y\psi_t\|_{L^2} \leq C_d\, e^{t-\mathrm{dist}(X,Y)} \langle \mathrm{dist}(X,Y)\rangle^d \|\psi_0\|_{L^2}. \tag{2.4}$$

The restriction to more regular initial states in Proposition 2.6, compared to Theorem 2.4, comes from the fact that, in order to use the covering argument, we need to write, similarly as in [1] (for the non-relativistic Hartree equation), the solution to (1.1) as $\psi_t = U_t\psi_0$, with $U_t = U_{t,0}$ the propagator generated by the time-dependent (and "initial state-dependent") Hamiltonian $H_t = \langle\nabla\rangle + w * |\psi_t|^2$. For the same reason we only consider the local solution to (1.1) given by Theorem 2.2, not the extension to a larger class of potentials provided by Theorem 2.3. Under these conditions we can apply the abstract results derived in [6] for time-dependent pseudo-relativistic Hamiltonians of the form $H_t = \langle\nabla\rangle + V_t$.

2.3 Global existence

As already mentioned before, for potentials $w \in L^\infty$ and initial states ψ_0 in L^2, Theorem 2.2 together with the conservation of mass (see Lemma 5.1 below) imply the existence of a unique global solution to (1.1) associated to ψ_0. For more general potentials (and more regular initial data), the next theorems provide the global existence of solutions to (1.1) in two distinct regimes, namely assuming that the convolution potential w has either a "long-range" or a "short-range" behavior.

For long-range potentials, similarly as in [45] (where the Yukawa-type interaction potential $w(x) = \kappa|x|^{-1}e^{-\mu|x|}$ is considered in dimension $d = 3$, with $\kappa \in \mathbb{R}$, $\mu \geq 0$), we can use the conservation of the energy (and of the mass) to obtain the global existence. To do that, we need that the energy defined in (1.3) is well-defined and real. Hence we need to assume that the initial data ψ_0 belong to H^s with $s \geq \frac{1}{2}$ and that w is even.

Theorem 2.7 (Global existence for long-range interaction potentials I). *Let $s \geq \frac{1}{2}$. There exists a universal constant $C_0 > 0$ such that, for all w even of the form $w = w_d + w_\infty \in L^{d,\infty} + L^\infty$ and $\psi_0 \in H^s$ verifying*

$$\|(w_d)_-\|_{L^{d,\infty}}\|\psi_0\|_{L^2}^2 < C_0, \tag{2.5}$$

Eq. (1.1) admits a unique solution

$$\psi \in \mathcal{C}^0([0,\infty), H^s) \cap \mathcal{C}^1([0,\infty), H^{s-1}).$$

Remark 2.8.

1. *The universal constant* C_0 *appearing in the statement of the previous theorem can be chosen as* $C_0 = 2C_S^{-1}$ *where* C_S *is the optimal constant in the Sobolev embedding* $H^{\frac{1}{2}} \hookrightarrow L^{\frac{2d}{d-1},2}$. *Here* $L^{\frac{2d}{d-1},2}$ *stands for the usual Lorentz space (see Subsection 3.2 for the definition).*

2. *The same continuity property with respect to the initial data as that stated in Theorem 2.2 holds.*

3. *The smallness condition* (2.5) *cannot be avoided since, as mentioned in the introduction, it is proven in [27] that finite time blow-up holds , in dimension* $d = 3$ *and for the Newtonian potential* $w(x) = -\kappa|x|^{-1}$ *(which belongs to* $L^{3,\infty}$*) with* $\kappa > 0$ *if the mass* $M(\psi_0) = \|\psi_0\|^2_{L^2}$ *of the initial state is larger than some critical value.*

Using the Strichartz estimates given in Appendix B and taking initial states with a small enough $H^{1/2}$-norm, the class of potentials $L^{d,\infty} + L^\infty$ considered in the previous theorem can be extended to $L^{(d+1)/2,\infty} + L^\infty$. More precisely, we have the following result.

Theorem 2.9 (Global existence for long-range interaction potentials II)**.** *Let* $s = \frac{1}{2}$. *There exists a universal constant* $C_0 > 0$ *such that, for all* w *even of the form* $w = w_{d/2} + w_\infty \in L^{(d+1)/2,\infty} + L^\infty \subset L^{d/2} + L^\infty$ *and* $\psi_0 \in H^{\frac{1}{2}}$ *verifying*

$$\|(w_{d/2})_-\|_{L^{\frac{d}{2}}}(E(\psi_0) + \|\psi_0\|^2_{H^{\frac{1}{2}}} + \|(w_\infty)_-\|_{L^\infty}\|\psi_0\|^4_{L^2}) < C_0,$$

Eq. (1.1) *admits a unique solution*

$$\psi \in \mathcal{C}^0([0,\infty), H^{\frac{1}{2}}) \cap \mathcal{C}^1([0,\infty), H^{-\frac{1}{2}}) \cap L^a_{\mathrm{loc}}([0,\infty), L^{b,2}),$$

where $\frac{1}{a} = \frac{1}{b} = \frac{d-1}{2d+2}$.

A more general version of Theorem 2.9 will be given in Theorem 5.2 below, involving any admissible pair (a, b) for the Strichartz estimates from Appendix B. A related result is proven in [12] for sums of potentials of the form $w_i(x) = \kappa_i|x|^{-\alpha_i}$ with $0 < \alpha_i < 2d/(d+1)$ (which corresponds to potentials in $L^{q_1} + L^{q_2}$ with $(d+1)/2 < q_1, q_2 < \infty$).

For short-range interaction potentials, the conservation of energy is not sufficient anymore to obtain the global existence of solutions to (1.1), but one can rely instead on dispersive estimates satisfied by the free half-Klein-Gordon equation $i\partial_t\psi_t = \langle\nabla\rangle\psi_t$, see (1.6) for the $L^1 \to L^\infty$ estimate. For $s > \frac{d}{2}$, $r \geq 0$ and $1 \leq p \leq \infty$, we introduce the subspace $S^{s,r,p}$ of $L^\infty([0,\infty), H^s)$ containing the functions φ such that

$$\|\varphi\|_{S^{s,r,p}} := \sup_{t\geq 0}\|\varphi_t\|_{H^s} + \sup_{t\geq 0}\langle t\rangle^{\frac{d}{2r}}\|\varphi_t\|_{L^\infty\cap L^p} < \infty\,. \tag{2.6}$$

We also use the notation $H^{s,p}$ for the Bessel-Sobolev space with norm $\|f\|_{H^{s,p}} = \|\langle\nabla\rangle^s f\|_{L^p}$ (see Subsection 3.1 for the precise definition).

Theorem 2.10 (Global existence for short-range interaction potentials)**.** *Let* $s \geq \frac{d}{2} + 1$, $1 \leq r < d$ *and* $1 \leq q < \frac{2d}{3}$ *be such that* $\frac{1}{d} + \frac{1}{2r} < \frac{1}{q}$. *Define* p *by the relation* $1 = \frac{1}{r} + \frac{2}{p}$.

There exists $\varepsilon_0 > 0$ *such that the following holds: for all* $w \in \mathcal{M} + L^q$ *and* $\psi_0 \in H^s \cap H^{s,p'}$ *satisfying*

$$\|w\|_{\mathcal{M}+L^q}\,\|\psi_0\|^2_{H^s\cap H^{s,p'}} \leq \varepsilon_0, \tag{2.7}$$

Eq. (1.1) *admits a unique solution* ψ *in* $S^{s,r,p}$.

This solution ψ *belongs to* $\mathcal{C}^0([0,\infty),H^s)\cap\mathcal{C}^1([0,\infty),H^{s-1})$ *and satisfies, for all* $t\geq 0$,

$$\|\psi_t\|_{L^\infty\cap L^p}\lesssim\langle t\rangle^{-\frac{d}{2r}}\|\psi_0\|_{H^s\cap H^{s,p'}},\tag{2.8}$$

$$\big\|\psi_t-\psi_t^{(0)}\big\|_{L^\infty\cap L^p}\lesssim\varepsilon_0\langle t\rangle^{-\frac{d}{2r}}\|\psi_0\|_{H^s\cap H^{s,p'}},\tag{2.9}$$

$$\|\psi_t\|_{H^s}\lesssim\|\psi_0\|_{H^s\cap H^{s,p'}},\tag{2.10}$$

$$\big\|\psi_t-\psi_t^{(0)}\big\|_{H^s}\lesssim\varepsilon_0\|\psi_0\|_{H^s\cap H^{s,p'}},\tag{2.11}$$

where $\psi_t^{(0)}:=e^{-it\langle\nabla\rangle}\psi_0$. *In particular, the map* $\psi_0\mapsto\psi\in S^{s,r,p}$, *defined on the subset of* ψ_0*'s in* $H^s\cap H^{s,p'}$ *satisfying* (2.7), *is continuous.*

Remark 2.11. *Theorem 2.10 gives global existence for all potentials* $w\in L^{q,\infty}$ *with* $1<q<\frac{2d}{3}$, *by applying the theorem with* $q+\varepsilon<\frac{2d}{3}$ *and observing that* $L^{q,\infty}\subset\mathcal{M}+L^{q+\varepsilon}$. *In Figure 3 we represent the class of admissible potentials for Theorem 2.10.*

As mentioned in the introduction, global existence – and small initial data scattering – are proven in [12] for (sums of) short-range interaction potentials of the form $w_i(x)=\kappa_i|x|^{-\alpha_i}$ with $2<\alpha_i<d$ (which corresponds to potentials in $L^{q_1}+L^{q_2}$ with $1<q_1,q_2<d/2$). The proof in [12] relies in particular on the use of Strichartz estimates with Schrödinger admissible pairs. See also [52] for a single potential w satisfying $|w(x)|\lesssim|x|^{-\alpha}$ with $\frac{3}{2}<\alpha<d$ and [62] for small data scattering for low regularity initial states and a smooth potential w.

2.4 Scattering states and scattering operators

In the case of short-range interaction potentials, one can show that the global solution given by Theorem 2.10 scatters to a free solution in H^s, in the sense of the following theorem. For any $\delta>0$, we denote by $\mathcal{B}_{\mathcal{E}}(\delta)$ the closed ball of radius δ in a normed vector space $\mathcal{E}$.

Theorem 2.12 (Scattering for short-range interaction potentials)**.** *Under the conditions of Theorem 2.10, the global solution* ψ *to* (1.1) *scatters to a free solution:*

$$\|\psi_t-e^{-it\langle\nabla\rangle}\psi_+\|_{H^s}\to 0,\qquad t\to\infty,\tag{2.12}$$

where the scattering state $\psi_+\in H^s$ *is defined by*

$$\psi_+:=\psi_0-i\int_0^\infty e^{i\tau\langle\nabla\rangle}(w*|\psi_\tau|^2)\psi_\tau\,\mathrm{d}\tau.\tag{2.13}$$

Moreover there exists $\delta_w>0$ *such that the "inverse" wave operator*

$$W_+:\mathcal{B}_{H^s\cap H^{s,p'}}(\delta_w)\to H^s,$$
$$\psi_0\mapsto W_+\psi_0:=\psi_0-i\int_0^\infty e^{i\tau\langle\nabla\rangle}(w*|\psi_\tau|^2)\psi_\tau\,\mathrm{d}\tau,$$

is continuous.

Remark 2.13. *Our proof gives an explicit rate of decay in* (2.12), *namely we have the bound*

$$\|\psi_t-e^{-it\langle\nabla\rangle}\psi_+\|_{H^s}\lesssim\langle t\rangle^{1-\frac{d}{r}\min\{1,\frac{r}{q}\}}\|\psi_0\|_{H^s\cap H^{s,p'}},\tag{2.14}$$

uniformly in $t\geq 0$.

As in previous works [11–13], the proof of Theorem 2.12 is a rather straightforward application of the regularity and decay properties of the solution to (1.1), see (2.8)–(2.11) in our context.

The wave operator Ω_+, mapping any scattering state ψ_+ to an initial state ψ_0 can be defined similarly as W_+, constructing a solution to the Cauchy problem at ∞ for the pseudo-relativistic Hartree equation,

$$\begin{cases} i\partial_t\psi_t = (\langle\nabla\rangle + w * |\psi_t|^2)\psi_t, \\ \lim_{t\to\infty} \|\psi_t - e^{-it\langle\nabla\rangle}\psi_+\|_{L^2} = 0. \end{cases} \tag{2.15}$$

See Theorems 8.4 and 8.5 below for precise statements. Formally, Ω_+ is the inverse of W_+ by definition; however, since W_+ maps $H^s \cap H^{s,p'}$ to H^s and Ω_+ is defined on (a small ball in) $H^s \cap H^{s,p'}$, the composition $\Omega_+ W_+$ is ill-defined in general. To overcome this difficulty, for suitable values of γ and s, we restrict W_+ to the weighted Sobolev space H^s_γ with norm

$$\|\varphi\|_{H^s_\gamma} := \|\langle x\rangle^\gamma \langle\nabla\rangle^s \varphi\|_{L^2}.$$

We will choose parameters ensuring that $H^s_\gamma \hookrightarrow H^s \cap H^{s,p'}$, and hence that W_+ is well-defined on a ball in H^s_γ with sufficiently small radius. Moreover, restricting the class of admissible potentials to $w \in \mathcal{M} + L^q$ with $1 \le q < \frac{d}{2}$, we will show that W_+ maps this ball in (a small ball in) $H^s \cap H^{s,p'}$, and likewise for Ω_+. We then have the following result on the invertibility of the wave operator.

Theorem 2.14 (Right invertibility of the wave operators for short-range interaction potentials)**.** *Let $s \ge \frac{d}{2} + 1$, $\max\{1, \frac{d}{4}\} \le r < \frac{d}{2}$, $1 \le q \le r$ and $\frac{d}{2r} < \gamma < \min\{2, \frac{d}{r} - 1\}$. Let $w \in \mathcal{M} + L^q$. There exists $\delta_w > 0$ such that, for all $\varphi \in \mathcal{B}_{H^s_\gamma}(\delta_w)$,*

$$\Omega_+ W_+ \varphi = W_+ \Omega_+ \varphi = \varphi. \tag{2.16}$$

Remark 2.15. *The operators Ω_+ and W_+ may not map $\mathcal{B}_{H^s_\gamma}(\delta_w)$ into itself. We only have that Ω_+ and W_+ map $\mathcal{B}_{H^s_\gamma}(\delta_w)$ into $\mathcal{B}_{H^s_\gamma}(\delta'_w)$ for some $\delta'_w > \delta_w$ (one can take $\delta'_w = \delta_w + C\varepsilon_0$ for some positive constant C and some small enough $\varepsilon_0 > 0$, see Theorem 8.6 below for a precise statement). Eq. (2.16) therefore shows that $W_+ : \operatorname{Ran}\Omega_+ \to \mathcal{B}_{H^s_\gamma}(\delta_w)$ is a right inverse of $\Omega_+ : \mathcal{B}_{H^s_\gamma}(\delta_w) \to \operatorname{Ran}\Omega_+$ and vice versa.*

The proof of Theorem 2.14 requires to control, in a rather precise way, the asymptotic behavior in weighted L^p spaces of global solutions to (1.1) (constructed in Theorem 2.12). For this reason, the class of allowed potentials in Theorem 2.14 is more restrictive than that considered in Theorem 2.12. Our proof will also provide estimates on the operators $W_+ - \mathrm{Id}$ and $\Omega_+ - \mathrm{Id}$, see Theorem 8.6 below for a precise statement.

Theorem 2.14 will be crucial in order to construct a suitable set of initial states leading to an asymptotic minimal velocity estimate, see Theorem 2.18 below.

2.5 Asymptotic propagation and minimal velocity estimates

Our last concern is the asymptotic behavior of the speed of propagation of solutions to (1.1). We introduce the "instantaneous" velocity operator

$$\Theta := [x, \langle\nabla\rangle] = -i\nabla\langle\nabla\rangle^{-1}.$$

The next theorem shows that, along the evolution associated to (1.1), the instantaneous velocity and the average velocity $\frac{x}{t}$ converge to each other.

Theorem 2.16 (Asymptotic phase-space propagation estimate for short-range interaction potentials)**.** *Let $f, g \in \mathcal{C}_0^\infty(\mathbb{R})$ be such that $\operatorname{supp}(g) \cap \operatorname{supp}(f) = \emptyset$. Under the conditions of Theorem 2.10, and assuming in addition that $\|\langle x\rangle\psi_0\|_{L^2} < \infty$, the global solution ψ to (1.1) given by Theorem 2.10 satisfies*

$$\Big\| g\Big(\frac{x^2}{t^2}\Big) f(\Theta^2)\psi_t \Big\|_{L^2} \to 0, \qquad t \to \infty. \tag{2.17}$$

Remark 2.17.

1. *Since $0 \le \Theta^2 \le 1$, taking $f \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $f = 1$ on $[0, 1]$, we have $f(\Theta^2) = \mathrm{Id}$. Hence the previous theorem implies that, for all $\varepsilon > 0$,*

$$\Big\| \mathbf{1}_{[1+\varepsilon,\infty)}\Big(\frac{x^2}{t^2}\Big)\psi_t \Big\|_{L^2} \to 0, \qquad t \to \infty.$$

We thus recover a maximal velocity estimate, in the sense that the boson star cannot propagate faster than the speed of light (equal to 1 in our unit), asymptotically as $t \to \infty$.

2. *Our proof gives an explicit rate of decay in (2.17), more precisely we will show that*

$$\Big\| g\Big(\frac{x^2}{t^2}\Big) f(\Theta^2)\psi_t \Big\|_{L^2} \lesssim \big(\langle t\rangle^{-1} + \langle t\rangle^{1-\frac{d}{r}\min\{1,\frac{r}{q}\}}\big)\|\langle x\rangle\psi_0\|_{L^2},$$

uniformly in $t \ge 0$.

Combining Theorems 2.12 and 2.16, one can deduce a minimal velocity estimate, in the sense that if the initial state ψ_0 is associated to a scattering state ψ_+ with an instantaneous velocity localized in $[\alpha, 1]$ with $0 < \alpha < 1$, then

$$\Big\| \mathbf{1}_{[0,\alpha)}\Big(\frac{x^2}{t^2}\Big)\psi_t \Big\|_{L^2} \to 0.$$

It is however unclear in general how to identify initial states ψ_0's associated to scattering states ψ_+ with an instantaneous velocity localized in $[\alpha, 1]$. To overcome this difficulty we restrict the class of admissible potentials and use Theorem 2.14. This leads to the following result.

Theorem 2.18 (Asymptotic minimal velocity estimate for short-range interaction potentials)**.** *Let $s \ge \frac{d}{2} + 1$, $\max\{1, \frac{d}{4}\} \le r < \frac{d}{2}$, $1 \le q \le r$ and $\frac{d}{2r} < \gamma < \min\{2, \frac{d}{r} - 1\}$. Let $w \in \mathcal{M} + L^q$ and $0 < \alpha < 1$. There exists δ_w such that, for all initial states*

$$\psi_0 = \Omega_+\psi_+ \quad \text{with} \quad \psi_+ \in \mathcal{B}_{H^s_\gamma}(\delta_w) \quad \text{and} \quad \psi_+ = \mathbf{1}_{[\alpha,1]}(\Theta^2)\psi_+,$$

the global solution ψ to (1.1) given by Theorem 2.10 satisfies

$$\Big\| \mathbf{1}_{[0,\alpha)}\Big(\frac{x^2}{t^2}\Big)\psi_t \Big\|_{L^2} \to 0, \qquad t \to \infty.$$

Remark 2.19. *It is not difficult to construct states $\psi_+ \in \mathcal{B}_{H^s_\gamma}(\delta_w)$ satisfying in addition $\psi_+ = \mathbf{1}_{[\alpha,1]}(\Theta^2)\psi_+$. Indeed, given a smooth function $f \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\operatorname{supp}(f) \subset (\alpha, 1)$, it suffices to choose $\tilde\psi_+$ such that $f(\Theta^2)\tilde\psi_+ = \tilde\psi_+$ and $\|\tilde\psi_+\|_{H^s_\gamma}$ is small enough. Since one can verify that $\|f(\Theta^2)\psi_+\|_{H^s_\gamma} \le C_f\|\psi_+\|_{H^s_\gamma}$ for some $C_f > 0$, the state given by $\psi_+ = f(\Theta^2)\tilde\psi^+$ satisfies the conditions of the theorem for $\|\tilde\psi_+\|_{H^s_\gamma}$ small enough.*

It should be noted that the maximal velocity estimate mentioned in Remark 2.17 holds for all initial states, without requiring a construction as in Theorem 2.18. This is due to the fact that the instantaneous velocity Θ is bounded. For the non-relativistic Hartree equation considered in [1], proving a maximal velocity estimate does require a suitable construction of initial states; instead of using "asymptotic energy cutoffs" as in [1], it would be interesting to follow the approach developed in the present paper based on invertibility properties of the wave operator.

2.6 Summary of results

For the reader's convenience, we summarize the conditions required to obtain our results in two tables and three figures. Table 1 collects the well-posedness results. The relationships between the regularity of the initial data and the class of admissible potentials in these results is illustrated in Figure 1 for a generic spatial dimension $d \geq 4$ while Figure 2 highlights features specific to dimension 3, including some endpoint cases. Figure 3 depicts the relation between the parameters p and q appearing in Theorem 2.10, which concerns the global existence in the case of short-range potentials; this existence theory is the one employed in our analysis of scattering and minimal velocity estimates. Finally, Table 2 summarizes our results on the maximal and minimal velocity estimates.

s in	w in	Example of w: $\lvert x\rvert^{-\alpha}$, α in	Example of w: δ_0	Well-posedness	Theorem
$[0, \frac{d}{2})$	$L^{\frac{d}{2s},\infty} + L^\infty$	$[0, 2s]$		Local	2.2
$\{\frac{d}{2}\}$	$L^{q,\infty} + L^\infty,\ q > 1$	$[0, d)$			2.2
$(\frac{d}{2}, \infty)$	$\mathcal{M} + L^\infty$	$[0, d)$	✓		2.2
$[0, \frac{d+1}{2d-2})$	$L^{\frac{d+1}{4s},\infty} + L^\infty$	$[0, \frac{4ds}{d+1}]$			2.3
$[\frac{d+1}{2d-2}, \infty)$	$L^{\frac{d-1}{2},\infty} + L^\infty$ †	$[0, \frac{2d}{d-1})$			2.3
$\{0\}$	L^∞	$\{0\}$		Global	2.2
$[\frac{1}{2}, \infty)$	$L^{d,\infty} + L^\infty$	$[0, 1]$			2.7
$\{\frac{1}{2}\}$	$L^{\frac{d+1}{2},\infty} + L^\infty$	$[0, \frac{2d}{d+1}]$			2.9
$[\frac{d}{2}+1, \infty)$	$\mathcal{M} + L^q,\ q < \frac{2d}{3}$	$(\frac{3}{2}, d)$	✓		2.10

Table 1: Well-posedness results for (1.1) with $\psi_0 \in H^s$. We underline that the result of Theorem 2.10 requires the additional condition $\psi_0 \in H^{s,p}$ for suitable p. Moreover, while at fixed $s < \frac{d}{2}$ Theorem 2.3 provides local existence for a larger class of potentials compared to Theorem 2.2, it guarantees uniqueness of a solution only in a smaller functional space.
†If $d = 3$, then one can actually only take w in $L^{q,\infty} + L^\infty$ for any $q > \frac{d-1}{2}$.

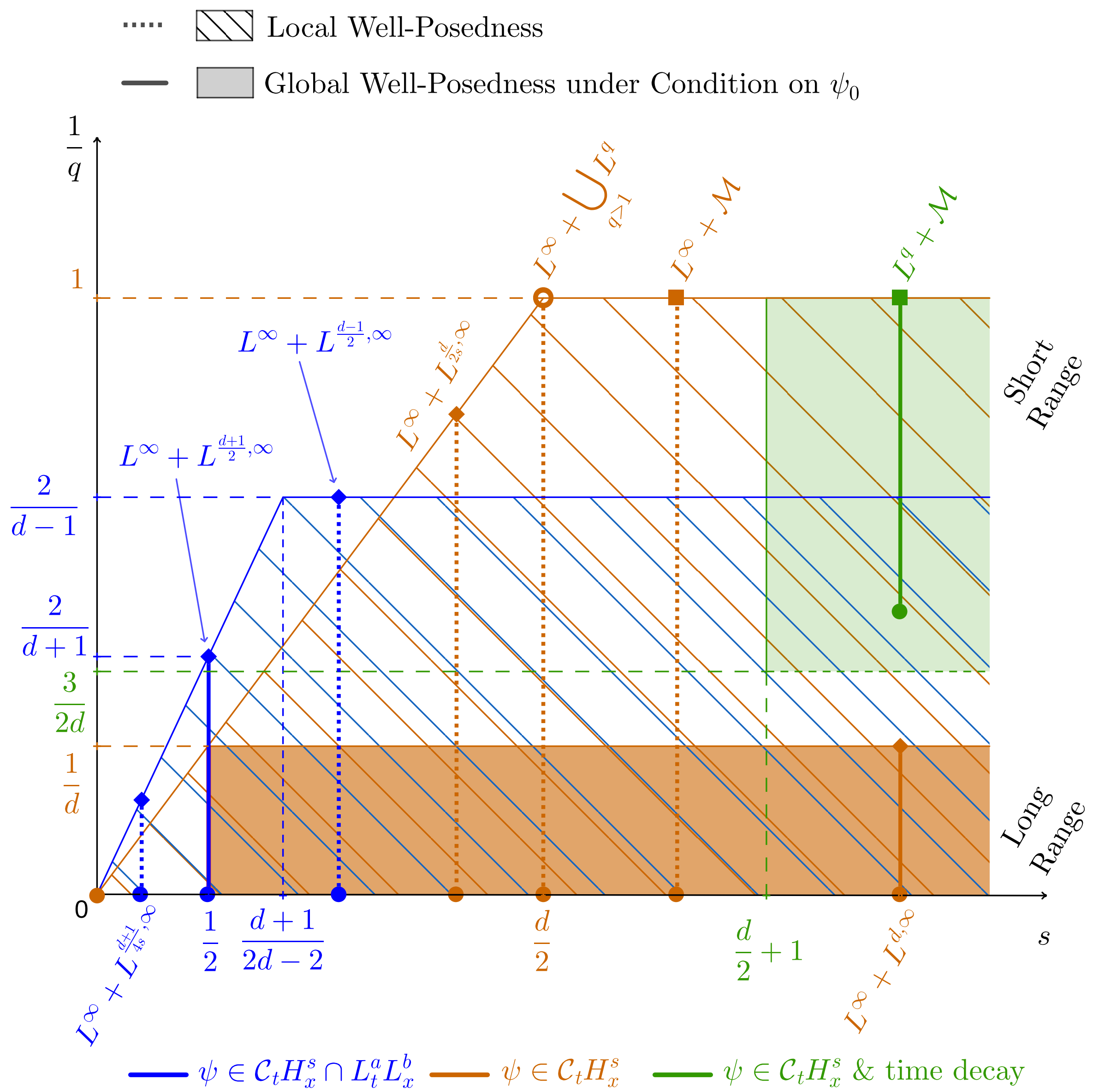

Figure 1: Dimension $d \geq 4$. Well-Posedness of (1.1): Admissible class of w depending on the regularity s of ψ_0.
A thick vertical line with extremities at $1/q_1$ and $1/q_2$ represents the space $L^{q_1,\tilde{q}_1} + L^{q_2,\tilde{q}_2}$, with $\tilde{q}_j = q_j$ if the extremity is a disk (●), $\tilde{q}_j = \infty$ if the extremity is diamond (◆). There are two special cases: the $L^{q,\tilde{q}}$ space is replaced by $\mathcal{M}$ if the extremity is a square (■), and by $\bigcup_{q>q_j} L^q$ if it is a circle (○). Solid lines correspond to global well-posedness while dotted lines correspond to local well-posedness.

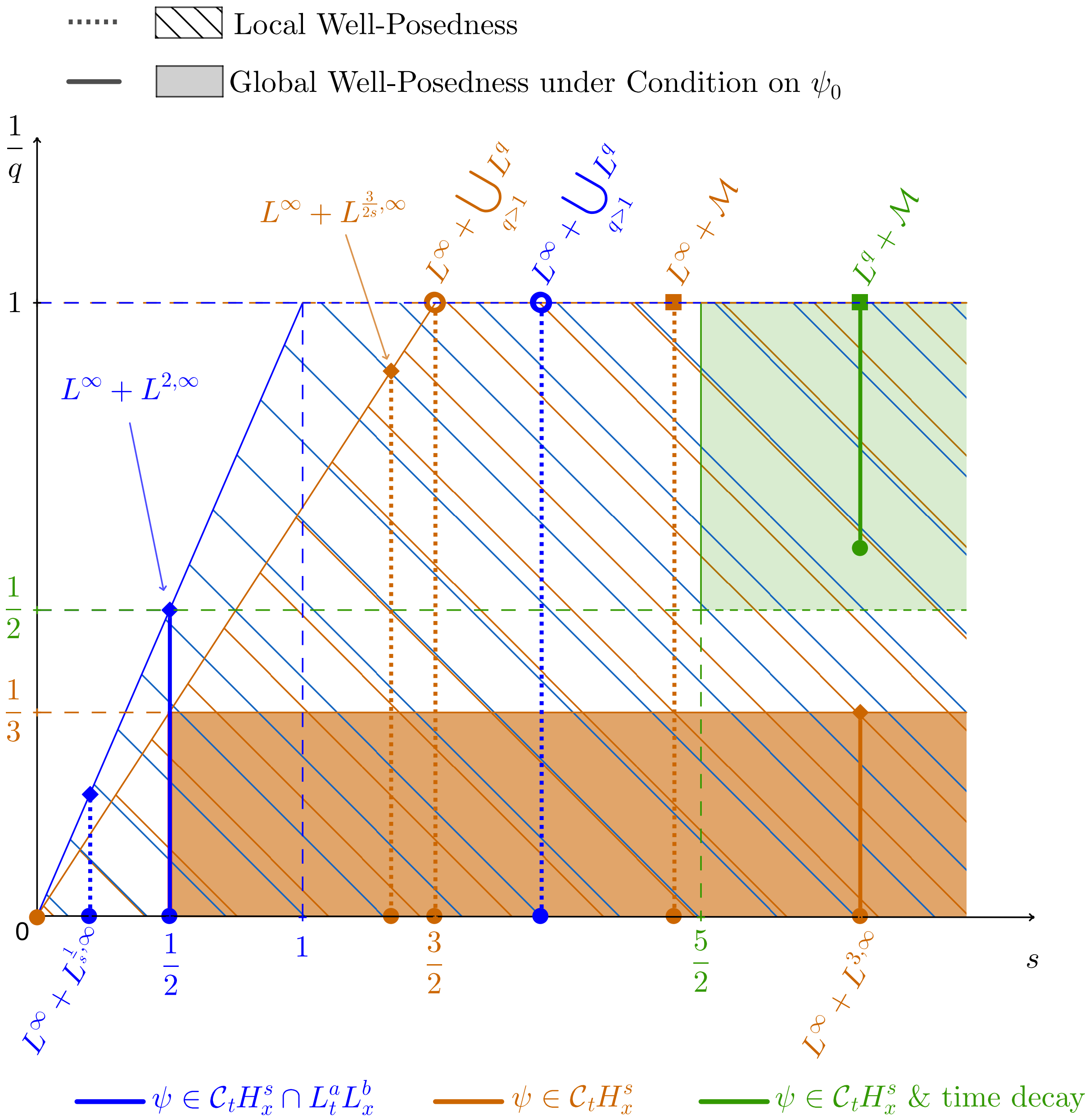

Figure 2: Dimension $d = 3$. Well-Posedness of (1.1): Admissible class of w depending on the regularity s of ψ_0.
A thick vertical line with extremities at $1/q_1$ and $1/q_2$ represents the space $L^{q_1,\tilde{q}_1} + L^{q_2,\tilde{q}_2}$, with $\tilde{q}_j = q_j$ if the extremity is a disk (●), $\tilde{q}_j = \infty$ if the extremity is diamond (◆). There are two special cases: the $L^{q,\tilde{q}}$ space is replaced by $\mathcal{M}$ if the extremity is a square (■), and by $\bigcup_{q>q_j} L^q$ if it is a circle (○). Solid lines correspond to global well-posedness while dotted lines correspond to local well-posedness.

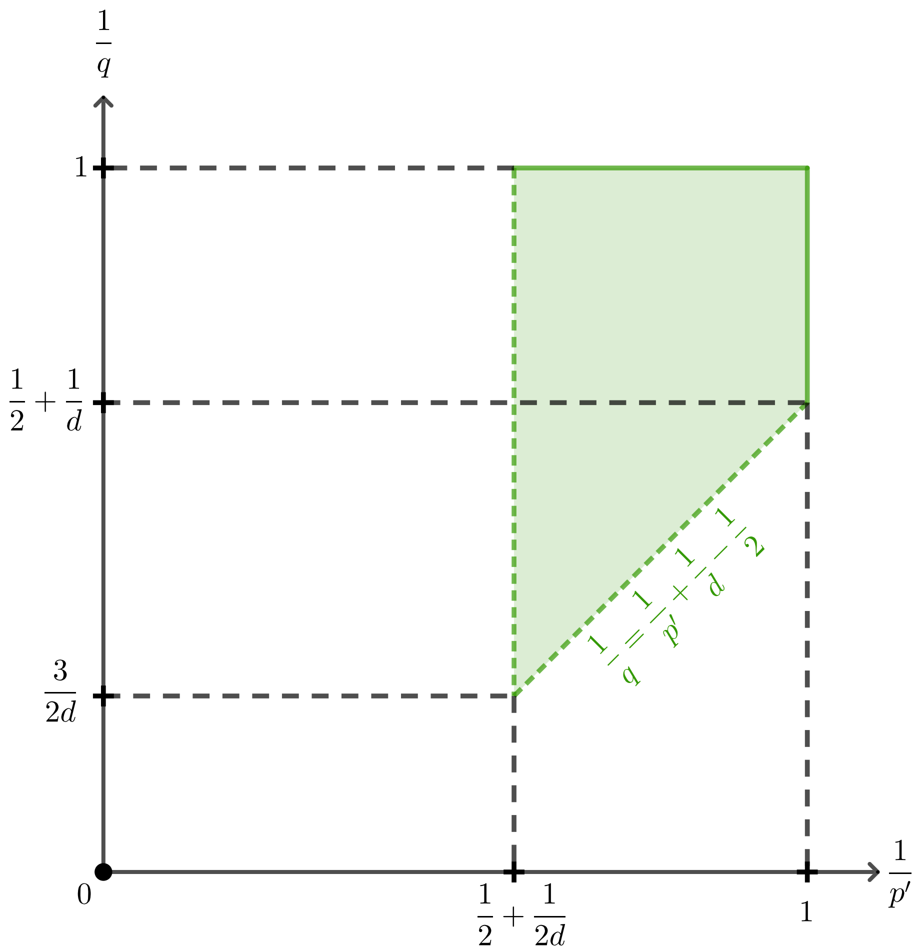

Figure 3: Global Well-Posedness in the Short range case: The green region represents the domain of admissible pairs $(\frac{1}{p'}, \frac{1}{q})$ such that, if ψ_0 is in $H^s \cap H^{s,p'}$ and w in $L^q + \mathcal{M}$ and their norms are sufficiently small, then (1.1) admits a global solution, provided by Theorem 2.10.

s in	w in	Example of w: $\lvert x\rvert^{-\alpha}$, α in	Example of w: δ_0	Velocity estimate	Theorem
$[0, \frac{d}{2})$	$L^{\frac{d}{2s},\infty} + L^\infty$	$[0, 2s]$		Maximal	2.4
$\{\frac{d}{2}\}$	$L^{q,\infty} + L^\infty,\ q > 1$	$[0, d)$		Maximal	2.4
$(\frac{d}{2}, \infty)$	$\mathcal{M} + L^\infty$	$[0, d)$	✓	Maximal	2.4
$[\frac{d}{2} + 1, \infty)$	$\mathcal{M} + L^q,\ q < \frac{d}{2}$	$(2, d)$	✓	Maximal & Minimal	2.4 & 2.18

Table 2: Speed of propagation results for (1.1) with $\psi_0 \in H^s$. We underline that the minimal velocity result holds under the additional condition that ψ_0 is in a weighted H^s space.

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3 Preliminaries

In this preliminary subsection, we first introduce functional spaces that will play an important role in our analysis, the Bessel-Sobolev spaces, and we recall a version of the fractional Leibniz rule in this context. Next we derive some multilinear estimates that will allow us to control the non-linearity appearing in the pseudo-relativistic Hartree equation (1.1).

3.1 Weighted Bessel-Sobolev spaces and fractional Leibniz rule

We denote by $\mathcal{S}$ the usual Schwartz space of rapidly decreasing functions from $\mathbb{R}^d$ to $\mathbb{C}$ and by $\mathcal{S}'$ the associated space of tempered distributions. For a tempered distribution f in $\mathcal{S}'$ and s in $\mathbb{R}$, we write $\langle \nabla \rangle^s f = \mathcal{F}^{-1}(\langle \xi \rangle^s \mathcal{F} f)$. Recall that $\mathcal{F}$ stands for the Fourier transform normalized such that $\mathcal{F}$ is unitary on L^2. For $p \in [1, \infty]$, the Bessel-Sobolev spaces are defined by

$$H^{s,p} := \{ f \in \mathcal{S}' \mid \langle \nabla \rangle^s f \in L^p \},$$

and endowed with the norm $\|f\|_{H^{s,p}} = \|\langle \nabla \rangle^s f\|_{L^p}$. If $p = 2$, we use the shorhand $H^s = H^{s,2}$. When $1 < p < \infty$ the Bessel-Sobolev spaces $H^{s,p}$ coincide with the fractional Sobolev spaces $W^{s,p}$ with equivalent norms, but it is not the case when $p = 1$ or $p = \infty$, which we we will also need in the sequel.

In the case of non integer exponents s, the Leibniz rule cannot be used, but a useful replacement is the fractional Leibniz rule, see e.g. [34].

Proposition 3.1. *Let $1 \le p_0 < \infty$, $1 < p_j, \tilde{p}_j \le \infty$ satisfying $\frac{1}{p_0} = \frac{1}{p_j} + \frac{1}{\tilde{p}_j}$ and $j \in \{1, 2\}$. If $s \ge 0$, there exists $C > 0$ such that, for all $f, g \in \mathcal{S}$,*

$$\|fg\|_{H^{s,p_0}} \le C \big(\|f\|_{H^{s,p_1}} \|g\|_{L^{\tilde{p}_1}} + \|f\|_{L^{p_2}} \|g\|_{H^{s,\tilde{p}_2}} \big) .$$

Remark 3.2. *We will use Proposition 3.1 with functions f in H^{s,p_1} and L^{p_2} and g in $H^{s,\tilde{p}_2}$ and $L^{\tilde{p}_1}$, which is fine if these spaces are included in the closure of $\mathcal{S}$ for the corresponding norms.*

We will sometime have $p_2 = \infty$ (respectively $\tilde{p}_1 = \infty$). In this case, we will have to make sure that f (respectively g) is in the closure of $\mathcal{S}$ with respect to the L^∞ norm, which is the space $\mathcal{C}^0_\infty$ of continuous functions vanishing at infinity.

We will also use the following weighted Bessel-Sobolev spaces, defined as

$$H^{s,p}_\gamma := \{ f \in \mathcal{S}' \mid \langle x \rangle^\gamma \langle \nabla \rangle^s f \in L^p \}, \quad p \in [1, \infty], \quad s \ge 0, \quad \gamma \ge 0, \tag{3.1}$$

and endowed with the norm $\|f\|_{H^{s,p}_\gamma} = \|\langle x \rangle^\gamma \langle \nabla \rangle^s f\|_{L^p}$. We observe that, for all $\gamma \ge 0$ and $s \ge 0$,

$$H^s_\gamma \hookrightarrow H^s,$$

while if $\gamma > \frac{d}{2r}$ wit $1 \le r \le \infty$, it follows from Hölder's inequality that

$$H^s_\gamma \hookrightarrow H^{s,p'},$$

with $p' = \frac{2r}{r+1}$. We also define the shorthands

$$H^s_\gamma := H^{s,2}_\gamma, \quad L^p_\gamma := H^{0,p}_\gamma. \tag{3.2}$$

In Appendix A we recall the following property (we only state and prove it for $0 \leq \gamma \leq 2$ for simplicity and since we only need it for such γ in our context).

Lemma 3.3. *Let $0 \leq \gamma \leq 2$, $s \geq 0$ and $1 \leq p \leq \infty$. There exist $c, c' > 0$ such that*

$$c' \, \|\langle\nabla\rangle^s \langle x\rangle^\gamma f\|_{L^p} \leq \|f\|_{H^{s,p}_\gamma} \leq c \, \|\langle\nabla\rangle^s \langle x\rangle^\gamma f\|_{L^p},$$

for all $f \in H^{s,p}_\gamma$.

Combined with Proposition 3.1, this lemma implies a weighted version of the fractional Leibniz rule.

Proposition 3.4. *Let $0 \leq \gamma \leq 2$, $1 \leq p_0 < \infty$, $1 < p_j, \tilde{p}_j \leq \infty$ satisfying $\frac{1}{p_0} = \frac{1}{p_j} + \frac{1}{\tilde{p}_j}$ and $j \in \{1, 2\}$. If $s \geq 0$, then there exists $C > 0$ such that, for all $f, g \in \mathcal{S}$,*

$$\|fg\|_{H^{s,p_0}_\gamma} \leq C(\|f\|_{H^{s,p_1}} \|g\|_{L^{\tilde{p}_1}_\gamma} + \|f\|_{L^{p_2}} \|g\|_{H^{s,\tilde{p}_2}_\gamma}).$$

3.2 Functional inequalities in Lorentz spaces

For p in $(1, \infty)$ and q in $[1, \infty]$ the Lorentz space $L^{p,q}$ can be defined as the real interpolation space $[L^1, L^\infty]_{1-1/p,q}$. The spaces $L^{p,\infty}$ coincide with the weak-Lebesgue spaces, $L^{p,p} = L^p$ and if $1 \leq q_1 \leq q_2 \leq \infty$ then L^{p,q_1} is continuously embedded in L^{p,q_2}.

The Lorentz spaces have several interesting features, in particular they include the weak spaces $L^{p,\infty}$, which themselves include $1/|x|^{d/p}$, Hölder and Young inequalities hold in Lorentz spaces, as well as a refined version of the usual Sobolev embeddings. More precisely we have the following three propositions, which are proven for instance in [42, Chapter 2].

Similarly as before, for f in $\mathcal{S}'$ (such that $\mathcal{F}f \in L^1_{\mathrm{loc}}$) and $s \geq 0$, we write $|\nabla|^s f = \mathcal{F}^{-1}(|\xi|^s \mathcal{F}f)$.

Proposition 3.5 (Sobolev inequality in Lorentz spaces)**.** *Assume that $0 < \frac{s}{d} < \frac{1}{p} < 1$ and $1 \leq q \leq \infty$. If $\frac{1}{p} - \frac{s}{d} = \frac{1}{p_0}$, then*

$$\|u\|_{L^{p_0,q}} \lesssim \||\nabla|^s u\|_{L^{p,q}} .$$

In particular

$$\|u\|_{L^{p_0,p}} \lesssim \|u\|_{H^{s,p}} .$$

Proposition 3.6 (Hölder inequality in Lorentz spaces)**.** *If $1 < p, p_1, p_2 < \infty$, $1 \leq q, q_1, q_2 \leq \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, and $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$, then*

$$\|f_1 f_2\|_{L^{p,q}} \lesssim \|f_1\|_{L^{p,q}} \|f_2\|_{L^\infty}$$
$$\|f_1 f_2\|_{L^1} \lesssim \|f_1\|_{L^{p,q}} \|f_2\|_{L^{p',q'}}$$
$$\|f_1 f_2\|_{L^{p,q}} \lesssim \|f_1\|_{L^{p_1,q_1}} \|f_2\|_{L^{p_2,q_2}}$$

provided that the right-hand sides are finite.

Proposition 3.7 (Young inequality in Lorentz spaces). *If $1 < p, p_1, p_2 < \infty$, $1 \le q, q_1, q_2 \le \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} + 1$, $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$, then*

$$\|f_1 * f_2\|_{L^{p,q}} \lesssim \|f_1\|_{L^{p,q}} \|f_2\|_{L^1}$$
$$\|f_1 * f_2\|_{L^\infty} \lesssim \|f_1\|_{L^{p,q}} \|f_2\|_{L^{p',q'}}$$
$$\|f_1 * f_2\|_{L^{p,q}} \lesssim \|f_1\|_{L^{p_1,q_1}} \|f_2\|_{L^{p_2,q_2}}$$

provided that the right-hand sides are finite.

In the next proposition, we remark that the Young inequalities in Lebesgue or Lorentz spaces involving the Lebesgue space L^1 can be generalized to the space $\mathcal{M}$ of finite, signed Radon measures (recall that $\mathcal{M}$ is equipped with the total variation norm $\|\mu\|_{\mathcal{M}} = |\mu|(\mathbb{R}^d)$).

Proposition 3.8. *If $1 < p < \infty$, $1 \le q \le \infty$, then*

$$\|f_1 * f_2\|_{L^1} \lesssim \|f_1\|_{L^1} \|f_2\|_{\mathcal{M}} ,$$
$$\|f_1 * f_2\|_{L^\infty} \lesssim \|f_1\|_{L^\infty} \|f_2\|_{\mathcal{M}} ,$$
$$\|f_1 * f_2\|_{L^{p,q}} \lesssim \|f_1\|_{L^{p,q}} \|f_2\|_{\mathcal{M}} ,$$

provided that the right-hand sides are finite.

Proof. If $f_1 \in L^1$ and $f_2 \in \mathcal{M}$, then $(f_1 * f_2)(x)$ is defined almost everywhere and defines an element of L^1. Indeed $\iint |f_1(x-y)| \mathrm{d}x \, \mathrm{d}|f_2|(y) = \int \|f_1\|_{L^1} \mathrm{d}|f_2|(y) = \|f_1\|_{L^1} |f_2|(\mathbb{R}^d) < \infty$, hence the Fubini theorem applies and

$$\iint |f_1(x-y)| \mathrm{d}|f_2|(y) \, \mathrm{d}x < \infty.$$

This implies that $y \mapsto |f_1(x-y)|$ is $|f_2|$-integrable for almost every x. We can thus define the function

$$(f_1 * f_2)(x) = \int f_1(x-y) \, \mathrm{d}f_2(y) \tag{3.3}$$

almost everywhere. Using again the Fubini theorem, this function is in L^1.

Actually (3.3) also makes sense when f_1 is in L^∞, and the inequality $\|f_1 * f_2\|_{L^\infty} \le \|f_1\|_{L^\infty} \|f_2\|_{\mathcal{M}}$ holds. Moreover if f_1 is in $L^\infty \cap L^1$ both definitions of $f_1 * f_2$ coincide. Hence, for f_2 in $\mathcal{M}$, the operator Φ_{f_2} defined on $L^1 + L^\infty$ by $\Phi_{f_2}(f_1) = f_1 * f_2$ is well defined, continuous from L^1 to L^1 and from L^∞ to L^∞ and thus, by real interpolation, it defines a continous operator from $L^{p,q}$ to $L^{p,q}$ for $1 < p < \infty$ and $1 \le q \le \infty$, with an operator norm smaller than $C_{p,q} \|f\|_{\mathcal{M}}$. This implies the stated Young inequality. □

To conclude this subsection we recall a result about derivatives of convolution products:

Proposition 3.9 (Young inequality in Bessel-Sobolev spaces). *If $0 \le s < \infty$, $1 \le p_j \le \infty$ for j in $\{0, 1, 2\}$, and $1 + \frac{1}{p_0} = \frac{1}{p_1} + \frac{1}{p_2}$ then*

$$\|f_1 * f_2\|_{H^{s,p_0}} \lesssim \|f_1\|_{\mathcal{M}} \|f_2\|_{H^{s,p_0}} \tag{3.4}$$
$$\|f_1 * f_2\|_{H^{s,p_0}} \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{H^{s,p_2}} \tag{3.5}$$

provided that the right-hand sides are finite.

If, moreover, $1 < p_j < \infty$ for j in $\{0, 1, 2\}$, then

$$\|f_1 * f_2\|_{H^{s,p_0}} \lesssim \|f_1\|_{L^{p_1,\infty}} \|f_2\|_{H^{s,p_2}} . \tag{3.6}$$

Proof. Using the definition of Bessel-Sobolev spaces, the property that the Fourier transform of a convolution product is a constant times the product of the Fourier transforms, we deduce that

$$\begin{aligned}\|f_1 * f_2\|_{H^{s,p_0}} &= \|\mathcal{F}^{-1}\langle\xi\rangle^s \mathcal{F}(f_1 * f_2)\|_{L^{p_0}} \\ &= (2\pi)^{\frac{d}{2}}\|\mathcal{F}^{-1}(\mathcal{F}(f_1)\langle\xi\rangle^s\mathcal{F}(f_2))\|_{L^{p_0}} \\ &= \|f_1 * (\langle\nabla\rangle^s f_2)\|_{L^{p_0}}.\end{aligned}$$

It just remains to apply the Young inequalities in Propositions 3.7 and 3.8 to obtain the results stated above. □

3.3 Multilinear estimates

To handle the non-linearity in the pseudo-relativistic Hartree equation (1.1), we will use several times the following estimates.

Lemma 3.10 (Multilinear estimates I). *Assume $s \geq 0$, $p > 2$ and $1 < q \leq \infty$. Let $w_1 \in \mathcal{M}$ and $w_q \in L^{q,m}$ with $m \in \{q, \infty\}$. For $u_j \in H^s \cap L^\sigma$ (or $u_j \in H^s \cap L^{\sigma,\alpha}$ for the last inequality), $j \in \mathbb{Z}_3 = \mathbb{Z}/(3\mathbb{Z})$, the following inequalities hold:*

$$\|(w_1 * (u_0u_1))u_2\|_{H^s} \lesssim \|w_1\|_{\mathcal{M}} \sum_{j\in\mathbb{Z}_3} \|u_j\|_{H^s}\|u_{j+1}\|_{L^\sigma}\|u_{j+2}\|_{L^\sigma} \text{ with } \sigma = \infty\,,\ s > \frac{d}{2}\,, \tag{3.7}$$

$$\|(w_1 * (u_0u_1))u_2\|_{H^{s,p'}} \lesssim \|w_1\|_{\mathcal{M}} \sum_{j\in\mathbb{Z}_3} \|u_j\|_{H^s}\|u_{j+1}\|_{L^\sigma}\|u_{j+2}\|_{L^\sigma} \text{ with } \frac{1}{p'} = \frac{1}{2} + \frac{2}{\sigma}\,, \tag{3.8}$$

$$\|(w_q * (u_0u_1))u_2\|_{H^{s,p'}} \lesssim \|w_q\|_{L^{q,\infty}} \sum_{j\in\mathbb{Z}_3} \|u_j\|_{H^s}\|u_{j+1}\|_{L^\sigma}\|u_{j+2}\|_{L^\sigma} \text{ with } \frac{1}{p'} + \frac{1}{2} = \frac{1}{q} + \frac{2}{\sigma}, \tag{3.9}$$

$$\|(w_q * (u_0u_1))u_2\|_{H^s} \lesssim \|w_q\|_{L^{q,m}} \sum_{j\in\mathbb{Z}_3} \|u_j\|_{H^s}\|u_{j+1}\|_{L^{\sigma,\alpha}}\|u_{j+2}\|_{L^{\sigma,\alpha}} \text{ with } \begin{cases} 1 = \frac{1}{q} + \frac{2}{\sigma} \\ 1 = \frac{1}{m} + \frac{2}{\alpha}. \end{cases} \tag{3.10}$$

Proof. Let us first remark that if u is in H^s with $s > d/2$, then u is in $\mathcal{C}^0_\infty$ which is the closure of $\mathcal{S}$ for the L^∞ norm. This will be important for our applications of the fractional Leibniz rule with L^∞. For u_j, u_{j+1} in H^s, as H^s is an algebra for $s > d/2$, u_ju_{j+1} is also in H^s and (3.4) shows that $w_1 * (u_ju_{j+1})$ lies in H^s which, again, is included in $\mathcal{C}^0_\infty$. This allows us to apply the fractional Leibniz rule with L^∞ below.

The fractional Leibniz rule, along with Young and Hölder's inequalities, yield both (3.7) and (3.8) by considering $p \geq 2$:

$$\begin{aligned}\|(w_1 * (u_0u_1))\,u_2\|_{H^{s,p'}} &\leq \|w_1 * (u_0u_1)\|_{H^{s,\frac{1}{\frac{1}{\sigma}+\frac{1}{2}}}}\|u_2\|_{L^\sigma} + \|w_1 * (u_0u_1)\|_{L^{\frac{\sigma}{2}}}\|u_2\|_{H^s} \\ &\leq \|w_1\|_{\mathcal{M}}\big(\|u_0u_1\|_{H^{s,\frac{1}{\frac{1}{\sigma}+\frac{1}{2}}}}\|u_2\|_{L^\sigma} + \|u_0u_1\|_{L^{\frac{\sigma}{2}}}\|u_2\|_{H^s}\big) \\ &\leq \|w_1\|_{\mathcal{M}} \sum_{j\in\mathbb{Z}_3}\|u_j\|_{H^s}\|u_{j+1}\|_{L^\sigma}\|u_{j+2}\|_{L^\sigma}\,.\end{aligned}$$

For (3.9), the same inequalities as above yield

$$\begin{aligned}\|(w_q*(u_0u_1))\,u_2\|_{H^{s,p'}} &\le \|w_q*(u_0u_1)\|_{H^{s,\frac{1}{\frac{1}{\sigma}+\frac{1}{q}-\frac{1}{2}}}}\|u_2\|_{L^\sigma}+\|w_q*(u_0u_1)\|_{L^{\frac{1}{\frac{2}{\sigma}+\frac{1}{q}-1}}}\|u_2\|_{H^s}\\ &\le \|w_q\|_{L^{q,\infty}}\big(\|u_0u_1\|_{H^{s,\frac{1}{\frac{1}{\sigma}+\frac{1}{2}}}}\|u_2\|_{L^\sigma}+\|u_0u_1\|_{L^{\frac{\sigma}{2}}}\|u_2\|_{H^s}\big)\\ &\le \|w_q\|_{L^{q,\infty}}\sum_{j\in\mathbb{Z}_3}\|u_j\|_{H^s}\|u_{j+1}\|_{L^\sigma}\|u_{j+2}\|_{L^\sigma}\,.\end{aligned}$$

For (3.10), the fractional Leibniz rule, along with Young and Hölder's inequalities in Lorentz spaces, yield

$$\begin{aligned}\|(w_q*(u_0u_1))\,u_2\|_{H^s} &\le \|w_q*(u_0u_1)\|_{H^{s,\frac{1}{\frac{1}{\sigma}+\frac{1}{q}-\frac{1}{2}}}}\|u_2\|_{L^\sigma}+\|w_q*(u_0u_1)\|_{L^\infty}\|u_2\|_{H^s}\\ &\le \|w_q\|_{L^{q,m}}\big(\|u_0u_1\|_{H^{s,\frac{1}{\frac{1}{\sigma}+\frac{1}{2}}}}\|u_2\|_{L^\sigma}+\|u_0u_1\|_{L^{\frac{\sigma}{2},\frac{\alpha}{2}}}\|u_2\|_{H^s}\big)\\ &\le \|w_q\|_{L^{q,m}}\sum_{j\in\mathbb{Z}_3}\|u_j\|_{H^s}\|u_{j+1}\|_{L^{\sigma,\alpha}}\|u_{j+2}\|_{L^{\sigma,\alpha}}\,.\end{aligned}$$

This concludes the proof. □

We will also need the following estimates in weighted Sobolev spaces.

Lemma 3.11 (Weighted multilinear estimates I)**.** *Assume* $0\le\gamma\le 2$, $s\ge 0$ *and* $1\le q<\infty$. *Let* $w_1\in\mathcal{M}$ *and* $w_q\in L^q$. *For* $u\in H^s_\gamma\cap L^\sigma_\gamma$ *it holds*

$$\|(w_1*|u|^2)u\|_{L^2_\gamma}\lesssim\|w_1\|_{\mathcal{M}}\|u\|^2_{L^\infty}\|u\|_{L^2_\gamma},\tag{3.11}$$
$$\|(w_q*|u|^2)u\|_{L^2_\gamma}\lesssim\|w_q\|_{L^q}\|u\|^2_{L^{2q'}}\|u\|_{L^2_\gamma},\tag{3.12}$$

as well as

$$\|(w_1*|u|^2)u\|_{H^s_\gamma}\lesssim\|w_1\|_{\mathcal{M}}\|u\|_{L^\infty}(\|u\|_{L^\infty}\|u\|_{H^s_\gamma}+\|u\|_{H^s}\|u\|_{L^\infty_\gamma})\quad\textit{if}\quad s>\frac{d}{2},\tag{3.13}$$
$$\|(w_q*|u|^2)u\|_{H^s_\gamma}\lesssim\|w_q\|_{L^q}\|u\|_{L^{2q'}}(\|u\|_{L^{2q'}}\|u\|_{H^s_\gamma}+\|u\|_{H^s}\|u\|_{L^{2q'}_\gamma}).\tag{3.14}$$

Proof. The first two inequalities are a direct consequence of Hölder and Young inequalities. Applying the weighted fractional Leibniz rule as well as Young inequality we have

$$\begin{aligned}\|(w_1*|u|^2)u\|_{H^s_\gamma}&\lesssim\|w_1*|u|^2\|_{L^\infty}\|u\|_{H^s_\gamma}+\|w_1*|u|^2\|_{H^s}\|u\|_{L^\infty_\gamma}\\ &\lesssim\|w_1\|_{\mathcal{M}}\|u\|_{L^\infty}(\|u\|_{L^\infty}\|u\|_{H^s_\gamma}+\|u\|_{H^s}\|u\|_{L^\infty_\gamma}),\end{aligned}$$

which proves the third inequality. We prove similarly the last inequality. Let $\sigma\ge 2$ such that $1=\frac{1}{q}+\frac{2}{\sigma}$, we write

$$\begin{aligned}\|(w_q*|u|^2)u\|_{H^s_\gamma}&\lesssim\|w_q*|u|^2\|_{L^\infty}\|u\|_{H^s_\gamma}\\ &\quad+\|w_q*(|u|^2)\|_{H^{s,\frac{1}{\frac{1}{\sigma}+\frac{1}{q}-\frac{1}{2}}}}\|u\|_{L^\sigma_\gamma}\\ &\lesssim\|w_q\|_{L^q}(\|u\|^2_{L^\sigma}\|u\|_{H^s_\gamma}+\||u|^2\|_{H^{s,\frac{1}{\frac{1}{\sigma}+\frac{1}{2}}}}\|u\|_{L^\sigma_\gamma})\\ &\lesssim\|w_q\|_{L^q}\|u\|_{L^\sigma}(\|u\|_{L^\sigma}\|u\|_{H^s_\gamma}+\|u\|_{H^s}\|u\|_{L^\sigma_\gamma}).\end{aligned}$$

This concludes the proof. □

In order to study the global existence and asymptotic properties of solutions to (1.1) in the case where w is short-range (see Sections 6 and 8), we will need slightly refined versions of the previous estimates that we state in the following two lemmas.

Lemma 3.12 (Multilinear estimates II)**.** *Assume $s \geq 0$, $p \geq 2$, $q, r \geq 1$ such that $q \leq 2r$, $\frac{2}{p} + \frac{1}{r} = 1$, $q \neq \infty$ and $w_1 \in \mathcal{M}$, $w_q \in L^q$. For $u_j \in H^s \cap L^\infty$, $j \in \mathbb{Z}_3 = \mathbb{Z}/(3\mathbb{Z})$, the inequalities*

$$\begin{aligned}\|(w_q * (u_0 u_1))u_2\|_{H^s} \lesssim \|w_q\|_{L^q} \sum_{j\in\mathbb{Z}_3} &\|u_j\|_{H^s} \|u_{j+1}\|_{H^s}^{1-\theta(\frac{r}{q})} \|u_{j+2}\|_{H^s}^{1-\theta(\frac{r}{q})} \\ &\|u_{j+1}\|_{L^p\cap L^\infty}^{\theta(\frac{r}{q})} \|u_{j+2}\|_{L^p\cap L^\infty}^{\theta(\frac{r}{q})}, \end{aligned} \tag{3.15}$$

$$\begin{aligned}\|(w_1 * (u_0 u_1))u_2\|_{H^{s,p'}} \lesssim \|w_1\|_{\mathcal{M}} \sum_{j\in\mathbb{Z}_3} &\|u_j\|_{H^s} \|u_{j+1}\|_{H^s}^{1-\theta(r-\frac{1}{2})} \|u_{j+2}\|_{H^s}^{1-\theta(r-\frac{1}{2})} \\ &\|u_{j+1}\|_{L^p\cap L^\infty}^{\theta(r-\frac{1}{2})} \|u_{j+2}\|_{L^p\cap L^\infty}^{\theta(r-\frac{1}{2})}, \end{aligned} \tag{3.16}$$

$$\begin{aligned}\|(w_q * (u_0 u_1))u_2\|_{H^{s,p'}} \lesssim \|w_q\|_{L^{q,\infty}} \sum_{j\in\mathbb{Z}_3} &\|u_j\|_{H^s} \|u_{j+1}\|_{H^s}^{1-\theta(\frac{r}{q}-\frac{1}{2})} \|u_{j+2}\|_{H^s}^{1-\theta(\frac{r}{q}-\frac{1}{2})} \\ &\|u_{j+1}\|_{L^p\cap L^\infty}^{\theta(\frac{r}{q}-\frac{1}{2})} \|u_{j+2}\|_{L^p\cap L^\infty}^{\theta(\frac{r}{q}-\frac{1}{2})} \end{aligned} \tag{3.17}$$

hold, with $\theta(x) = \min\{x, 1\}$.

Proof. Eq. (3.15) follows from (3.10) with $m = q$. This implies $\alpha = \sigma$, where σ is defined by $1 - \frac{1}{q} = \frac{2}{\sigma}$. Hence,

- if $r \leq q$, then $2 \leq \sigma \leq p$ and we set $\frac{1}{\sigma} = \frac{\theta}{p} + \frac{1-\theta}{2} \Leftrightarrow \theta = \frac{r}{q}$, so that $\|u\|_{L^\sigma} \leq \|u\|_{L^p}^{\theta} \|u\|_{L^2}^{1-\theta}$,
- else $r > q$, and then $p < \sigma$ and $\|u\|_{L^\sigma} \leq \|u\|_{L^p\cap L^\infty}$.

Eq. (3.16), follows from (3.8) where σ is defined through $1 + \frac{1}{p'} = 1 + \frac{1}{2} + \frac{2}{\sigma}$ observing that

- If $r \leq \frac{3}{2}$ then $2 \leq \sigma \leq p$ and we set $\theta = r - \frac{1}{2}$, which gives $\|u\|_{L^\sigma} \leq \|u\|_{L^p}^{\theta} \|u\|_{L^2}^{1-\theta}$,
- else $3/2 < r$ and then $p \leq \sigma$ and $\|u\|_{L^\sigma} \leq \|u\|_{L^p\cap L^\infty}$.

Finally, (3.17), follows from (3.9), with $1 + \frac{1}{p'} - \frac{1}{2} - \frac{1}{q} = \frac{2}{\sigma}$, as

- if $r \leq \frac{3}{2}q$, then $2 \leq \sigma \leq p$ and we set $\theta = \frac{r}{q} - \frac{1}{2}$, so that $\|u\|_{L^\sigma} \leq \|u\|_{L^p}^{\theta} \|u\|_{L^2}^{1-\theta}$,
- else $\frac{3}{2}q < r$, and then $p \leq \sigma$ and $\|u\|_{L^\sigma} \leq \|u\|_{L^p\cap L^\infty}$. □

Lemma 3.13 (Weighted multilinear estimates II)**.** *Assume $s \geq 0$, $0 \leq \gamma \leq 2$, $p \geq 2$, $q, r \geq 1$, such that $q \leq 2r$, $\frac{2}{p} + \frac{1}{r} = 1$ and $w_1 \in \mathcal{M}$, $w_q \in L^q$. For $u \in H^s_\gamma$, it holds*

$$\|(w * |u|^2)u\|_{L^2_\gamma} \lesssim \|w\|_{\mathcal{M}+L^q} \|u\|_{L^2_\gamma} \|u\|_{L^p\cap L^\infty}^{2\theta(\frac{r}{q})} (\|u\|_{L^2}^{2-2\theta(\frac{r}{q})} + \|u\|_{L^\infty}^{2-2\theta(\frac{r}{q})}), \tag{3.18}$$

where $w = w_1 + w_q$ and

$$\begin{aligned}\|(w_q * |u|^2)u\|_{H^s_\gamma} \lesssim &\|w_q\|_{L^q} \|u\|_{H^s}^{2-2\theta(\frac{r}{q})} \|u\|_{L^p\cap L^\infty}^{\theta(\frac{r}{q})} \\ &(\|u\|_{L^p\cap L^\infty}^{\theta(\frac{r}{q})} \|u\|_{H^s_\gamma} + \|u\|_{H^s}^{\theta(\frac{r}{q})} \|u\|_{L^{2q'}_\gamma}), \end{aligned} \tag{3.19}$$

with $\theta(x) = \min\{x, 1\}$.

Proof. To prove the inequalities we reason analogously to the proof of Lemma 3.12, by using the bound $\|u\|_{L^\sigma} \leq \|u\|_{L^2}^{1-\theta(\frac{r}{q})} \|u\|_{L^p \cap L^\infty}^{\theta(\frac{r}{q})}$ with $\sigma = 2q'$, as in the proof of (3.15). Applying this to (3.11)-(3.12) or (3.14) we obtain, respectively, (3.18) and (3.19). □

4 Local existence

In this section we prove the local existence results stated in Theorems 2.2 and 2.3. We actually establish more precise and more general versions of these theorems, stated in Theorems 4.1 and 4.2, respectively. The proofs rely on a standard fixed point argument in $L^\infty_t H^s_x$ for the first result, and in addition the Strichartz estimates of Proposition B.5 for the second result. We give some details for the sake of completeness.

For $s \geq 0$ and $T > 0$, we say that ψ is a solution to (1.1) on the time interval $[0, T]$ if ψ belongs to $\mathcal{C}^0([0,T], H^s) \cap \mathcal{C}^1([0,T], H^{s-1})$ and satisfies (1.1) in H^{s-1}. In particular, a function ψ in $\mathcal{C}^0([0,T], H^s) \cap \mathcal{C}^1([0,T], H^{s-1})$ is a solution to (1.1) if and only if it satisfies $\partial_t\big(e^{it\langle\nabla\rangle}\psi_t\big) = -ie^{it\langle\nabla\rangle}\big((w * |\psi_t|^2)\psi_t\big)$ in H^{s-1} and thus if and only if it satisfies the Duhamel equation

$$\psi_t = e^{-it\langle\nabla\rangle}\psi_0 - i\int_0^t e^{i(\tau-t)\langle\nabla\rangle}(w * |\psi_\tau|^2)\psi_\tau \, \mathrm{d}\tau \tag{4.1}$$

in $\mathcal{C}^0([0,T], H^s)$.

We introduce the notation $X_T := \mathcal{C}^0([0,T], H^s)$, endowed with the norm

$$\|\psi\|_{X_T} := \sup_{0\leq t\leq T} \|\psi_t\|_{H^s}.$$

Moreover, for $w \in \mathcal{W}_{d,s}$ (see (2.1)) we set

$$\|w\| := \begin{cases} \|w\|_{L^{\frac{d}{2s},\infty}+L^\infty} & \text{if } s < \frac{d}{2}, \\ \inf_{q>1: w\in L^q+L^\infty} \|w\|_{L^q+L^\infty} & \text{if } s = \frac{d}{2}, \\ \|w\|_{\mathcal{M}+L^\infty} & \text{if } s > \frac{d}{2}. \end{cases} \tag{4.2}$$

The next result readily implies Theorem 2.2.

Theorem 4.1. *Let $s \geq 0$ and w in $\mathcal{W}_{d,s}$. For every initial datum ψ_0 in H^s, there exists $T_{\max}$ in $(0,\infty]$ such that the Duhamel equation (4.1) associated with (1.1) admits a solution ψ in $\mathcal{C}^0([0,T_{\max}), H^s) \cap \mathcal{C}^1([0,T_{\max}), H^{s-1})$ where either $T_{\max} = \infty$ or $\lim_{t\to T_{\max}} \|\psi_t\|_{H^s} = \infty$.*

If $T > 0$ and ψ, $\tilde\psi$ are two solutions in X_T of (4.1) with initial data ψ_0 and $\tilde\psi_0$ in H^s, then, for some $C_d > 0$,

$$\|\psi - \tilde\psi\|_{X_T} \leq \|\psi_0 - \tilde\psi_0\|_{H^s} \exp\big(C_d\|w\|(\|\psi\|_{X_T} + \|\tilde\psi\|_{X_T})^2\big). \tag{4.3}$$

In particular, for any $0 < T < T_{\max}$, the solution ψ to (1.1) on $[0,T]$ associated to the initial datum ψ_0 in H^s is unique, and the map

$$H^s \ni \psi_0 \mapsto \psi \in \mathcal{C}^0([0,T], H^s) \cap \mathcal{C}^1([0,T], H^{s-1})$$

is continuous.

Proof. We first show the existence of a solution for short times T by a standard contraction argument. Let $T > 0$. The operator $\langle \nabla \rangle$ is selfadjoint with form domain $H^{1/2}$ and hence generates a strongly continuous unitary group $e^{-it\langle\nabla\rangle}$ acting on L^2. For ψ_0 in H^s and ψ in $\mathcal{C}^0([0,T), H^s)$ we write

$$\psi_t^{(0)} = e^{-it\langle\nabla\rangle}\psi_0\,, \quad I(t) = \int_0^t e^{i(\tau - t)\langle\nabla\rangle}(w * |\psi(\tau)|^2)\psi(\tau)\,\mathrm{d}\tau\,, \quad \mathcal{T}(\psi) = \psi^{(0)} - iI(\psi)\,.$$

The Duhamel formula (4.1) then reads $\psi = \mathcal{T}(\psi)$. The proof relies on usual contraction arguments.

We consider first the case $0 \le s < \frac{d}{2}$. We define σ_q by $1 = \frac{1}{q} + \frac{2}{\sigma_q}$. For $q \in \{\infty, \frac{d}{2s}\}$, as $\frac{1}{2} = \frac{s}{d} + \frac{1}{\sigma_{d/(2s)}}$, we get $\|u\|_{L^{\sigma_q,2}} \lesssim \|u\|_{H^s}$ and thus, using (3.10), we get, for u_j in H^s, $j \in \mathbb{Z}_3 = \mathbb{Z}/(3\mathbb{Z})$,

$$\begin{aligned} \|(w * (u_0 u_1))u_2\|_{H^s} &\le \sum_{q \in \{\infty, \frac{d}{2s}\}} \|w\|_{L^{q,\infty}} \sum_{j \in \mathbb{Z}_3} \|u_j\|_{H^s} \|u_{j+1}\|_{L^{\sigma_q,2}} \|u_{j+2}\|_{L^{\sigma_q,2}} \\ &\lesssim \|w\|_{L^{\frac{d}{2s},\infty} + L^\infty} \prod_{j \in \mathbb{Z}_3} \|u_j\|_{H^s}\,. \end{aligned} \tag{4.4}$$

Thus, for $R = \|\psi^{(0)}\|_{X_T} = \|\psi_0\|_{H^s}$ and ψ in $\overline{B(\psi^{(0)}, R)}$ in X_T, (4.4) yields

$$\begin{aligned} \|\mathcal{T}(\psi)_t - \psi_t^{(0)}\|_{H^s} = \|I(\psi)_t\|_{H^s} &\le \int_0^t \|(w * |\psi_\tau|^2)\psi_\tau\|_{H^s} \mathrm{d}\tau \\ &\lesssim T\,\|w\|_{L^{\frac{d}{2s},\infty} + L^\infty} \sup_{0 \le \tau \le T} \|\psi_\tau\|_{H^s}^3 \\ &\lesssim TR^3 \|w\|_{L^{\frac{d}{2s},\infty} + L^\infty}\,. \end{aligned}$$

Hence, for T sufficiently small, $\mathcal{T}$ sends $\overline{B(\psi^{(0)}, R)}$ into itself.

Consider ψ_1 and ψ_2 in $\overline{B(\psi^{(0)}, R)}$, then again by (4.4),

$$\begin{aligned} \|\mathcal{T}(\psi_1)_t - \mathcal{T}(\psi_2)_t\|_{H^s} &\le \|I(\psi_1)_t - I(\psi_2)_t\|_{H^s} \\ &\le \int_0^t \|(w * |\psi_{1,\tau}|^2)\psi_{1,\tau} - (w * |\psi_{2,\tau}|^2)\psi_{2,\tau}\|_{H^s} \mathrm{d}\tau \\ &\lesssim T\|w\|_{L^{\frac{d}{2s},\infty} + L^\infty} \|\psi_{1,\tau} - \psi_{2,\tau}\|_{H^s} (\|\psi_{1,\tau}\|_{H^s} + \|\psi_{2,\tau}\|_{H^s})^2 \\ &\lesssim TR^2 \|w\|_{L^{\frac{d}{2s},\infty} + L^\infty} \|\psi_{1,\tau} - \psi_{2,\tau}\|_{H^s}\,. \end{aligned}$$

Hence, for T sufficiently small, $\mathcal{T}$ is strictly contractive on $\overline{B(\psi^{(0)}, R)}$.

Similarly, for $s = \frac{d}{2}$ by assumption $w \in L^{q,\infty} + L^\infty, q > 1$. We write $q = \frac{d}{2v}$ for some $v < \frac{d}{2}$. By definition of v and σ_q (which we recall is such that $1 = \frac{1}{q} + \frac{2}{\sigma_q}$), for $q \in \{\infty, \frac{d}{2v}\}$ it holds $\|u\|_{L^{\sigma_q,2}} \lesssim \|u\|_{H^v} \lesssim \|u\|_{H^{\frac{d}{2}}}$. Hence applying (3.10) again for $q \in \{\infty, \frac{d}{2v}\}$, we have

$$\|(w * (u_0 u_1))u_2\|_{H^{\frac{d}{2}}} \lesssim \|w\|_{L^{\frac{d}{2v},\infty} + L^\infty} \prod_{j \in \mathbb{Z}_3} \|u_j\|_{H^{\frac{d}{2}}} \tag{4.5}$$

for some $v < \frac{d}{2}$. We can then repeat the arguments of the case $s < \frac{d}{2}$ to obtain the stability and contraction properties of $\mathcal{T}$, where $\|w\|_{L^{\frac{d}{2s},\infty} + L^\infty}$ will be replaced by $\|w\|_{L^{q,\infty} + L^\infty}, q = \frac{d}{2v}$.

Finally, for $s > d/2$, using (3.10), (3.7) and $\|u\|_{L^\infty} \lesssim \|u\|_{H^s}$ we obtain

$$\begin{aligned}
&\|(w * (u_0 u_1)) u_2\|_{H^s} \\
&\quad \leq \|w\|_{\mathcal{M}} \sum_{j\in\mathbb{Z}_3} \|u_j\|_{H^s} \|u_{j+1}\|_{L^\infty} \|u_{j+2}\|_{L^\infty} + \|w\|_{L^\infty} \sum_{j\in\mathbb{Z}_3} \|u_j\|_{H^s} \|u_{j+1}\|_{L^2} \|u_{j+2}\|_{L^2} \\
&\quad \lesssim \|w\|_{\mathcal{M}+L^\infty} \prod_{j\in\mathbb{Z}_3} \|u_j\|_{H^s} \,. \qquad (4.6)
\end{aligned}$$

The proofs of the stability and contraction properties then go along the same lines as for $0 \leq s < \frac{d}{2}$ but with $\|w\|_{L^{\frac{d}{2s},\infty}+L^\infty}$ replaced by $\|w\|_{\mathcal{M}+L^\infty}$.

By standard arguments a solution ψ of (4.1) can be extended to a solution (again denoted by ψ) in $\mathcal{C}^0([0, T_{\max}), H^s)$ with a "maximal" time of existence $T_{\max}$, such that either $T_{\max} = \infty$, or $\lim_{t\to T_{\max}} \|\psi_t\|_{H^s} = \infty$.

We now prove the uniqueness of the solution and the continuity with respect to the initial data. Consider two initial data ψ_0 and $\tilde{\psi}_0$ and corresponding solutions ψ and $\tilde{\psi}$ with maximal times $T_{\max}$ and $\tilde{T}_{\max}$, and $0 < T < \min\{T_{\max}, \tilde{T}_{\max}\}$. Then, using the Duhamel formula, (4.4) or (4.6),

$$\begin{aligned}
\|\psi_t - \tilde{\psi}_t\|_{H^s} &\lesssim \|\psi_0 - \tilde{\psi}_0\|_{H^s} + \|w\| \int_0^t (\|\psi_\tau\|_{H^s} + \|\tilde{\psi}_\tau\|_{H^s})^2 \|\psi_\tau - \tilde{\psi}_\tau\|_{H^s} \mathrm{d}\tau \\
&\lesssim \|\psi_0 - \tilde{\psi}_0\|_{H^s} + \|w\| (\|\psi\|_{X_T} + \|\tilde{\psi}\|_{X_T})^2 \int_0^t \|\psi_\tau - \tilde{\psi}_\tau\|_{H^s} \mathrm{d}\tau \,.
\end{aligned}$$

Grönwall's inequality then yields for t in $[0, T]$:

$$\|\psi_t - \tilde{\psi}_t\|_{H^s} \leq \|\psi_0 - \tilde{\psi}_0\|_{H^s} \exp\left(C_d \|w\| (\|\psi\|_{X_T} + \|\tilde{\psi}\|_{X_T})^2\right).$$

Hence (4.3) holds for any $0 < T < \min\{T_{\max}, \tilde{T}_{\max}\}$.

Then it makes sense to define the maximal time of existence $T_{\max}$ of the solution with a given initial datum.

Finally we show that ψ belongs to $\mathcal{C}^1([0, T_{\max}), H^{s-1})$. We first remark that for any $f \in H^s$ we have

$$\partial_t (e^{-i\langle\nabla\rangle t} f) = -i e^{-i\langle\nabla\rangle t} \langle\nabla\rangle f$$

is continuous and bounded as a function of t with values in H^{s-1}, as $\langle\nabla\rangle f \in H^{s-1}$ and $t \mapsto e^{-i\langle\nabla\rangle t}$ is strongly continuous and bounded on H^{s-1}. We have then found $e^{-i\langle\nabla\rangle t} f \in \mathcal{C}^1([0, T), H^{s-1})$. By assumption $\psi_0 \in H^s$ and therefore $e^{-i\langle\nabla\rangle t}\psi_0 \in \mathcal{C}^1([0, T], H^{s-1})$. Similarly, we know that $\psi \in \mathcal{C}^0([0, T], H^s)$, hence $(w * |\psi_\tau|^2)\psi_\tau \in H^s$ uniformly in $\tau \in [0, T]$ for any $T < T_{\max}$ by (4.4), (4.5) or (4.6), and as $H^s \subset H^{s-1}$ and

$$\partial_t \int_0^t e^{i(\tau-t)\langle\nabla\rangle} (w * |\psi_\tau|^2) \psi_\tau \, \mathrm{d}\tau = (w * |\psi_t|^2)\psi_t + \int_0^t \partial_t \big(e^{i(\tau-t)\langle\nabla\rangle} (w * |\psi_\tau|^2) \psi_\tau \big) \mathrm{d}\tau \in H^{s-1}$$

we get that $I(\psi) \in \mathcal{C}^1([0, T_{\max}), H^{s-1})$. We conclude that both terms in (4.1) are then in $\mathcal{C}^1([0, T_{\max}), H^{s-1})$ and therefore ψ belongs to $\mathcal{C}^1([0, T_{\max}), H^{s-1})$ as claimed. □

Now we turn to the proof of Theorem 2.3. The argument is similar to that used in the previous proof, using in addition the Strichartz estimates given in Proposition B.5. We have the following result involving any admissible pair (a, b) for the Strichartz estimates (either wave admissible or Schrödinger admissible). For $s < (d+1)/(2d+2)$, it implies Theorem 2.3 by taking $\beta = 0$, $\frac{1}{a} = \frac{(d-1)s}{d+1}$ and $\frac{1}{b} = \frac{1}{2} - \frac{2s}{d+1}$, and likewise for $s \geq (d+1)/(2d+2)$.

Theorem 4.2. *Let $0 \leq \beta \leq 1$ and (a, b) be an admissible pair for the Strichartz estimates of Proposition B.5 and $s \geq \frac{1}{2} + \frac{1}{a} - \frac{1}{b}$. Let $w \in L^{b/(b-2),\infty} + L^\infty$. For every initial datum ψ_0 in H^s there exists $T_{\max}$ in $(0, \infty]$ such that, for all $0 < T < T_{\max}$, the Duhamel equation (4.1) associated with (1.1) admits a unique solution*

$$\psi \in \mathcal{C}^0([0, T], H^s) \cap \mathcal{C}^1([0, T], H^{s-1}) \cap L^a([0, T], L^{b,2}),$$

where either $T_{\max} = \infty$ or $\lim_{t\to T_{\max}} \|\psi_t\|_{H^s} + \|\psi\|_{L^a([0,T_{\max}),L^{b,2})} = \infty$. Moreover, for any $0 < T < T_{\max}$, the solution ψ to (1.1) on $[0, T]$ associated to the initial datum ψ_0 in H^s is unique, and the map

$$H^s \ni \psi_0 \mapsto \psi \in \mathcal{C}^0([0, T], H^s) \cap \mathcal{C}^1([0, T], H^{s-1}) \cap L^a([0, T], L^{b,2})$$

is continuous.

Proof. As in the proof of the previous theorem, it suffices to solve the Duhamel equation (4.1) using a fixed point argument. More precisely one verifies that the map $\mathcal{T}$ defined in the proof of Theorem 4.1 is a contraction in a suitable ball contained in the space

$$Y_T := \mathcal{C}^0([0, T], H^s) \cap L^a([0, T], L^{b,2})$$

endowed with the norm

$$\|\psi\|_{Y_T} := \sup_{0\leq t\leq T} \|\psi_t\|_{H_s} + \Big[\int_0^T \|\psi_\tau\|^a_{L^{b,2}} d\tau \Big]^{1/a}.$$

Using the Strichartz estimates from Proposition B.5, we see that (with the notations used in the proof of Theorem 4.1)

$$\|\mathcal{T}(\psi) - \psi^{(0)}\|_{L^a([0,T],L^{b,2})} \lesssim \int_0^T \|(w * |\psi_\tau|^2)\psi_\tau\|_{H^s} \mathrm{d}\tau.$$

We can then proceed exactly as in the proof of Theorem 4.1. The only difference is the estimate of the part $w_{b/(b-2)}$ of w belonging to $L^{b/(b-2),\infty}$. Using Hölder and Young inequalities in Lorentz spaces recalled in Section 3.2, it can be handled as follows:

$$\int_0^T \|(w_{b/(b-2)} * |\psi_\tau|^2)\psi_\tau\|_{H^s} \mathrm{d}\tau \lesssim \|w_{b/(b-2)}\|_{L^{b/(b-2),\infty}} \int_0^T \|\psi_\tau\|_{H^s} \|\psi_\tau\|^2_{L^{b,2}} \mathrm{d}\tau \lesssim T^{1-\frac{2}{a}} \|\psi\|^3_{Y_T}. \tag{4.7}$$

The contraction property of $\mathcal{T}$ can be proven similarly. The rest of the proof follows the arguments used at the end of the proof of Theorem 4.1. □

5 Global existence for long-range potentials

In this section we prove Theorems 2.7 and 2.9. To do so, we use the conservation of the mass and energy (see (1.2) and (1.3) for the definitions).

The proof of the next lemma is analogous to that in [45, Lemma 2]. We do not reproduce it here.

Lemma 5.1. *Let $s \geq 0$, w as in Theorem 4.1 or 4.2, and let ψ be a local solution to* (1.1) *given by Theorem 4.1 or 4.2. Then, for all $t \in [0, T_{\max})$ it holds*

$$M(\psi_t) = M(\psi_0).$$

Moreover, if $s \geq \frac{1}{2}$, for all $t \in [0, T_{\max})$ it holds

$$E(\psi_t) = E(\psi_0).$$

Now we can prove the global existence of solutions to (1.1) stated in Theorem 2.7.

Proof of Theorem 2.7. Let $s \geq \frac{1}{2}$, $\psi_0 \in H^s$ and $w \in \mathcal{W}_{d,1/2} = L^{d,\infty} + L^\infty$. Since $\mathcal{W}_{d,1/2} \subset \mathcal{W}_{d,s}$, Theorem 4.1 shows that (1.1) admits a unique local solution $\psi \in \mathcal{C}^0([0, T_{\max}), H^s) \cap \mathcal{C}^1([0, T_{\max}), H^{s-1})$, for some $T_{\max} > 0$, and that either $T_{\max} = \infty$ or $\|\psi_t\|_{H^s} \to \infty$ as $t \to T_{\max}$. Therefore is suffices to show that $\|\psi_t\|_{H^s}$ is uniformly bounded for $t \in [0, T_{\max})$.

Since ψ_t is a local solution in H^s with $s \geq \frac{1}{2}$, we can apply Lemma 5.1. This yields

$$E(\psi_0) = E(\psi_\tau) \geq \frac{1}{2}\|\langle\nabla\rangle^{\frac{1}{2}}\psi_\tau\|_{L^2}^2 - \frac{1}{4}\int (w_- * |\psi_\tau|^2)|\psi_\tau|^2 \,. \tag{5.1}$$

Writing w_- in the form $w_- = (w_d)_- + (w_\infty)_- \in L^{d,\infty} + L^\infty$, we use the inequalities

$$\|(w_\infty)_- * |\psi_\tau|^2\|_{L^\infty} \leq \|(w_\infty)_-\|_{L^\infty}\|\psi_\tau\|_{L^2}^2 = \|(w_\infty)_-\|_{L^\infty}\|\psi_0\|_{L^2}^2$$

and

$$\begin{aligned}\|(w_d)_- * |\psi_\tau|^2\|_{L^\infty} &\leq \|(w_d)_-\|_{L^{d,\infty}}\||\psi_\tau|^2\|_{L^{\frac{1}{1-\frac{1}{d}},1}} = \|(w_d)_-\|_{L^{d,\infty}}\|\psi_\tau\|^2_{L^{\frac{2d}{d-1},2}} \\ &\leq C_S\|(w_d)_-\|_{L^{d,\infty}}\|\psi_\tau\|^2_{H^{\frac{1}{2}}}\end{aligned}$$

to bound the integral

$$\left|\int (w_- * |\psi_\tau|^2)|\psi_\tau|^2\right| \leq \|(w_\infty)_-\|_{L^\infty}\|\psi_0\|_{L^2}^4 + C_S\|(w_d)_-\|_{L^{d,\infty}}\|\psi_\tau\|^2_{H^{\frac{1}{2}}}\|\psi_0\|_{L^2}^2,$$

where C_S is the constant in the Sobolev embedding $H^{\frac{1}{2}} \hookrightarrow L^{\frac{2d}{d-1},2}$. Applying this bound in (5.1), we obtain

$$E(\psi_0) + \frac{1}{4}\|(w_\infty)_-\|_{L^\infty}\|\psi_0\|_{L^2}^4 \geq \frac{1}{2}\|\psi_\tau\|^2_{H^{\frac{1}{2}}}(1 - \frac{1}{2}C_S\|(w_d)_-\|_{L^{d,\infty}}\|\psi_0\|_{L^2}^2).$$

Hence, if $\|(w_d)_-\|_{L^{d,\infty}}\|\psi_0\|_{L^2}^2 < 2C_S^{-1}$, we have

$$\sup_{\tau\geq 0}\|\psi_\tau\|^2_{H^{\frac{1}{2}}} \lesssim E(\psi_0) + \frac{1}{4}\|(w_\infty)_-\|_{L^\infty}\|\psi_0\|_{L^2}^4. \tag{5.2}$$

Now, by Duhamel's formula (4.1) and the estimates as above with w instead of w_-, we have, for all $t \in [0, T]$ with $T < T_{\max}$

$$\|\psi_t\|_{H^s} \lesssim \|\psi_0\|_{H^s} + \|w\|_{L^{d,\infty}+L^\infty}\int_0^t \|\psi_\tau\|^2_{H^{\frac{1}{2}}}\|\psi_\tau\|_{H^s}\,\mathrm{d}\tau. \tag{5.3}$$

Together with (5.2), this yields

$$\|\psi_t\|_{H^s} \lesssim \|\psi_0\|_{H^s} + \|w\|_{L^{d,\infty}+L^\infty} \int_0^t \|\psi_\tau\|_{H^s}\, \mathrm{d}\tau,$$

and by Gronwall's inequality we obtain that for some $C > 0$ and all $t \in [0, T]$, it holds

$$\|\psi_t\|_{H^s} \lesssim \|\psi_0\|_{H^s} e^{Ct}.$$

Therefore, we have a uniform bound on $\|\psi_t\|_{H^s}$ on any arbitrary time interval $[0, T]$. As argued above, this implies that the solution ψ is globally defined. □

For small energy initial data, the class of admissible potentials can be extended using the Strichartz estimates of Proposition B.5. The next result implies Theorem 2.9 by taking $\beta = 0$ and $\frac{1}{a} = \frac{1}{b} = \frac{d-1}{2d+2}$.

Theorem 5.2. *Let $s = \frac{1}{2}$. Let $0 \le \beta \le 1$ and let (a, b) be an admissible pair for the Strichartz estimates of Proposition B.5 with $b \le a$. There exists a universal constant $C_0 > 0$ such that, for all w even of the form $w = w_{d/2} + w_\infty \in L^{b/(b-2),\infty} + L^\infty \subset L^{d/2} + L^\infty$ and $\psi_0 \in H^{\frac{1}{2}}$ verifying*

$$\|(w_{d/2})_-\|_{L^{\frac{d}{2}}} (E(\psi_0) + \|(w_\infty)_-\|_{L^\infty} \|\psi_0\|_{L^2}^4) < C_0, \tag{5.4}$$

and

$$\|(w_{d/2})_-\|_{L^{\frac{d}{2}}} \|\psi_0\|_{H^{\frac{1}{2}}}^2 < C_0, \tag{5.5}$$

Eq. (1.1) admits a unique solution

$$\psi_t \in \mathcal{C}^0([0,\infty), H^{\frac{1}{2}}) \cap \mathcal{C}^1([0,\infty), H^{-\frac{1}{2}}) \cap L^a_{\mathrm{loc}}([0,\infty), L^{b,2}).$$

Proof. Let $w \in L^{b/(b-2),\infty} + L^\infty$ and let ψ be a local solution to (1.1) on $[0, T_{\max})$ given by Theorem 4.2 (note that ψ exists since $\psi_0 \in H^{\frac{1}{2}}$ and $\frac{1}{2} \ge \frac{1}{2} + \frac{1}{a} - \frac{1}{b}$). By the blow-up alternative stated in Theorem 4.2, it suffices to show that $t \mapsto \|\psi_t\|_{H^{1/2}}$ is uniformly bounded on $[0, T_{\max})$ and that $t \mapsto \|\psi_t\|_{L^b}$ belongs to $L^a([0, T_{\max}))$.

Using the energy estimate (5.1) from the previous proof and Hölder's and Young's inequality, we obtain, for all $t \in [0, T_{\max})$,

$$\begin{aligned} \|\psi_t\|_{H^{\frac{1}{2}}}^2 &\le E(\psi_t) + \|(w_{d/2})_-\|_{L^{d/2}} \|\psi_t\|_{H^{\frac{1}{2}}}^4 + \|(w_\infty)_-\|_{L^\infty} \|\psi_t\|_{L^2}^4 \\ &= E(\psi_0) + \|(w_{d/2})_-\|_{L^{d/2}} \|\psi_t\|_{H^{\frac{1}{2}}}^4 + \|(w_\infty)_-\|_{L^\infty} \|\psi_0\|_{L^2}^4, \end{aligned}$$

where we used the conservation of the mass and energy (see Lemma 5.1) in the equality.

Setting

$$X_t := \|\psi_t\|_{H^{\frac{1}{2}}}^2, \quad A := \|(w_{d/2})_-\|_{L^{d/2}} \quad C := E(\psi_0) + \|(w_\infty)_-\|_{L^\infty} \|\psi_0\|_{L^2}^4,$$

the previous inequality gives that $AX_t^2 - X_t + C \ge 0$ for all $t \in [0, T_{\max})$. If $(w_{d/2})_- = 0$, then $\|\psi_t\|_{H^{\frac{1}{2}}}^2$ is uniformly bounded on $[0, T_{\max})$ by C. Otherwise, letting $\delta := 1 - 4AC$ and choosing $C_0 > 0$ small enough, (5.4) implies that $0 < \delta < 1$. Hence, for all $t \in [0, T]$, X_t belongs to $[0, (1 - \delta^{\frac{1}{2}})/(2A)] \cup [(1 + \delta^{\frac{1}{2}})/(2A), \infty)$. Now if C_0 is small enough, one easily verifies using (5.5) that $X_0 < (1-\delta^{\frac12})/(2A)$. By continuity of $t \mapsto X_t$, this implies that $X_t \in [0,(1-\delta^{\frac12})/(2A)]$ for all $t \in [0,T_{\max})$.

We have thus shown that $\|\psi_t\|_{H^{\frac12}}$ is uniformly bounded by $(2\|(w_{d/2})_-\|_{L^{d/2}})^{-1}$ on $[0,T_{\max})$ (or by C if $(w_{d/2})_- = 0$). To prove that $\|\psi_t\|_{L^{b,2}}$ belongs to $L^a([0,T_{\max}))$, we use again the Strichartz estimates from Proposition B.5 together with (4.7), obtaining for all $0<T<T_{\max}$,

$$\begin{aligned}\|\psi\|_{L^a([0,T],L^{b,2})} &\lesssim \|\psi_0\|_{H^{\frac12}} + \int_0^T \|(w*|\psi_\tau|^2)\psi_\tau\|_{H^{\frac12}}\mathrm{d}\tau \\ &\lesssim \|\psi_0\|_{H^{\frac12}} + \int_0^T \big(\|w_\infty\|_{L^\infty}\|\psi_0\|_{L^2}^2\|\psi_\tau\|_{H^{\frac12}} + \|w\|_{L^{b/(b-2),\infty}}\|\psi_\tau\|_{L^b}^2\|\psi_\tau\|_{H^{\frac12}}\big)\mathrm{d}\tau \\ &\lesssim C_1(w,M(\psi_0),E(\psi_0))(1+T) + C_2(w,M(\psi_0),E(\psi_0))T^{1-\frac2a}\|\psi\|^2_{L^a([0,T],L^{b,2})},\end{aligned}$$

for some positive constants $C_j(w,M(\psi_0),E(\psi_0))$ depending on w and the mass and energy of ψ_0. Fixing $T_0>0$ small enough (depending only again on w and $M(\psi_0)$, $E(\psi_0)$) and arguing as before, we deduce that

$$\|\psi\|_{L^a([0,T_0],L^{b,2})} \lesssim C_3(w,M(\psi_0),E(\psi_0)).$$

Choosing an integer n large enough so that $T_0 > \frac1n T_{\max} =: T_1$ yields

$$\|\psi\|^a_{L^a([0,T_{\max}),L^{b,2})} = \sum_{j=0}^{n-1}\|\psi\|^a_{L^a([jT_1,(j+1)T_1),L^{b,2})} < \infty,$$

which by the blowup alternative stated in Theorem 4.2, conludes the proof. □

6 Global existence and pointwise time-decay for short range potentials

In this section we prove the global well-posedness and time-decay properties of (1.1) stated in Theorem 2.10. We use the time-decay properties of the free flow associated to (1.1). This will allow us to consider interacting potentials $w \in \mathcal{M} + L^q$, $1 \le q < \frac{2d}{3}$ for $d \ge 3$.

In Appendix B (see Lemma B.3), we recall that the solutions of the linear equation corresponding to the free dynamics associated to (1.1) satisfy the following time-decay estimates for all $f \in H^{s,p'} \cap H^s$:

$$\|e^{-it\langle\nabla\rangle} f\|_{L^p\cap L^\infty} \lesssim \langle t\rangle^{-\frac{d}{2r}}\|f\|_{H^{s,p'}\cap H^s}, \tag{6.1}$$

for any $2 \le p \le \infty$, $s \ge \frac d2 + 1$, with $\frac1p + \frac1{p'} = 1$ and $\frac1p + \frac1{2r} = \frac12$.

Now we are ready to prove Theorem 2.10.

Proof of Theorem 2.10. We will construct a solution to the Duhamel equation

$$\psi_t = \psi_t^{(0)} - i\int_0^t e^{i(\tau-t)\langle\nabla\rangle}(w*|\psi_\tau|^2))\,\psi_\tau\,\mathrm{d}\tau\,. \tag{6.2}$$

To simplify the notations, given s,r,p as in the statement of the theorem, we set $S := S^{s,r,p}$ where $S^{s,r,p}$ was defined in (2.6).

Eq. (6.1) ensures that the free evolution $\psi_t^{(0)}$ is in S. For $\varphi_j \in S$, $j \in \mathbb{Z}_3 = \{0,1,2\}$, we set

$$I(\varphi_0,\varphi_1,\varphi_2)(t) = \int_0^t e^{i(\tau-t)\langle\nabla\rangle}(w*(\varphi_{0,\tau}\varphi_{1,\tau}))\,\varphi_{2,\tau}\,\mathrm{d}\tau\,.$$

We first show that I is well-defined on S^3, with values in S. It follows from (6.1) that

$$\|I(\varphi_0,\varphi_1,\varphi_2)\|_S \le \sup_{t>0}\int_0^t \|(w*(\varphi_{0,\tau}\varphi_{1,\tau}))\,\varphi_{2,\tau}\|_{H^s}\,\mathrm{d}\tau \\ + \sup_{t>0}\langle t\rangle^{\frac{d}{2r}}\int_0^t \langle t-\tau\rangle^{-\frac{d}{2r}}\|(w*(\varphi_{0,\tau}\varphi_{1,\tau}))\,\varphi_{2,\tau}\|_{H^s\cap H^{s,p'}}\,\mathrm{d}\tau\,. \tag{6.3}$$

Let $w_1 \in \mathcal{M}$ and $w_q \in L^q$ be such that $w = w_1 + w_q$. Using (3.7), we obtain

$$\int_0^t \|(w_1*(\varphi_{0,\tau}\varphi_{1,\tau}))\,\varphi_{2,\tau}\|_{H^s}\mathrm{d}\tau \lesssim \|w_1\|_{\mathcal{M}}\int_0^t \sum_{j\in\mathbb{Z}_3}\|\varphi_{j,\tau}\|_{H^s}\|\varphi_{j+1,\tau}\|_{L^\infty}\|\varphi_{j+2,\tau}\|_{L^\infty}\mathrm{d}\tau \\ \lesssim \|w_1\|_{\mathcal{M}}\prod_{j\in\mathbb{Z}_3}\|\varphi_j\|_S\int_0^t\langle\tau\rangle^{-\frac{d}{r}}\mathrm{d}\tau$$

and similarly, using (3.15) and the fact that $\|\varphi_{j,\tau}\|_{H^s}^{1-\theta(\frac{r}{q})}\|\varphi_{j,\tau}\|_{L^p\cap L^\infty}^{\theta(\frac{r}{q})} \lesssim \langle\tau\rangle^{-\frac{d}{2r}\theta(\frac{r}{q})}\|\varphi_j\|_S$ we have

$$\int_0^t \|(w_q*(\varphi_{0,\tau}\varphi_{1,\tau}))\,\varphi_{2,\tau}\|_{H^s}\mathrm{d}\tau \lesssim \|w_q\|_{L^q}\prod_{j\in\mathbb{Z}_3}\|\varphi_j\|_S\int_0^t\langle\tau\rangle^{-\frac{d}{r}\theta(\frac{r}{q})}\,\mathrm{d}\tau\,,$$

where we recall that $\theta(u) = \min\{1,u\}$. We can proceed in the same way to estimate the second term in the right-hand side of (6.3). Using the bound (3.16) twice, we obtain

$$\langle t\rangle^{\frac{d}{2r}}\int_0^t\langle t-\tau\rangle^{-\frac{d}{2r}}\|(w_1*(\varphi_{0,\tau}\varphi_{1,\tau}))\,\varphi_{2,\tau}\|_{H^s\cap H^{s,p'}}\mathrm{d}\tau \\ \lesssim \|w_1\|_{\mathcal{M}}\prod_{j\in\mathbb{Z}_3}\|\varphi_j\|_S\,\langle t\rangle^{\frac{d}{2r}}\int_0^t\langle t-\tau\rangle^{-\frac{d}{2r}}\left(\langle\tau\rangle^{-\frac{d}{r}}+\langle\tau\rangle^{-\frac{d}{r}\theta(r-\frac12)}\right)\,\mathrm{d}\tau,$$

while (3.15) and (3.17) lead to

$$\langle t\rangle^{\frac{d}{2r}}\int_0^t\|(w_q*(\varphi_{0,\tau}\varphi_{1,\tau}))\,\varphi_{2,\tau}\|_{H^s\cap H^{s,p'}}\mathrm{d}\tau \\ \lesssim \|w_q\|_{L^q}\prod_{j\in\mathbb{Z}_3}\|\varphi_j\|_S\,\langle t\rangle^{\frac{d}{2r}}\int_0^t\langle t-\tau\rangle^{-\frac{d}{2r}}\left(\langle\tau\rangle^{-\frac{d}{r}\theta(\frac{r}{q})}+\langle\tau\rangle^{-\frac{d}{r}\theta(\frac{r}{q}-\frac12)}\right)\,\mathrm{d}\tau\,.$$

We then observe that if $a \ge 0, b > 1$, then

$$\sup_{t>0}\langle t\rangle^a\int_0^t\langle t-\tau\rangle^{-a}\langle\tau\rangle^{-b}\,\mathrm{d}\tau < \infty\,, \tag{6.4}$$

which we can apply with $a = \frac{d}{2r}$ and $b \in \left\{\frac{d}{r}, \frac{d}{r}\theta(\frac{r}{q}), \frac{d}{r}\theta(\frac{r}{q}-\frac12), \frac{d}{r}\theta(r-\frac12)\right\}$. Indeed, since $r \ge 1$, we have

$$\min\left\{\frac{d}{r}, \frac{d}{r}\theta(\frac{r}{q}), \frac{d}{r}\theta(\frac{r}{q}-\frac12), \frac{d}{r}\theta(r-\frac12)\right\} > 1 \Leftrightarrow \min\left\{\frac1r, \frac1q, \frac1q-\frac1{2r}, 1-\frac1{2r}\right\} > \frac1d$$

$$\Leftrightarrow \min\left\{\frac{1}{r},\frac{1}{q},\frac{1}{q}-\frac{1}{2r}\right\} > \frac{1}{d}$$
$$\Leftrightarrow \frac{3}{2d} < \frac{1}{2r}+\frac{1}{d} < \frac{1}{q},$$

which is true by assumption. Therefore we have shown that I is well-defined on S^3, with values in S. Moreover, taking the infimum over all the decompositions $w = w_1 + w_q$ with $w_1 \in \mathcal{M}$ and $w_q \in L^q$, we deduce that

$$\|I(\varphi_0,\varphi_1,\varphi_2)\|_S \le C\|w\|_{\mathcal{M}+L^q} \prod_{j\in\mathbb{Z}_3} \|\varphi_j\|_S\,, \tag{6.5}$$

for some positive constant C.

Now we seek for a fixed point of the map $\mathcal{T} : S \to S$ defined by

$$\mathcal{T}(\psi) = \psi^{(0)} - iI(\psi,\bar{\psi},\psi).$$

We consider the restriction of $\mathcal{T}$ to the closed ball

$$\bar{B} := \overline{B(\psi^{(0)}, \|\psi^{(0)}\|_S)}.$$

The bound (6.5), for ψ in $\bar{B}$, gives

$$\|\mathcal{T}(\psi)-\psi^{(0)}\|_S = \|I(\psi,\bar{\psi},\psi)\|_S \le C\|w\|_{\mathcal{M}+L^q}\|\psi\|_S^3 \le C\|w\|_{\mathcal{M}+L^q} 8\|\psi^{(0)}\|_S^3\,.$$

It follows that if $\|w\|_{\mathcal{M}+L^q}\|\psi\|_S^2 \le \frac{1}{8C}$, then $\mathcal{T}$ sends $\bar{B}$ into itself.

Now we verify that $\mathcal{T}$ is a contraction on $\bar{B}$. Using again (6.5) for $\psi_1,\psi_2 \in \bar{B}$ yields

$$\begin{aligned}\|\mathcal{T}(\psi_1)-\mathcal{T}(\psi_2)\|_S &\le \|I(\psi_1,\bar{\psi}_1,\psi_1-\psi_2)\|_S + \|I(\psi_1,\bar{\psi}_1-\bar{\psi}_2,\psi_2)\|_S + \|I(\psi_1-\psi_2,\bar{\psi}_2,\psi_2)\|_S \\ &\le 3C\|w\|_{\mathcal{M}+L^q}\|\psi_1-\psi_2\|_S(\|\psi_1\|_S+\|\psi_2\|_S)^2 \\ &\le 48C\|w\|_{\mathcal{M}+L^q}\|\psi_1-\psi_2\|_S\|\psi^{(0)}\|_S^2.\end{aligned}$$

It is thus sufficient to have $\|w\|_{\mathcal{M}+L^q}\|\psi^{(0)}\|_S^2 \le \frac{1}{48C}$ in order for $\mathcal{T}$ to be a contraction on $\bar{B}$, this is true for $\|w\|_{\mathcal{M}+L^q}\|\psi_0\|^2_{H^s\cap H^{s,p'}}$ sufficiently small as the assumption requires.

We have thus proven that $\mathcal{T}$ has a fixed point in $\bar{B}$, which gives the existence of a solution ψ to Duhamel's equation (6.2). The fact that this solution ψ belongs to $\mathcal{C}^0([0,\infty),H^s) \cap \mathcal{C}^1([0,\infty),H^{s-1})$ follows from the previous estimates and standard arguments. Inequalities (2.8)–(2.11) are also direct consequences of the previous estimates.

Uniqueness of the solution follows from the blow-up alternative in Theorem 2.2. □

7 Maximal velocity estimates

In this section we prove our first result on the speed of propagation of the solution to (1.1). We first prove Theorem 2.4, which holds under a convexity assumption, in Subsection 7.1 and then Proposition 2.6 in the case of general sets in Subsection 7.2.

7.1 Convex sets

In this section we prove Theorem 2.4. If the initial data ψ_0 were in H^s and the potential w in $\mathcal{W}_{d,s}$ with $s \geq 1$, then Theorem 2.4 would be a corollary of our article on maximal velocity estimate for non-autonomous pseudo-relativistic Schrödinger equation [6], since in such case we could write a solution to (1.1) as $\psi_t = U_t\psi_0$ with $U_t = U_{t,0}$ the propagator generated by $\langle\nabla\rangle + w * |\psi_t|^2$. In this section we prove that the maximal velocity estimate for (1.1) as stated in Theorem 2.4 holds even for $s \geq 1/2$, assuming that the sets X and Y are convex and using several lemmata from [6] together with a final argument which avoids the use of the concept of propagator. Restricting to more regular initial states ψ_0 in H^s with $s \geq 1$ will allow us, in the next subsection, to prove the maximal velocity estimate for general subsets X and Y as stated in Proposition 2.6 by using the results of [6].

The main idea of the proof of Theorem 2.4 is that, for a well chosen function ℓ it holds

$$\|\mathbf{1}_Y\psi_t\|_{L^2} \leq \underbrace{\|\mathbf{1}_Y e^{\ell(x)}\|_{\mathcal{B}(L^2)}}_{\leq \exp\left(-\frac{\mathrm{dist}(X,Y)}{2}\right)} \quad \underbrace{\|e^{-\ell(x)}\psi_t\|_{L^2}}_{\leq \exp(t)\exp\left(-\frac{\mathrm{dist}(X,Y)}{2}\right)\|\psi_0\|_{L^2}}, \tag{7.1}$$

where the $e^{-\ell(x)}\psi_t$ part is estimated through a Gronwall argument. To reach those estimates, we need several lemmata from [6] that we recall without proof. The first one is a quantitative separation lemma which allows us to introduce the function ℓ appearing in (7.1).

Lemma 7.1 ([6]). *Let X, Y two convex subsets of $\mathbb{R}^d$ such that $\mathrm{dist}(X,Y) > 0$. There exist x_0 in $\mathbb{R}^d$ and a unit vector n in $\mathbb{R}^d$ such that the affine functional $\ell(x) = n\cdot(x - x_0)$ satisfies*

$$\forall x \in X, \quad \ell(x) \geq \frac{1}{2}\mathrm{dist}(X,Y) \qquad \text{and} \qquad \forall x \in Y, \quad \ell(x) \leq -\frac{1}{2}\mathrm{dist}(X,Y)\,.$$

From now on, in this section, we consider ℓ as in Lemma 7.1 and for all $\varepsilon > 0$, we introduce a bounded regularization of ℓ by setting

$$\ell_\varepsilon(x) := f_\varepsilon(\ell(x)) = f_\varepsilon(n\cdot(x - x_0)),$$

where $f_\varepsilon(r) = f(\varepsilon r)$, $f \in C^\infty(\mathbb{R})$, $f(r) = r$ on $[-1,1]$, $0 \leq f' \leq 1$ and f' is compactly supported.

For all $\varepsilon > 0$, we define the operator G_ε on $H^1(\mathbb{R}^d)$ by

$$G_\varepsilon := \mathrm{Im}\big(e^{\ell_\varepsilon(x)}\langle\nabla\rangle e^{-\ell_\varepsilon(x)}\big) = \frac{1}{2i}\big(e^{\ell_\varepsilon(x)}\langle\nabla\rangle e^{-\ell_\varepsilon(x)} - e^{-\ell_\varepsilon(x)}\langle\nabla\rangle e^{\ell_\varepsilon(x)}\big). \tag{7.2}$$

Lemma 7.2 ([6]). *For all $\varepsilon > 0$, G_ε extends to a bounded operator on L^2, with*

$$\sup_{\varepsilon>0}\|G_\varepsilon\|_{\mathcal{B}(L^2)} < \infty.$$

For all $z \in \mathbb{C}\setminus(-\infty,0)$, we write $\sqrt{z} = \sqrt{|z|}e^{\frac{i}{2}\mathrm{Arg}(z)}$ with $-\pi < \mathrm{Arg}(z) < \pi$ and for all $\xi \in \mathbb{R}^d$, we set

$$f_\pm(\xi) := \sqrt{|\xi \pm in|^2 + 1} = \sqrt{|\xi|^2 \pm 2in\cdot\xi}. \tag{7.3}$$

Lemma 7.3 ([6]). *For all $\xi \in \mathbb{R}^d$, the bound*

$$\big|\mathrm{Im} f_\pm(\xi)\big| \leq 1$$

holds.

We define the operator G_0 on L^2 by

$$G_0 := \mathrm{Im}(f_+(-i\nabla)) = \mathcal{F}\,\mathrm{Im}(f_+(\xi))\,\mathcal{F}^{-1}. \tag{7.4}$$

It then follows from Lemma 7.3 that

$$\|G_0\|_{\mathcal{B}(L^2)} \leq 1. \tag{7.5}$$

The next lemma shows that G_0 is the weak limit of G_ε (defined in (7.2)) as $\varepsilon \to 0$.

Lemma 7.4 ([6]). *We have*

$$G_\varepsilon \to G_0, \quad \varepsilon \to 0,$$

weakly in $\mathcal{B}(L^2)$.

We are now in the position to prove Theorem 2.4.

Proof of Theorem 2.4. For $\tilde{\psi}_0$ in $\mathcal{C}_0^\infty$ whose support is included in X, the integral form of Hartree's equation (1.1) yields:

$$\begin{aligned}
\|e^{-\ell_\varepsilon}\tilde{\psi}_t\|_{L^2}^2 &= \|e^{-\ell_\varepsilon}\tilde{\psi}_0\|_{L^2}^2 + 2\,\mathrm{Im}\int_0^t \langle e^{-\ell_\varepsilon}\tilde{\psi}_\tau, e^{-\ell_\varepsilon}[\langle\nabla\rangle + w * |\tilde{\psi}_\tau|^2]\tilde{\psi}_\tau\rangle_{L^2}\mathrm{d}\tau \\
&= \|e^{-\ell_\varepsilon}\tilde{\psi}_0\|_{L^2}^2 + 2\,\mathrm{Im}\int_0^t \langle e^{-\ell_\varepsilon}\tilde{\psi}_\tau, e^{-\ell_\varepsilon}\langle\nabla\rangle e^{\ell_\varepsilon} e^{-\ell_\varepsilon}\tilde{\psi}_\tau\rangle_{L^2}\mathrm{d}\tau \\
&= \|e^{-\ell_\varepsilon}\mathbf{1}_X\tilde{\psi}_0\|_{L^2}^2 + 2\int_0^t \langle e^{-\ell_\varepsilon}\tilde{\psi}_\tau, G_\varepsilon e^{-\ell_\varepsilon}\tilde{\psi}_\tau\rangle_{L^2}\mathrm{d}\tau\,.
\end{aligned} \tag{7.6}$$

Lemma 7.2 provides a uniform control of $\|G_\varepsilon\|_{\mathcal{B}(L^2)}$, and thus there exists $C > 0$ such that

$$C = \sup_{\varepsilon\in(0,1)} \|G_\varepsilon\|_{\mathcal{B}(L^2)} < +\infty\,.$$

This, along with Lemma 7.1 about the separation of the convex sets X and Y gives

$$\|e^{-\ell_\varepsilon}\tilde{\psi}_t\|_{L^2}^2 \leq e^{-\mathrm{dist}(X,Y)}\|\tilde{\psi}_0\|_{L^2}^2 + 2C\int_0^t \|e^{-\ell_\varepsilon}\tilde{\psi}_\tau\|_{L^2}^2\mathrm{d}\tau\,.$$

It is then possible to use Gronwall's Lemma to get

$$\|e^{-\ell_\varepsilon}\tilde{\psi}_t\|_{L^2}^2 \leq e^{2Ct-\mathrm{dist}(X,Y)}\|\tilde{\psi}_0\|_{L^2}^2 \tag{7.7}$$

which resembles the estimate in (7.1). It remains to take the limit as ε goes to 0, obtain 1 instead of C and replace $\tilde{\psi}_0$ by ψ_0 in H^s.

The square of the L^2 norm of $e^{-\ell_\varepsilon}\tilde{\psi}_t$ can be split into two terms:

$$\int_{\mathbb{R}^d} e^{-2\ell_\varepsilon}|\tilde{\psi}_t|^2 = \int_{\ell(x)\geq 0} e^{-2\ell_\varepsilon}|\tilde{\psi}_t|^2 + \int_{\ell(x)<0} e^{-2\ell_\varepsilon}|\tilde{\psi}_t|^2\,.$$

The first term on the right-hand side converges to $\int_{\ell(x)\geq 0} e^{-2\ell}|\tilde{\psi}_t|^2$ by Lebesgue's dominated convergence Theorem while the second term on the right-hand side converges to

$\int_{\ell(x)<0} e^{-2\ell}|\tilde{\psi}_t|^2$ by Beppo–Levi's monotone convergence Theorem. Hence, the uniform bound in (7.7) implies that $e^{-\ell}\tilde{\psi}_t$ lies in L^2, with bound

$$\|e^{-\ell}\tilde{\psi}_t\|_{L^2}^2 \leq e^{2Ct-\mathrm{dist}(X,Y)}\|\tilde{\psi}_0\|_{L^2}^2$$

and

$$\|e^{-\ell_\varepsilon}\tilde{\psi}_t - e^{-\ell}\tilde{\psi}_t\|_{L^2} \xrightarrow[\varepsilon\to 0]{} 0\,. \tag{7.8}$$

We wish to take the limit ε to 0 in (7.6). We remark that

$$\begin{aligned} &|\langle e^{-\ell_\varepsilon}\tilde{\psi}_\tau, G_\varepsilon e^{-\ell_\varepsilon}\tilde{\psi}_\tau\rangle_{L^2} - \langle e^{-\ell}\tilde{\psi}_\tau, G_0 e^{-\ell}\tilde{\psi}_\tau\rangle_{L^2}| \\ &\quad\leq \|e^{-\ell_\varepsilon}\tilde{\psi}_\tau - e^{-\ell}\tilde{\psi}_\tau\|_{L^2}\|G_\varepsilon\|_{\mathcal{B}(L^2)}\|e^{-\ell_\varepsilon}\tilde{\psi}_\tau\|_{L^2} + \|e^{-\ell}\tilde{\psi}_\tau\|_{L^2}\|G_\varepsilon\|_{\mathcal{B}(L^2)}\|e^{-\ell_\varepsilon}\tilde{\psi}_\tau - e^{-\ell}\tilde{\psi}_\tau\|_{L^2} \\ &\qquad + |\langle e^{-\ell}\tilde{\psi}_\tau, (G_\varepsilon - G_0)e^{-\ell}\tilde{\psi}_\tau\rangle_{L^2}|\,. \end{aligned}$$

The first two terms on the right-hand side converge to 0 as ε goes to 0 using (7.8), Lemma 7.2, which provides a uniform bound on G_ε, and (7.7), while the third term goes to 0 by Lemma 7.4, which states the convergence of G_ε towards G_0 for the weak operator topology.

As a result, we can take the limit as ε goes to 0 in (7.6) and obtain

$$\|e^{-\ell}\tilde{\psi}_t\|_{L^2}^2 = \|e^{-\ell}\mathbf{1}_X\tilde{\psi}_0\|_{L^2}^2 + 2\int_0^t \langle e^{-\ell}\tilde{\psi}_\tau, G_0 e^{-\ell}\tilde{\psi}_\tau\rangle_{L^2}\mathrm{d}\tau\,.$$

A new application of Gronwall's Lemma along with (7.5) to control $\|G_0\|_{\mathcal{B}(L^2)}$ yields:

$$\|e^{-\ell}\tilde{\psi}_t\|_{L^2}^2 \leq e^{2t-\mathrm{dist}(X,Y)}\|\tilde{\psi}_0\|_{L^2}^2\,.$$

Using a second time Lemma 7.1, we obtain

$$\|1_Y\tilde{\psi}_t\|_{L^2} \leq \|1_Y e^{\ell}\|_{\mathcal{B}(L^2)}\|e^{-\ell}\tilde{\psi}_t\|_{L^2} \leq e^{t-\mathrm{dist}(X,Y)}\|\tilde{\psi}_0\|_{L^2}\,.$$

We now consider ψ_0 in H^s such that $\mathbf{1}_X\psi_0 = \psi_0$. Then, for all $\delta > 0$ and all $T < T_{\max}(\psi_0)$, Theorem 4.1 implies that we can find $\tilde{\psi}_0$ in $\mathcal{C}_0^\infty$ such that $T_{\max}(\tilde{\psi}_0) > T$ and $\sup_{t\in[0,T]}\|\psi_t - \tilde{\psi}_t\|_{H^s} \leq \delta$, hence

$$\|\mathbf{1}_Y\psi_t\|_{L^2} \leq \|\mathbf{1}_Y\tilde{\psi}_t\|_{L^2} + \delta \leq e^{t-\mathrm{dist}(X,Y)}\|\tilde{\psi}_0\|_{L^2} + \delta \leq e^{t-\mathrm{dist}(X,Y)}(\|\psi_0\|_{L^2} + \delta) + \delta,$$

for any $\delta > 0$ which implies the bound in Theorem 2.4. □

7.2 General sets

We now turn to the proof of Proposition 2.6 giving a maximal velocity estimate in the case of non-convex sets X and Y. As mentioned before, we need to impose more regularity on the initial states ψ_0 in order to apply results from our companion paper [6]. More precisely we must work in a setting where the family of operators $(\langle\nabla\rangle + w * |\psi_t|^2)_t$ defines a unitary propagator in the following sense.

Definition 7.5. *Let I a compact interval of $\mathbb{R}$ and $(A_t)_{t\in I}$ a family of self-adjoint operators on L^2 such that $\mathcal{D}(A_t)\cap H^{1/2}$ is dense in $H^{1/2}$ and A_t are continuously extendable to $\mathcal{B}(H^{1/2}, H^{-1/2})$. The map $I\times I \ni (t,s)\mapsto U(t,s)$ is a* unitary propagator *associated to*

$$i\partial_t\psi_t = A_t\psi_t\,,\quad t\in I, \tag{7.9}$$

if and only if

1. *$U(t,s)$ is unitary on L^2 for all t,s in I.*

2. *$U(t,t) = \mathbf{1}_{L^2}$ for all t in I and $U(t,s)U(s,r) = U(t,r)$ for all t,s,r in I.*

3. *For all s in I, the map $t \ni I \mapsto U(t,s)$ belongs to $\mathcal{C}^0(I,\mathcal{B}(H^{1/2})_{\mathrm{str}}) \cap \mathcal{C}^1(I,\mathcal{B}(H^{1/2},H^{-1/2})_{\mathrm{str}})$ and satisfies*
$$\forall t,s \in I\,,\ \forall \psi \in H^{1/2}\,,\quad i\partial_t U(t,s)\psi = A_t U(t,s)\psi\,,$$

as an equality in $H^{-1/2}$. The index "str" indicates that the considered topology is the strong operator topology.

Proposition 2.6 will then be a consequence of the following theorem from [6].

Theorem 7.6 (Corollary 1.4 in [6])**.** *There exists $C_d > 0$ such that, if $(V_t)_{0\le t\le T}$, with $T > 0$, is a family of real-valued potentials such that $(U(t,s))_{0\le t,s\le T}$ is a unitary propagator associated to $(\langle\nabla\rangle + V_t)_{0\le t\le T}$, then, for any measurable subsets X and Y of $\mathbb{R}^d$, it holds*
$$\forall t \in [0,T]\,,\qquad \|\mathbf{1}_Y U(t,0)\mathbf{1}_X\|_{\mathcal{B}(L^2)} \le C_d\, e^{t-\mathrm{dist}(X,Y)}\, \langle \mathrm{dist}(X,Y)\rangle^d\,,$$
with $\langle r\rangle = \sqrt{1+r^2}$.

To apply Theorem 7.6 with $V_t = w * |\psi_t|^2$, it is sufficient to check that it satisfies the assumptions of the following result from [6]. (Note that this result remains true in dimension $d = 2$.)

Proposition 7.7 (Remark 1.8 in [6])**.** *There exists $c_d > 0$, such that for any $T > 0$, the following holds: Suppose that for all t in $[0,T]$, $V_t = V_{\infty,t} + V_{d,t}$ with $V_{\infty,t}$ in L^∞, $V_{d,t}$ in $L^{d,\infty}$ and*

1. *for all t in $[0,T]$, $\|V_{d,t}\|_{L^{d,\infty}} < c_d$,*

2. *$\sup_{t\in[0,T]} \|V_{\infty,t}\|_{L^\infty} < \infty$,*

3. *$\sup_{t\in[0,T]} \|\partial_t V_t\|_{L^{d,\infty}+L^\infty} < \infty$.*

Then the non-autonomous equation (7.9), with the self-adjoint operator $A_t = \langle\nabla\rangle + V_t$ defined through the KLMN theorem, admits a unique unitary propagator.

It is thus sufficient to prove the following estimates.

Proposition 7.8. *Let $s \ge 1$, ψ_0 in H^s and w in $\mathcal{W}_{d,s}$. Let ψ in $\mathcal{C}^0([0,T_{\max}),H^s) \cap \mathcal{C}^1([0,T_{\max}),H^{s-1})$ be the local solution to (1.1) given by Theorem 2.2. Then, for any $0 < T < T_{\max}$, it holds*

1. *$\sup_{t\in[0,T]} \|w * |\psi_t|^2\|_{L^\infty} < \infty$,*

2. *$\sup_{t\in[0,T]} \|\partial_t (w * |\psi_t|^2)\|_{L^{d,\infty}+L^\infty} < \infty$.*

As a consequence, for all T in $[0,T_{\max})$, $(\langle\nabla\rangle + w*|\psi_t|^2)_{t\in[0,T]}$ generates a unitary propagator.

Proof. Let us first remark that if w_∞ is in L^∞, then:

- $\|w_\infty * |\psi_t|^2\|_{L^\infty}$ is uniformly bounded on $[0,T]$ as
$$\|w_\infty * |\psi_t|^2\|_{L^\infty} \lesssim \|w_\infty\|_{L^\infty}\|\psi_t\|_{L^2}^2 = \|w_\infty\|_{L^\infty}\|\psi_0\|_{L^2}^2\,. \tag{7.10}$$

- $\|\partial_t(w_\infty * |\psi_t|^2)\|_{L^\infty}$ is uniformly bounded on $[0,T]$ as

$$\begin{aligned}\|\partial_t(w_\infty * |\psi_t|^2)\|_{L^\infty} &\lesssim \|w_\infty * (\bar{\psi}_t \partial_t \psi_t)\|_{L^\infty}\\ &\lesssim \|w_\infty\|_{L^\infty}\|\psi_t\|_{L^2}\|\partial_t\psi_t\|_{L^2}\\ &\lesssim \|w_\infty\|_{L^\infty}\|\psi_t\|_{H^s}\|\partial_t\psi_t\|_{H^{s-1}}\end{aligned}$$

  and the right-hand side is uniformly bounded on the compact interval $[0,T]$ using the continuity of $t \mapsto \|\psi_t\|_{H^s}$ and $t \mapsto \|\partial_t\psi_t\|_{H^{s-1}}$ on $[0,T]$ given by Theorem 2.2.

We will now treat the singular part of w distinguishing three cases, depending on the parameter s, as in the definition of $\mathcal{W}_{d,s}$. In any case we will have that $w * |\psi_t|^2$ belongs to L^∞ and therefore the first point of Proposition 7.7 obviously holds.

Case $1 \le s < \frac{d}{2}$. We consider a decomposition $w = w_{d/(2s)} + w_\infty$ with $w_{d/(2s)}$ in $L^{d/(2s),\infty}$ and w_∞ in L^∞. The convolution product with w_∞ can be estimated as above while for the convolution involving $w_{d/(2s)}$ we use the Young inequality in Lorentz spaces (Proposition 3.7) in the special case of a L^∞ function, followed by the Hölder and Sobolev inequalities in Lorentz spaces (Propositions 3.6 and 3.5):

$$\begin{aligned}\|w_{\frac{d}{2s}} * |\psi_t|^2\|_{L^\infty} &\lesssim \|w_{\frac{d}{2s}}\|_{L^{\frac{d}{2s},\infty}}\||\psi_t|^2\|_{L^{\frac{d}{d-2s},1}}\\ &\lesssim \|w_{\frac{d}{2s}}\|_{L^{\frac{d}{2s},\infty}}\|\psi_t\|^2_{L^{\frac{2d}{d-2s},2}}\\ &\lesssim \|w_{\frac{d}{2s}}\|_{L^{\frac{d}{2s},\infty}}\|\psi_t\|^2_{H^s}\end{aligned}$$

and the right hand side is uniformly bounded on $[0,T]$ by Theorem 2.2.

We proceed similarly for the time derivative of the term involving the $L^{d/(2s),\infty}$ part of the potential:

$$\begin{aligned}\|\partial_t(w_{\frac{d}{2s}} * |\psi_t|^2)\|_{L^{d,\infty}} &\lesssim \|w_{\frac{d}{2s}} * (\bar{\psi}_t\partial_t\psi_t)\|_{L^{d,\infty}}\\ &\lesssim \|w_{\frac{d}{2s}}\|_{L^{\frac{d}{2s},\infty}}\|\bar{\psi}_t\partial_t\psi_t\|_{L^{\frac{2d}{2d-4s+2},1}}\\ &\lesssim \|w_{\frac{d}{2s}}\|_{L^{\frac{d}{2s},\infty}}\|\psi_t\|_{L^{\frac{2d}{d-2s},2}}\|\partial_t\psi_t\|_{L^{\frac{2d}{d-2(s-1)},2}}\\ &\lesssim \|w_{\frac{d}{2s}}\|_{L^{\frac{d}{2s},\infty}}\|\psi_t\|_{H^s}\|\partial_t\psi_t\|_{H^{s-1}}\end{aligned}$$

and the last line is uniformly bounded on $[0,T]$ by Theorem 2.2.

Case $s = \frac{d}{2}$. In this case w is in $L^q + L^\infty$ for some $q > 1$ and we can assume without loss of generality that q is in $(1, d/2]$. Thus we can consider $1 \le s' < \frac{d}{2}$ such that $\frac{d}{2s'} = q$. In particular ψ_0 is in $H^{s'}$ and w is in $\mathcal{W}_{d,s'}$. Since s' is in $[1, \frac{d}{2})$, it suffices to apply the previous case.

Case $\frac{d}{2} < s$. For the $\mathcal{M}$ part w_1 of the potential w, we have

$$\|w_1 * |\psi_t|^2\|_{L^\infty} \lesssim \|w_1\|_{\mathcal{M}}\|\psi_t\|^2_{L^\infty} \lesssim \|w_1\|_{\mathcal{M}}\|\psi_t\|^2_{H^s} \tag{7.11}$$

and the right hand side is uniformly bounded in $[0,T]$ by Theorem 2.2.

Moreover, with $0 < \varepsilon < \min\{s - \frac{d}{2}, 1\}$, we deduce from Propositions 3.8, 3.6 and 3.5 that

$$\|\partial_t(w_1 * |\psi_t|^2)\|_{L^{d,\infty}+L^\infty} \lesssim \|\partial_t(w_1 * |\psi_t|^2)\|_{L^{\frac{d}{1-\varepsilon},\infty}}$$

$$\begin{aligned}
&\lesssim \|w_1 * (\bar{\psi}_t \partial_t \psi_t)\|_{L^{\frac{d}{1-\varepsilon},\infty}} \\
&\lesssim \|w_1\|_{\mathcal{M}} \|\bar{\psi}_t \partial_t \psi_t\|_{L^{\frac{d}{1-\varepsilon},\infty}} \\
&\lesssim \|w_1\|_{\mathcal{M}} \|\psi_t\|_{L^\infty} \|\partial_t \psi_t\|_{L^{\frac{d}{1-\varepsilon},\infty}} \\
&\lesssim \|w_1\|_{\mathcal{M}} \|\psi_t\|_{H^s} \|\partial_t \psi_t\|_{H^{\frac{d}{2}-1+\varepsilon}} \\
&\lesssim \|w_1\|_{\mathcal{M}} \|\psi_t\|_{H^s} \|\partial_t \psi_t\|_{H^{s-1}} .
\end{aligned}$$

Again the last line is uniformly bounded on $[0, T]$ by Theorem 2.2.

Applying Proposition 7.7 with $V_t = w * |\psi_t|^2$ concludes the proof. □

We can now conclude the proof of Proposition 2.6.

Proof of Proposition 2.6. Let ψ be the local solution to (1.1) on $[0, T_{\max})$ given by Theorem 2.2. By Proposition 7.8, $(\langle\nabla\rangle + w * |\psi_t|^2)_{t\in[0,T]}$ generates a unitary propagator $U(t,s)_{t,s\in[0,T]}$, for any T in $(0, T_{\max})$. By uniqueness of the solution ψ to (1.1), we have

$$\psi_t = U(t,0)\psi_0 = U(t,0)\mathbf{1}_X \psi_0,$$

for all t in $[0, T]$. Applying then Theorem 7.6 yields Proposition 2.6. □

8 Scattering and asymptotic minimal velocity estimate

In this section we prove Theorems 2.12, 2.14, 2.16 and 2.18 on the scattering and long-time behavior of solutions to the pseudo-relativistic Hartree equation (1.1) in the case of short-range interaction potentials.

8.1 Estimates on the solution in weighted Sobolev spaces

We begin with a preliminary subsection where we prove several estimates on the norm of the solutions to (1.1) in weighted Sobolev spaces. These estimates subsequently play an important role in our analysis.

We first estimate the growth of the first moment of a solution in the L^2-norm. The proof is rather straightforward, see [27, Lemma A.1] for a slightly different argument. We recall the notation $\theta(x) = \min\{1, x\}$.

Lemma 8.1. *Let $0 \le \gamma \le 2$. Under the conditions of Theorem 2.10, assuming in addition that $\psi_0 \in L^2_\gamma$, the global solution ψ to (1.1) given by Theorem 2.10 satisfies*

$$\|\psi_t\|_{L^2_\gamma} \lesssim \|\psi_0\|_{L^2_\gamma} + \langle t\rangle^\gamma \|\psi_0\|_{L^2}, \tag{8.1}$$

uniformly in $t \in [0, \infty)$.

Proof. From Duhamel's formula

$$\psi_t = e^{-it\langle\nabla\rangle}\psi_0 - i\int_0^t e^{i(\tau-t)\langle\nabla\rangle}(w * |\psi_\tau|^2))\, \psi_\tau \, \mathrm{d}\tau\,, \tag{8.2}$$

and Lemma B.6, we obtain

$$\|\psi_t\|_{L^2_\gamma} \le \left\|e^{-it\langle\nabla\rangle}\psi_0\right\|_{L^2_\gamma} + \int_0^t \left\|e^{-i(t-\tau)\langle\nabla\rangle}(w * |\psi_\tau|^2)\psi_\tau\right\|_{L^2_\gamma} \mathrm{d}\tau$$

$$\lesssim \|\psi_0\|_{L^2_\gamma} + \langle t\rangle^\gamma \|\psi_0\|_{L^2}$$
$$+ \int_0^t \Big(\big\|(w * |\psi_\tau|^2)\psi_\tau\big\|_{L^2_\gamma} + \langle t-\tau\rangle^\gamma \big\|(w * |\psi_\tau|^2)\psi_\tau\big\|_{L^2}\Big)\,\mathrm{d}\tau.$$

Using (3.18) and Theorem 2.10, we estimate the first term in the integral as

$$\begin{aligned}\big\|(w * |\psi_\tau|^2)\psi_\tau\big\|_{L^2_\gamma} &\lesssim \|w\|_{\mathcal{M}+L^q} \|\psi_\tau\|^{2\theta(\frac{r}{q})}_{L^p\cap L^\infty} \|\psi_\tau\|_{L^2_\gamma} \|\psi_0\|^{2-2\theta(\frac{r}{q})}_{H^s\cap H^{s,p'}} \\ &\lesssim \langle\tau\rangle^{-\frac{d}{r}\theta(\frac{r}{q})} \|w\|_{\mathcal{M}+L^q} \|\psi_0\|^2_{H^s\cap H^{s,p'}} \|\psi_\tau\|_{L^2_\gamma} \\ &\lesssim \varepsilon_0 \langle\tau\rangle^{-\frac{d}{r}\theta(\frac{r}{q})} \|\psi_\tau\|_{L^2_\gamma},\end{aligned} \tag{8.3}$$

and likewise, applying Lemma 3.12 and Theorem 2.10 to the second term we obtain

$$\big\|(w * |\psi_\tau|^2)\psi_\tau\big\|_{L^2} \lesssim \varepsilon_0 \langle\tau\rangle^{-\frac{d}{r}\theta(\frac{r}{q})} \|\psi_\tau\|_{L^2} = \varepsilon_0 \langle\tau\rangle^{-\frac{d}{r}\theta(\frac{r}{q})} \|\psi_0\|_{L^2}. \tag{8.4}$$

Since $r < d$, we thus obtain that

$$\|\psi_t\|_{L^2_\gamma} \lesssim \|\psi_0\|_{L^2_\gamma} + \langle t\rangle^\gamma (1+\varepsilon_0)\|\psi_0\|_{L^2} + \varepsilon_0 \int_0^t \langle\tau\rangle^{-\frac{d}{r}\theta(\frac{r}{q})} \|\psi_\tau\|_{L^2_\gamma} \mathrm{d}\tau.$$

Using again that $r < d$ and $q < d$ imply $\frac{d}{r}\theta(\frac{r}{q}) > 1$, the statement of the lemma follows from Gronwall's inequality. ☐

We now need to estimate the norm of a solution to (1.1) in the weighted Sobolev space H^s_γ defined in (3.1)–(3.2).

In the remainder of this subsection we restrict the class of admissible interaction potentials to $w \in \mathcal{M} + L^q$ with $1 \le q < \frac{d}{2}$. Recall that, for $\gamma > \frac{d}{2r}$, $H^s_\gamma \hookrightarrow H^s \cap H^{s,p'}$ (where, as before $p = \frac{2r}{r-1}$, $p' = \frac{2r}{r+1}$). Recall also that Lemma 3.3 implies $\|\langle\nabla\rangle^s \langle x\rangle^\gamma \varphi\|_{L^2} \lesssim \|\varphi\|_{H^s_\gamma}$.

Lemma 8.2. *Let $s \ge \frac{d}{2}+1$, $\max\{\frac{d}{4},1\} \le r < \frac{d}{2}$ and $1 \le q < \frac{d}{2}$. Let $\gamma \ge 0$ be such that $\frac{d}{2r} < \gamma \le 2$. There exists $\varepsilon_0 > 0$ such that, for all $w \in \mathcal{M}+L^q$ and $\psi_0 \in H^s_\gamma \hookrightarrow H^s \cap H^{s,p'}$ (with $p = \frac{2r}{r-1}$) satisfying*

$$\|w\|_{\mathcal{M}+L^q} \|\psi_0\|^2_{H^s\cap H^{s,p'}} \le \varepsilon_0,$$

the global solution ψ to (1.1) given by Theorem 2.10 satisfies

$$\|\psi_t\|_{H^s_\gamma} \lesssim \|\psi_0\|_{H^s_\gamma} + \langle t\rangle^\gamma \|\psi_0\|_{H^s\cap H^{s,p'}},$$

uniformly in $t \in [0,\infty)$.

Proof. Duhamel's formula (8.2) and Lemma B.6 give

$$\begin{aligned}\|\psi_t\|_{H^s_\gamma} &= \big\|\langle x\rangle^\gamma \langle\nabla\rangle^s \psi_t\big\|_{L^2} \\ &\le \big\|e^{-it\langle\nabla\rangle}\langle\nabla\rangle^s\psi_0\big\|_{L^2_\gamma} + \int_0^t \big\|e^{-i(t-\tau)\langle\nabla\rangle}\langle\nabla\rangle^s\big[(w*|\psi_\tau|^2)\psi_\tau\big]\big\|_{L^2_\gamma}\,\mathrm{d}\tau \\ &\lesssim \|\psi_0\|_{H^s_\gamma} + t^\gamma\|\psi_0\|_{H^s} \\ &\quad + \int_0^t \Big(\big\|(w*|\psi_\tau|^2)\psi_\tau\big\|_{H^s_\gamma} + \langle t-\tau\rangle^\gamma \big\|(w*|\psi_\tau|^2)\psi_\tau\big\|_{H^s}\Big)\mathrm{d}\tau.\end{aligned}$$

Using (3.13) from Lemma 3.11 and (3.19) from Lemma 3.13 and the embeddings $H^s \hookrightarrow L^2$, $H^s \hookrightarrow L^{2q'}$, $H^s \hookrightarrow L^p \cap L^\infty \hookrightarrow L^\infty$ (since $p, 2q' \geq 2$), we estimate the first term in the integral as

$$\big\|(w * |\psi_\tau|^2)\psi_\tau\big\|_{H^s_\gamma} \lesssim \|w\|_{\mathcal{M}+L^q} \|\psi_\tau\|_{H^s}^{2-\theta(\frac{r}{q})} \|\psi_\tau\|_{L^p \cap L^\infty}^{\theta(\frac{r}{q})} \|\psi_\tau\|_{H^s_\gamma}.$$

Applying (2.8) and (2.10) from Theorem 2.10 therefore implies that

$$\begin{aligned}\big\|(w * |\psi_\tau|^2)\psi_\tau\big\|_{H^s_\gamma} &\lesssim \langle\tau\rangle^{-\frac{d}{2r}\theta(\frac{r}{q})} \|w\|_{\mathcal{M}+L^q} \|\psi_0\|^2_{H^s \cap H^{s,p'}} \|\psi_\tau\|_{H^s_\gamma} \\ &\lesssim \varepsilon_0 \langle\tau\rangle^{-\frac{d}{2r}\theta(\frac{r}{q})} \|\psi_\tau\|_{H^s_\gamma}.\end{aligned}$$

Likewise, applying Lemma 3.12 and Theorem 2.10 to the second term in the integral above we have

$$\begin{aligned}\big\|(w * |\psi_\tau|^2)\psi_\tau\big\|_{H^s} &\lesssim \|w\|_{\mathcal{M}+L^q} \|\psi_\tau\|_{L^p \cap L^\infty}^{2\theta(\frac{r}{q})} \|\psi_\tau\|_{H^s}^{3-2\theta(\frac{r}{q})} \\ &\lesssim \langle\tau\rangle^{-\frac{d}{r}\theta(\frac{r}{q})} \|w\|_{\mathcal{M}+L^q} \|\psi_0\|^3_{H^s \cap H^{s,p'}}.\end{aligned} \tag{8.5}$$

Hence, since $r < d$ and $q < d$, it holds

$$\begin{aligned}\int_0^t \langle t-\tau\rangle^\gamma \big\|(w * |\psi_\tau|^2)\psi_\tau\big\|_{H^s} \mathrm{d}\tau &\lesssim \langle t\rangle^\gamma \|w\|_{\mathcal{M}+L^q} \|\psi_0\|^3_{H^s \cap H^{s,p'}} \int_0^t \langle\tau\rangle^{-\frac{d}{r}\theta(\frac{r}{q})} \mathrm{d}\tau \\ &\lesssim \varepsilon_0 \langle t\rangle^\gamma \|\psi_0\|_{H^s \cap H^{s,p'}}.\end{aligned}$$

Therefore we have shown that

$$\|\psi_t\|_{H^s_\gamma} \lesssim \|\psi_0\|_{H^s_\gamma} + (1+\varepsilon_0)\langle t\rangle^\gamma \|\psi_0\|_{H^s \cap H^{s,p'}} + \varepsilon_0 \int_0^t \langle\tau\rangle^{-\frac{d}{2r}\theta(\frac{r}{q})} \|\psi_\tau\|_{H^s_\gamma} \mathrm{d}\tau.$$

Since $r < \frac{d}{2}$ and $q < \frac{d}{2}$ imply $\frac{d}{2r}\theta(\frac{r}{q}) > 1$, the statement of the lemma follows from Gronwall's inequality. □

We will need yet another related, auxiliary result, which will be used in our proof of the invertibility of the wave operators in Theorem 2.14.

Lemma 8.3. *Let $s \geq \frac{d}{2}+1$, $\max\{\frac{d}{4}, 1\} \leq r < \frac{d}{2}$ and $1 \leq q \leq r$. Let $\gamma \geq 0$ be such that $\frac{d}{2r} < \gamma \leq 2$. There exists $\varepsilon_0 > 0$ such that, for all $w \in \mathcal{M} + L^q$ and $\psi_0 \in H^s_\gamma \hookrightarrow H^s \cap H^{s,p'}$ (with $p = \frac{2r}{r-1}$) satisfying*

$$\|w\|_{\mathcal{M}+L^q} \|\psi_0\|^2_{H^s \cap H^{s,p'}} \leq \varepsilon_0,$$

the global solution ψ to (1.1) given by Theorem 2.10 satisfies

$$\|\psi_t\|_{L^p_\gamma \cap L^\infty_\gamma} \lesssim \langle t\rangle^{\gamma - \frac{d}{2r}} \|\psi_0\|_{H^s_\gamma}, \tag{8.6}$$

uniformly in $t \in [0,\infty)$.

Proof. Starting with Duhamel's formula (8.2) and applying Lemma B.7, we estimate

$$\|\psi_t\|_{L^p_\gamma \cap L^\infty_\gamma} \leq \big\|e^{-it\langle\nabla\rangle}\psi_0\big\|_{L^p_\gamma \cap L^\infty_\gamma} + \int_0^t \big\|e^{-i(t-\tau)\langle\nabla\rangle}\big[(w * |\psi_\tau|^2)\psi_\tau\big]\big\|_{L^p_\gamma \cap L^\infty_\gamma} \mathrm{d}\tau$$

$$\lesssim \|\psi_0\|_{H^s_\gamma} + \langle t\rangle^{\gamma-\frac{d}{2r}}\|\psi_0\|_{H^s\cap H^{s,p'}}$$
$$+ \int_0^t \Big(\big\|(w*|\psi_\tau|^2)\psi_\tau\big\|_{H^s_\gamma} + \langle t-\tau\rangle^{\gamma-\frac{d}{2r}}\big\|(w*|\psi_\tau|^2)\psi_\tau\big\|_{H^s\cap H^{s,p'}}\Big)\,\mathrm{d}\tau. \tag{8.7}$$

Using (3.13) from Lemma 3.11 and (3.19) from Lemma 3.13 we estimate the first term in the integral as

$$\big\|(w*|\psi_\tau|^2)\psi_\tau\big\|_{H^s_\gamma} \lesssim \|w\|_{\mathcal{M}+L^q}\|\psi_\tau\|_{L^p\cap L^\infty}\big(\|\psi_\tau\|_{L^p\cap L^\infty}\|\psi_\tau\|_{H^s_\gamma} + \|\psi_\tau\|_{H^s}\|\psi_\tau\|_{L^p_\gamma\cap L^\infty_\gamma}\big),$$

where we used that $r \geq q$ and $2q' \geq p$ under our assumptions. Now, as mentioned above, $\gamma > \frac{d}{2r}$ implies $\|\psi_0\|_{H^{s,p'}\cap H^s} \lesssim \|\psi_0\|_{H^s_\gamma}$ and hence Lemma 8.2 yields $\|\psi_\tau\|_{H^s_\gamma} \lesssim \langle\tau\rangle^\gamma\|\psi_0\|_{H^s_\gamma}$. Applying this property and (2.8) and (2.10) from Theorem 2.10 to the previous inequality we obtain

$$\begin{aligned}\big\|(w*|\psi_\tau|^2)\psi_\tau\big\|_{H^s_\gamma} &\lesssim \|w\|_{\mathcal{M}+L^q}\|\psi_0\|^2_{H^s\cap H^{s,p'}}\big(\langle\tau\rangle^{\gamma-\frac{d}{r}}\|\psi_0\|_{H^s_\gamma} + \langle\tau\rangle^{-\frac{d}{2r}}\|\psi_\tau\|_{L^p_\gamma\cap L^\infty_\gamma}\big)\\ &\lesssim \varepsilon_0\big(\langle\tau\rangle^{\gamma-\frac{d}{r}}\|\psi_0\|_{H^s_\gamma} + \langle\tau\rangle^{-\frac{d}{2r}}\|\psi_\tau\|_{L^p_\gamma\cap L^\infty_\gamma}\big).\end{aligned} \tag{8.8}$$

We can proceed similarly to estimate the second term of the integral in (8.7). Indeed, applying (3.15)-(3.17) from Lemma 3.12 we have

$$\begin{aligned}\|(w*|\psi_\tau|^2)\psi_\tau\|_{H^s} &\lesssim \|w\|_{\mathcal{M}+L^q}\|\psi_\tau\|_{H^s}\|\psi_\tau\|^2_{L^p\cap L^\infty}\\ &\lesssim \|w\|_{\mathcal{M}+L^q}\langle\tau\rangle^{-\frac{d}{r}}\|\psi_0\|^3_{H^s\cap H^{s,p'}}\\ &\lesssim \varepsilon_0\langle\tau\rangle^{-\frac{d}{r}}\|\psi_0\|_{H^s_\gamma},\end{aligned} \tag{8.9}$$

where we applied again (2.8) and (2.10) from Theorem 2.10 in the second inequality, and used the embedding $H^s_\gamma \hookrightarrow H^s \cap H^{s,p'}$ in the last one. Analogously, thanks to (3.16)-(3.17) from Lemma 3.12 and Theorem 2.10 we obtain

$$\|(w*|\psi_\tau|^2)\psi_\tau\|_{H^{s,p'}} \lesssim \|w\|_{\mathcal{M}+L^q}\|\psi_0\|^3_{H^s\cap H^{s,p'}}\big(\langle\tau\rangle^{-\frac{d}{r}\theta(r-\frac12)} + \langle\tau\rangle^{-\frac{d}{r}\theta(\frac{r}{q}-\frac12)}\big).$$

Now we remark that, for $d > 3$, we have $r - \frac12 > 1$ and $\frac{r}{q} - \frac12 > 1$, while if $d = 3$, then by assumption it holds $\frac12 \leq r - \frac12 < 1$ and $\frac12 \leq \frac{r}{q} - \frac12 \leq 1$ and therefore $\langle\tau\rangle^{-\frac{d}{r}\theta(r-\frac12)} + \langle\tau\rangle^{-\frac{d}{r}\theta(\frac{r}{q}-\frac12)} \lesssim \langle\tau\rangle^{-\frac{d}{2r}}$. With these properties and the embedding $H^s_\gamma \hookrightarrow H^s\cap H^{s,p'}$ we conclude that

$$\|(w*|\psi_\tau|^2)\psi_\tau\|_{H^{s,p'}} \lesssim \varepsilon_0\langle\tau\rangle^{-\frac{d}{2r}}\|\psi_0\|_{H^s_\gamma}. \tag{8.10}$$

Combining (8.7) with (8.8)-(8.10), we obtain

$$\begin{aligned}\|\psi_t\|_{L^p_\gamma\cap L^\infty_\gamma} &\lesssim \langle t\rangle^{\gamma-\frac{d}{2r}}\|\psi_0\|_{H^s_\gamma} + \varepsilon_0\|\psi_0\|_{H^s_\gamma}\Big(\int_0^t\langle t-\tau\rangle^{\gamma-\frac{d}{2r}}\langle\tau\rangle^{-\frac{d}{2r}}\mathrm{d}\tau + \int_0^t\langle\tau\rangle^{\gamma-\frac{d}{r}}\mathrm{d}\tau\Big)\\ &\quad + \varepsilon_0\int_0^t\langle\tau\rangle^{-\frac{d}{2r}}\|\psi_\tau\|_{L^p_\gamma\cap L^\infty_\gamma}\mathrm{d}\tau\\ &\lesssim \langle t\rangle^{\gamma-\frac{d}{2r}}\|\psi_0\|_{H^s_\gamma} + \varepsilon_0\int_0^t\langle\tau\rangle^{-\frac{d}{2r}}\|\psi_\tau\|_{L^p_\gamma\cap L^\infty_\gamma}\mathrm{d}\tau,\end{aligned}$$

where in the second inequality, we used the property

$$\sup_{t>0}\langle t\rangle^a\int_0^t\langle t-\tau\rangle^{-a}\langle\tau\rangle^{-b}\,\mathrm{d}\tau < \infty \qquad \text{for any } a \geq 0,\ b > 1$$

with $b = \frac{d}{2r}$. Using again that $\frac{d}{2r} > 1$, the statement of the lemma follows from Gronwall's inequality. □

8.2 Scattering states and wave operators

In this subsection we establish the scattering results stated in Theorems 2.12 and 2.14. We begin with showing that, for short-range interaction potentials, any global solutions to (1.1) given by Theorem 2.10 scatters to a free solution. The proof straightforwardly follows from the multinilear estimates stated in Lemma 3.12 and the regularity and decay properties of a solution obtained in (2.8)–(2.11).

Proof of Theorem 2.12. We recall that ψ_+ is defined as

$$\psi_+ := \psi_0 - i\int_0^\infty e^{i\tau\langle\nabla\rangle}(w * |\psi_\tau|^2)\psi_\tau \,\mathrm{d}\tau.$$

First we note that ψ_+ is well-defined in H^s since $\psi_0 \in H^s$ and, using Lemma 3.12 and Theorem 2.10, we have

$$\|e^{i\tau\langle\nabla\rangle}(w * |\psi_\tau|^2)\psi_\tau\|_{H^s} \lesssim \|w\|_{\mathcal{M}+L^q}\|\psi_\tau\|^{2\theta(\frac{r}{q})}_{L^p\cap L^\infty}\|\psi_\tau\|^{3-2\theta(\frac{r}{q})}_{H^s} \lesssim \varepsilon_0\langle\tau\rangle^{-\frac{d}{r}\theta(\frac{r}{q})}\|\psi_0\|_{H^s\cap H^{s,p'}}, \tag{8.11}$$

which is integrable over $[0,\infty)$ since $r<d$ and $q<d$ imply $\frac{d}{r}\theta(\frac{r}{q})>1$. Next it follows from Duhamel's formula that

$$\psi_t - e^{-it\langle\nabla\rangle}\psi_+ = i\int_t^\infty e^{i\tau\langle\nabla\rangle}(w * |\psi_\tau|^2)\psi_\tau \,\mathrm{d}\tau.$$

Applying (8.11) gives (2.12) (or more precisely (2.14) in Remark 2.13).

To prove the continuity of W_+, let ψ_0, $\tilde\psi_0$ in $\mathcal{B}_{H^s\cap H^{s,p'}}(\delta_w)$ and let ψ_t, $\tilde\psi_t$ be the corresponding solutions to (1.1) given by Theorem 2.10. Then applying again Lemma 3.12 and arguing as in the proof of Theorem 2.10, we obtain

$$\begin{aligned}&\left\|e^{i\tau\langle\nabla\rangle}(w * |\psi_\tau|^2)\psi_\tau - e^{i\tau\langle\nabla\rangle}(w * |\tilde\psi_\tau|^2)\tilde\psi_\tau\right\|_{H^s}\\ &\lesssim \varepsilon_0\langle\tau\rangle^{-\frac{d}{r}\theta(\frac{r}{q})}\|\psi_0-\tilde\psi_0\|_{H^s\cap H^{s,p'}},\end{aligned}$$

which in turn implies that

$$\|W_+\psi_0 - W_+\tilde\psi_0\|_{H^s} \lesssim \|\psi_0-\tilde\psi_0\|_{H^s\cap H^{s,p'}}.$$

This implies the continuity of W_+. □

In order to prove the invertibility properties of W_+ stated in Theorem 2.14, we first have to justify the existence of the wave operator Ω_+. As mentioned in the introduction, the Cauchy problem at ∞ for the pseudo-relativistic Hartree equation, Eq. (2.15), can be solved exactly as in Theorem 2.10, considering the integral equation

$$\psi_t = e^{-it\langle\nabla\rangle}\psi_+ + i\int_t^\infty e^{-i(t-\tau)\langle\nabla\rangle}(w * |\psi_\tau|^2)\psi_\tau \,\mathrm{d}\tau,$$

and then applying a fixed point argument in $S^{s,r,p}$ (recall that the norm in $S^{s,r,p}$ has been defined in (2.6)). This leads to the following result.

Proposition 8.4 (Solutions to (2.15))**.** *Let* $s \geq \frac{d}{2}+1$*,* $1 \leq r < d$ *and* $1 \leq q < \frac{2d}{3}$ *be such that* $\frac{1}{d}+\frac{1}{2r} < \frac{1}{q}$*. Define* p *by the relation* $1 = \frac{1}{r}+\frac{2}{p}$*.*

There exists $\varepsilon_0 > 0$ *such that the following holds: for all* $w \in \mathcal{M}+L^q$ *and* $\psi_+ \in H^s \cap H^{s,p'}$ *satisfying*

$$\|w\|_{\mathcal{M}+L^q} \|\psi_+\|^2_{H^s \cap H^{s,p'}} \leq \varepsilon_0, \tag{8.12}$$

Eq. (2.15) *has a unique solution* ψ *in* $S^{s,r,p}$*.*

Given this theorem one can mimic the proof of Theorem 2.12 to obtain the existence and continuity of the wave operator mapping any scattering state ψ_+ to the corresponding initial state ψ_0.

Theorem 8.5 (Wave operator)**.** *Under the conditions of Proposition 8.4, there exists* $\delta_w > 0$ *such that the (non-linear) wave operator*

$$\begin{aligned} &\Omega_+ : \mathcal{B}_{H^s \cap H^{s,p'}}(\delta_w) \to H^s, \\ &\qquad \psi_+ \mapsto \Omega_+\psi_+ = \psi_+ + i\int_0^\infty e^{i\tau\langle\nabla\rangle}(w * |\psi_\tau|^2)\psi_\tau \,\mathrm{d}\tau, \end{aligned}$$

is well-defined and continuous.

Our next purpose is to identify a subset

$$\mathcal{A} \subset \mathcal{B}_{H^s \cap H^{s,p'}}(\delta_w)$$

such that $W_+ : \mathcal{A} \to \mathcal{B}_{H^s \cap H^{s,p'}}(\delta_w)$ and $\Omega_+ : \mathcal{A} \to \mathcal{B}_{H^s \cap H^{s,p'}}(\delta_w)$, which allows us to give a meaning to the expressions $\Omega_+ W_+ \varphi$ and $W_+\Omega_+\varphi$ for $\varphi \in \mathcal{A}$. The next theorem shows that, provided we restrict the class of potentials to $w \in \mathcal{M}+L^q$ with $1 \leq q < \frac{d}{2}$, it suffices to consider for $\mathcal{A}$ a closed ball with sufficiently small radius in H^s_γ, where we recall that

$$\|\varphi\|_{H^s_\gamma} = \|\langle x\rangle^\gamma \langle\nabla\rangle^s \varphi\|_{L^2}.$$

Note that Theorem 8.6 readily implies Theorem 2.14. The proof relies on the estimates obtained in Section 8.1.

Theorem 8.6. *Let* $s \geq \frac{d}{2}+1$*,* $\max\{1,\frac{d}{4}\} \leq r < \frac{d}{2}$*,* $1 \leq q \leq r$ *and* $\frac{d}{2r} < \gamma < \min\{2, \frac{d}{r}-1\}$*. There exists* $\varepsilon_0 > 0$ *such that, for all* $w \in \mathcal{M}+L^q$ *and* $\psi_0 \in H^s_\gamma$ *such that*

$$\|w\|_{\mathcal{M}+L^q}\|\psi_0\|^2_{H^s \cap H^{s,p'}} \leq \varepsilon_0,$$

we have

$$\|W_+\psi_0 - \psi_0\|_{H^s_\gamma} \lesssim \varepsilon_0 \|\psi_0\|_{H^s_\gamma}. \tag{8.13}$$

Likewise, for all $\psi_+ \in H^s_\gamma$ *such that*

$$\|w\|_{\mathcal{M}+L^q}\|\psi_+\|^2_{H^s \cap H^{s,p'}} \leq \varepsilon_0,$$

we have

$$\|\Omega_+\psi_+ - \psi_+\|_{H^s_\gamma} \lesssim \varepsilon_0 \|\psi_+\|_{H^s_\gamma}. \tag{8.14}$$

In particular, there exists $\delta_w > 0$ *such that*

$$\Omega_+ W_+ \varphi = W_+ \Omega_+ \varphi \quad \text{for all} \quad \varphi \in \mathcal{B}_{H^s_\gamma}(\delta_w).$$

Proof. Since $1 \leq q \leq r < \frac{d}{2}$, the conditions in Theorem 2.12 are satisfied and hence the expression of $W_+\psi_0$ gives

$$\|W_+\psi_0 - \psi_0\|_{H^s_\gamma} \leq \int_0^\infty \big\|e^{i\tau\langle\nabla\rangle}\langle\nabla\rangle^s\big[(w * |\psi_\tau|^2)\psi_\tau\big]\big\|_{L^2_\gamma}\,\mathrm{d}\tau.$$

Using first Lemma B.6, we estimate the integrated norm as

$$\big\|e^{i\tau\langle\nabla\rangle}\langle\nabla\rangle^s\big[(w * |\psi_\tau|^2)\psi_\tau\big]\big\|_{L^2_\gamma} \lesssim \big\|(w * |\psi_\tau|^2)\psi_\tau\big\|_{H^s_\gamma} + \langle\tau\rangle^\gamma\big\|(w * |\psi_\tau|^2)\psi_\tau\big\|_{H^s}.$$

Combining (8.6) and (8.8), we obtain

$$\big\|(w * |\psi_\tau|^2)\psi_\tau\big\|_{H^s_\gamma} \lesssim \varepsilon_0\langle\tau\rangle^{\gamma-\frac{d}{r}}\|\psi_0\|_{H^s_\gamma}.$$

Similarly, by (8.9) we have

$$\langle\tau\rangle^\gamma\|(w * |\psi_\tau|^2)\psi_\tau\|_{H^s} \lesssim \varepsilon_0\langle\tau\rangle^{\gamma-\frac{d}{r}}\|\psi_0\|_{H^s_\gamma}.$$

Since $\gamma < \frac{d}{r} - 1$, this implies (8.13). The proof of (8.14) is identical.

Now since $H^s_\gamma \hookrightarrow H^s \cap H^{s,p'}$, we deduce from (8.13) and Theorem 8.5 that, for $\delta_w > 0$ small enough and for all $\varphi \in \mathcal{B}_{H^s_\gamma}(\delta_w)$, $\Omega_+W_+\varphi$ is well-defined. The fact that $\Omega_+W_+\varphi = \varphi$ then follows from the definitions of W_+ and Ω_+. The same holds if one inverts the roles of W_+ and Ω_+. □

8.3 Average velocity and instantaneous velocity

This subsection contains first justifications that the average velocity and the "instantaneous" velocity converge to each other as $t \to \infty$. The results obtained here will then be used in the next subsection to prove the propagation estimates stated in Theorems 2.16 and 2.18.

We recall the definition of the instantaneous velocity operator $\Theta := -i\nabla\langle\nabla\rangle^{-1}$. Note the following identity

$$e^{it\langle\nabla\rangle}x^2e^{-it\langle\nabla\rangle} = (x + t\Theta)^2, \tag{8.15}$$

which follows from direct computations, and recall that L^2_1 is the weighted L^2 space, with weight $\langle x\rangle$.

Proposition 8.7. *Under the conditions of Theorem 2.10, assuming in addition that $\psi_0 \in L^2_1$, the global solution ψ to (1.1) given by Theorem 2.10 satisfies*

$$\Big\langle\psi_t, \Big(\frac{x}{t} - \Theta\Big)^2\psi_t\Big\rangle_{L^2} \lesssim \big(\langle t\rangle^{-2} + \langle t\rangle^{2-\frac{2d}{r}\theta(\frac{r}{q})}\big)\|\psi_0\|^2_{L^2_1} \to 0, \quad \text{as } t \to \infty.$$

Proof. Using (8.15), we rewrite

$$\Big\langle\psi_t, \Big(\frac{x}{t} - \Theta\Big)^2\psi_t\Big\rangle_{L^2} = \frac{1}{t^2}\big\langle e^{it\langle\nabla\rangle}\psi_t, x^2e^{it\langle\nabla\rangle}\psi_t\big\rangle_{L^2}. \tag{8.16}$$

Duhamel's formula gives

$$e^{it\langle\nabla\rangle}\psi_t = \psi_0 - i\int_0^t e^{i\tau\langle\nabla\rangle}(w * |\psi_\tau|^2)\psi_\tau\,\mathrm{d}\tau, \tag{8.17}$$

and we can therefore estimate

$$\left\|e^{it\langle\nabla\rangle}\psi_t\right\|_{L^2_1} \leq \|\psi_0\|_{L^2_1} + \int_0^t \left\|e^{i\tau\langle\nabla\rangle}(w*|\psi_\tau|^2)\psi_\tau\right\|_{L^2_1} \mathrm{d}\tau. \tag{8.18}$$

Applying Lemma B.6 gives

$$\left\|e^{i\tau\langle\nabla\rangle}(w*|\psi_\tau|^2)\psi_\tau\right\|_{L^2_1} \lesssim \left\|(w*|\psi_\tau|^2)\langle x\rangle\psi_\tau\right\|_{L^2} + \langle\tau\rangle\left\|(w*|\psi_\tau|^2)\psi_\tau\right\|_{L^2}.$$

Combining (8.1), (8.3) and (8.4), we then obtain

$$\left\|e^{i\tau\langle\nabla\rangle}(w*|\psi_\tau|^2)\psi_\tau\right\|_{L^2_1} \lesssim \varepsilon_0\langle\tau\rangle^{1-\frac{d}{r}\theta(\frac{r}{q})}\|\psi_0\|_{L^2_1}.$$

Since $\frac{d}{r}\theta(\frac{r}{q}) > 1$, this implies that

$$\int_0^t \left\|e^{i\tau\langle\nabla\rangle}(w*|\psi_\tau|^2)\psi_\tau\right\|_{L^2_1} \mathrm{d}\tau \lesssim \varepsilon_0\langle t\rangle^{2-\frac{d}{r}\theta(\frac{r}{q})}\|\psi_0\|_{L^2_1}.$$

Together with (8.16)–(8.18), this gives the statement of the proposition. □

The previous proposition, combined with functional calculus, implies the following result.

Proposition 8.8. *Under the conditions of Theorem 2.10, assuming in addition that* $\psi_0 \in L^2_1$*, the global solution* ψ *to* (1.1) *given by Theorem 2.10 satisfies*

$$\left\|\Big(f\Big(\frac{x^2}{t^2}\Big) - f(\Theta^2)\Big)\psi_t\right\|_{L^2} \lesssim (\langle t\rangle^{-1} + \langle t\rangle^{1-\frac{d}{r}\theta(\frac{r}{q})})\|\psi_0\|_{L^2_1} \to 0, \quad \textit{as } t\to\infty,$$

for all $f \in \mathcal{C}_0^\infty(\mathbb{R})$.

Proof. Let F be an almost analytic extension of f. This means that $F \in \mathcal{C}_0^\infty(\mathbb{C})$, $F|_{\mathbb{R}} = f$ and, for all $n \in \mathbb{N}$, $|\frac{\partial F}{\partial \bar z}(z)| \leq C_n|\mathrm{Im}(z)|^n$, with $C_n > 0$. Using the Helffer-Sjöstrand representation (see e.g. [16]), we write

$$\begin{aligned}
&f\Big(\frac{x^2}{t^2}\Big) - f(\Theta^2)\\
&= \frac{1}{\pi}\int \frac{\partial F}{\partial \bar z}(z)\Big(\frac{x^2}{t^2} - z\Big)^{-1}\Big(\Theta^2 - \frac{x^2}{t^2}\Big)(\Theta^2 - z)^{-1}\mathrm{dRe}(z)\mathrm{dIm}(z)\\
&= \int \frac{\partial F}{\partial \bar z}(z)\Big(\frac{x^2}{t^2} - z\Big)^{-1}\Big(\Theta + \frac{x}{t}\Big)\cdot\Big(\Theta - \frac{x}{t}\Big)(\Theta^2 - z)^{-1}\mathrm{dRe}(z)d\mathrm{Im}(z)\\
&\quad - \frac{1}{t}\int \frac{\partial F}{\partial \bar z}(z)\Big(\frac{x^2}{t^2} - z\Big)^{-1}[\Theta, x](\Theta^2 - z)^{-1}\mathrm{dRe}(z)\mathrm{dIm}(z),
\end{aligned}$$

where $[\Theta, x] := \sum_j [\Theta_j, x_j]$ is a bounded operator on L^2. In particular, from the properties of the almost analytic extension F, we see that the last term is a bounded operator, with bound $C_f t^{-1}$. For the first term on the right-hand side, we commute $\Theta - \frac{x}{t}$ through $(\Theta^2 - z)^{-1}$, obtaining

$$\int \frac{\partial F}{\partial \bar z}(z)\Big(\frac{x^2}{t^2} - z\Big)^{-1}\Big(\Theta + \frac{x}{t}\Big)\cdot\Big(\Theta - \frac{x}{t}\Big)(\Theta^2 - z)^{-1}\mathrm{dRe}(z)\mathrm{dIm}(z)$$

$$= \int \frac{\partial F}{\partial \bar z}(z)\Big(\frac{x^2}{t^2}-z\Big)^{-1}\Big(\Theta+\frac{x}{t}\Big)\cdot(\Theta^2-z)^{-1}\Big(\Theta-\frac{x}{t}\Big)\mathrm{dRe}(z)\mathrm{dIm}(z)$$
$$+\frac{1}{t}\int \frac{\partial F}{\partial \bar z}(z)\Big(\frac{x^2}{t^2}-z\Big)^{-1}$$
$$\Big(\Theta+\frac{x}{t}\Big)\cdot(\Theta^2-z)^{-1}(\Theta[\Theta,x]+[\Theta,x]\Theta)(\Theta^2-z)^{-1}\mathrm{dRe}(z)\mathrm{dIm}(z).$$

As before, using that Θ and $[\Theta,x]$ are bounded operators on L^2, one easily sees that the last term is a bounded operator, with bound $C_f t^{-1}$. Taking the expectation in ψ_t and using again the properties of the almost analytic extension F, it follows that

$$\Big\|\Big(f\Big(\frac{x^2}{t^2}\Big)-f(\Theta^2)\Big)\psi_t\Big\|_{L^2}\le C_f t^{-1}\|\psi_t\|_{L^2}+C\Big\|\Big(\Theta-\frac{x}{t}\Big)\psi_t\Big\|_{L^2}.$$

Applying Proposition 8.7 concludes the proof. □

8.4 Phase-space and minimal velocity estimates

We are now in position to prove Theorems 2.16 and 2.18. We begin with the following proposition, which implies Theorem 2.16 and provides a further result.

Proposition 8.9. *Let $f,g\in\mathcal{C}_0^\infty(\mathbb{R})$ be such that* $\mathrm{supp}(g)\cap\mathrm{supp}(f)=\emptyset$*. Under the conditions of Theorem 2.10, assuming in addition that $\psi_0\in L^2_1$, the global solution ψ to (1.1) given by Theorem 2.10 satisfies*

$$\Big\|g\Big(\frac{x^2}{t^2}\Big)f(\Theta^2)\psi_t\Big\|_{L^2}\lesssim(\langle t\rangle^{-1}+\langle t\rangle^{1-\frac{d}{r}\theta(\frac{r}{q})})\|\psi_0\|_{L^2_1}. \tag{8.19}$$

Additionally, if ψ_0 is chosen such that the associated scattering state ψ_+ given by Theorem 2.12 satisfies $f(\Theta^2)\psi_+=\psi_+$, then

$$\Big\|g\Big(\frac{x^2}{t^2}\Big)\psi_t\Big\|_{L^2}\lesssim(\langle t\rangle^{-1}+\langle t\rangle^{1-\frac{d}{r}})\|\psi_0\|_{L^2_1}+\langle t\rangle^{1-\frac{d}{r}\theta(\frac{r}{q})}\|\psi_0\|_{H^s\cap H^{s,p'}}. \tag{8.20}$$

Proof. To prove (8.19), it suffices to use that $\mathrm{supp}(g)\cap\mathrm{supp}(f)=\emptyset$, which gives

$$\Big\|g\Big(\frac{x^2}{t^2}\Big)f(\Theta^2)\psi_t\Big\|_{L^2}=\Big\|g\Big(\frac{x^2}{t^2}\Big)\Big(f(\Theta^2)-f\Big(\frac{x^2}{t^2}\Big)\Big)\psi_t\Big\|_{L^2},$$

and then apply Proposition 8.8.

To prove (8.20), we consider a function $\tilde f\in\mathcal{C}_0^\infty(\mathbb{R})$ such that $\tilde f=1$ on $\mathrm{supp}(f)$ and $\mathrm{supp}(\tilde f)\cap\mathrm{supp}(g)=\emptyset$. Then we write

$$\Big\|g\Big(\frac{x^2}{t^2}\Big)\psi_t\Big\|_{L^2}\le\Big\|g\Big(\frac{x^2}{t^2}\Big)\tilde f(\Theta^2)\psi_t\Big\|_{L^2}+\Big\|g\Big(\frac{x^2}{t^2}\Big)(1-\tilde f)(\Theta^2)\psi_t\Big\|_{L^2}.$$

The first term is estimated by (8.19), while for the second term, we have

$$\begin{aligned}\Big\|g\Big(\frac{x^2}{t^2}\Big)(1-\tilde f)(\Theta^2)\psi_t\Big\|_{L^2}&\lesssim\big\|(1-\tilde f)(\Theta^2)\psi_t\big\|_{L^2}\\&=\big\|(1-\tilde f)(\Theta^2)e^{it\langle\nabla\rangle}\psi_t\big\|_{L^2}\\&=\big\|(1-\tilde f)(\Theta^2)(e^{it\langle\nabla\rangle}\psi_t-\psi_+)\big\|_{L^2},\end{aligned}$$

where we used the unitarity of $e^{it\langle\nabla\rangle}$, and that $f(\Theta^2)\psi_+=\psi_+$ in the last inequality. This gives the result by the unitarity of $e^{it\langle\nabla\rangle}$ and Theorem 2.12. □

Using the previous proposition, we can now easily prove Theorem 2.18.

Proof of Theorem 2.18. Let ψ_0 be as in the statement of Theorem 2.18. By Theorem 8.6, we have $\psi_0 \in \mathcal{B}_{H^s_\gamma}(2\delta_w)$ provided that δ_w is small enough. Since $H^s_\gamma \hookrightarrow H^s \cap H^{s,p'}$, we can apply Theorem 2.10 to obtain a solution ψ to (1.1) associated to the initial state ψ_0. Now we can consider $f, g \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\mathrm{supp}(f) \cap \mathrm{supp}(g) = \emptyset$, $\mathbf{1}_{[\alpha,1]} = \mathbf{1}_{[\alpha,1]} f$ and $\mathbf{1}_{[0,\alpha)} = \mathbf{1}_{[0,\alpha)} g$. Theorem 2.18 then follows from Proposition 8.9. □

A Commutation of weights and derivatives

In this appendix we prove the boundedness in L^p spaces of some Fourier multipliers used in the main text, and we next deduce from it the equivalence of the norms $\|f\|_{H^{s,p}_\gamma} = \|\langle x\rangle^\gamma \langle\nabla\rangle^s f\|_{L^p}$ and $\|\langle\nabla\rangle^s \langle x\rangle^\gamma f\|_{L^p}$, for all $0 \le \gamma \le 2$, $s \ge 0$ and $1 \le p \le \infty$. Most of the results recalled here are well-known (see e.g. [48]), especially in the case where $1 < p < \infty$. We provide some details for self-containedness.

We will first need the following representation formulas.

Lemma A.1. *Let $0 \le \alpha < 2$. Then there exists $c_\alpha > 0$ such that*

$$\langle\nabla\rangle^\alpha = \frac{1}{c_\alpha}\int_0^\infty \big(u^{\frac{\alpha}{2}-1} - u^{\frac{\alpha}{2}}(-\Delta+1+u)^{-1}\big)\mathrm{d}u,$$
$$\langle x\rangle^\alpha = \frac{1}{c_\alpha}\int_0^\infty \big(u^{\frac{\alpha}{2}-1} - u^{\frac{\alpha}{2}}(x^2+1+u)^{-1}\big)\mathrm{d}u,$$

as operators on $\mathcal{S}'(\mathbb{R}^d)$.

Proof. Let $b > 0$, then it holds

$$\int_0^\infty \frac{1}{u^{1-\frac{\alpha}{2}}(b+u)}\mathrm{d}u = \int_0^\infty \frac{1}{u^{1-\frac{\alpha}{2}}(1+u/b)}\frac{\mathrm{d}u}{b} = \frac{1}{b^{1-\frac{\alpha}{2}}}\int_0^\infty \frac{1}{t^{1-\frac{\alpha}{2}}(1+t)}\mathrm{d}t$$

from which we obtain

$$b^{\alpha/2} = \frac{b}{c_\alpha}\int_0^\infty \frac{1}{u^{1-\frac{\alpha}{2}}(b+u)}\mathrm{d}u = \frac{1}{c_\alpha}\int_0^\infty \big(u^{\frac{\alpha}{2}-1} - u^{\frac{\alpha}{2}}(b+u)^{-1}\big)\mathrm{d}u$$

where $c_\alpha = \int_0^\infty \frac{1}{t^{1-\alpha/2}(1+t)}\mathrm{d}t$. The statement follows replacing b by $\langle x\rangle$ and $\langle\nabla\rangle$ (working in Fourier space for $\langle\nabla\rangle$). □

Lemma A.2. *Let $1 \le p \le \infty$ and $\lambda, s > 0$, then it holds*

$$\|(-\Delta+\lambda)^{-1}\|_{\mathcal{B}(L^p)} \lesssim \lambda^{-1}, \quad \|\partial_k^m(-\Delta+\lambda)^{-1}\|_{\mathcal{B}(L^p)} \lesssim \lambda^{-m/2}, \quad m = 1,2 \tag{A.1}$$

and

$$\|\langle\nabla\rangle^{-s}\|_{\mathcal{B}(L^p)} \lesssim 1, \quad \|\partial_k\langle\nabla\rangle^{-s-1}\|_{\mathcal{B}(L^p)} \lesssim 1 \tag{A.2}$$

for all $k = 1, \dots, d$. If $1 < p < \infty$, we also have

$$\|\partial_k\langle\nabla\rangle^{-1}\|_{\mathcal{B}(L^p)} \lesssim 1. \tag{A.3}$$

Proof. The properties of the resolvent can be derived by direct computations, using the explicit expression of its kernel, namely

$$(-\Delta+\lambda)^{-1}f = G_\lambda * f, \qquad G_\lambda(x) = \int_0^\infty (4\pi t)^{-\frac{d}{2}} \exp\Big(-\frac{x^2}{4t}-\lambda t\Big)\mathrm{d}t,$$

see e.g. [46, Theorem 6.23]. The first inequality in (A.1) then directly follows from Young's inequality, since

$$\|G_\lambda\|_{L^1} = \int_{\mathbb{R}^d} G_\lambda(x)\mathrm{d}x = (2\pi)^{\frac{d}{2}}\mathcal{F}(G_\lambda)(0) = (2\pi)^{\frac{d}{2}}\lambda^{-1}.$$

When $m=1$, to prove the second bound in (A.1), we estimate

$$\|\partial_k G_\lambda\|_{L^1} \lesssim \int_{\mathbb{R}^d}\int_0^\infty |x_k| t^{-\frac{d}{2}} \exp\Big(-\frac{x^2}{4t}-\lambda t\Big)\frac{\mathrm{d}t}{t}.$$

Changing variables $x=\lambda^{-\frac{1}{2}}y$ next $t=\lambda^{-1}u$, we obtain $\|\partial_k G_\lambda\|_{L^1} \lesssim \lambda^{-\frac{1}{2}}$, which implies the second bound in (A.1) by Young's inequality. The case $m=2$ can be proved similarly by estimating $\|\partial_k^2 G_\lambda\|_{L^1}$.

Now we prove (A.2). For $1\le p\le\infty$, the operator $\langle\nabla\rangle^{-s}$ is bounded on L^p by [32, Corollary 6.1.6]. The proof is based as before on the explicit expression

$$\langle\nabla\rangle^{-s} f = J_s * f, \qquad J_s(x) = \frac{1}{\Gamma(\frac{s}{2})}\int_0^\infty (4\pi t)^{-\frac{d}{2}} \exp\Big(-\frac{|x|^2}{4t}-t\Big) t^{\frac{s}{2}-1}\mathrm{d}t.$$

Using

$$\|J_s\|_{L^1} = \int_{\mathbb{R}^d} J_s(x)\mathrm{d}x = (2\pi)^{\frac{d}{2}}\mathcal{F}(J_s)(0) = (2\pi)^{\frac{d}{2}},$$

we obtain the first bound in (A.2) by Young's inequality. Moreover we can estimate

$$\|\partial_k J_{s+1}\|_{L^1} \lesssim \int_{\mathbb{R}^d}\int_0^\infty |x_j| t^{-\frac{d}{2}} \exp\Big(-\frac{x^2}{4t}-t\Big) t^{\frac{s+1}{2}-2}\mathrm{d}t.$$

Changing variables $x=t^{\frac{1}{2}}y$ in the integral over x and using that $s>0$, we deduce that $\|\partial_k J_{s+1}\|_{L^1}\lesssim 1$, from which the second bound in (A.2) follows by Young's inequality.

Finally, since $1<p<\infty$, (A.3) follows from the Mihlin–Hörmander Multiplier Theorem, see e.g. [33, Theorem 5.2.7]. □

Lemma A.3. *Let $0\le\gamma\le 2$, $s\ge 0$ and $1\le p\le\infty$. Then it holds*

$$\|[\langle x\rangle^\gamma,\langle\nabla\rangle^s]\langle x\rangle^{-\gamma}\langle\nabla\rangle^{-s}\|_{\mathcal{B}(L^p)}\lesssim 1, \quad \|[\langle x\rangle^\gamma,\langle\nabla\rangle^s]\langle\nabla\rangle^{-s}\langle x\rangle^{-\gamma}\|_{\mathcal{B}(L^p)}\lesssim 1.$$

As a direct consequence of the previous lemma we have the following.

Corollary A.4. *Let $0\le\gamma\le 2$, $s\ge 0$ and $1\le p\le\infty$. There exist $c,c'>0$ such that*

$$c'\,\|\langle\nabla\rangle^s\langle x\rangle^\gamma f\|_{L^p} \le \|f\|_{H^{s,p}_\gamma} \le c\,\|\langle\nabla\rangle^s\langle x\rangle^\gamma f\|_{L^p}.$$

Proof of Lemma A.3. We will prove the lemma for the operator $[\langle x\rangle^\gamma,\langle\nabla\rangle^s]\langle x\rangle^{-\gamma}\langle\nabla\rangle^{-s}$, the other case can be proved analogously. We distinguish different cases:

- $s=2$, $\gamma\in\mathbb{R}^+$.

  We compute explicitly the commutator yielding

$$\begin{aligned}[\langle x\rangle^\gamma,\langle\nabla\rangle^2]\langle x\rangle^{-\gamma}\langle\nabla\rangle^{-2} =&[\langle x\rangle^\gamma,-\Delta]\langle x\rangle^{-\gamma}\langle\nabla\rangle^{-2}\\ =&\Delta(\langle x\rangle^\gamma)\langle x\rangle^{-\gamma}\langle\nabla\rangle^{-2}+\langle x\rangle^{-\gamma}\nabla(\langle x\rangle^\gamma)\cdot\nabla\langle\nabla\rangle^{-2}\\ &+\nabla(\langle x\rangle^\gamma)\cdot\nabla(\langle x\rangle^{-\gamma})\langle\nabla\rangle^{-2},\end{aligned}$$

  which is a sum of bounded operators on L^p thanks to Lemma A.2.

- $s\in[0,2)$, $\gamma=2$.

  Applying Lemma A.1 we obtain the expression

$$[\langle x\rangle^2,\langle\nabla\rangle^s]\langle x\rangle^{-2}\langle\nabla\rangle^{-s}=\int_0^\infty u^{s/2}R_{-\Delta}(u)(\Delta(x^2)+2\nabla(x^2)\cdot\nabla)R_{-\Delta}(u)\,\langle x\rangle^{-2}\langle\nabla\rangle^{-s}\,\mathrm{d}u \tag{A.4}$$

  where for a non negative selfadjoint operator A we define $R_A(u)=(A+1+u)^{-1}$. For the second term in the integral we can argue as follows:

$$\begin{aligned}&\Big\|\int_0^\infty u^{s/2}R_{-\Delta}(u)\nabla(x^2)\cdot\nabla R_{-\Delta}(u)\,\langle x\rangle^{-2}\langle\nabla\rangle^{-s}\,\mathrm{d}u\Big\|_{\mathcal{B}(L^p)}\\ &\qquad=2\Big\|\sum_{k=1}^d\int_0^\infty u^{s/2}R_{-\Delta}(u)^2\,x_k\langle x\rangle^{-2}\partial_k\langle\nabla\rangle^{-s}\,\mathrm{d}u\\ &\qquad\qquad+\int_0^\infty u^{s/2}R_{-\Delta}(u)^2\,x_k\partial_k(\langle x\rangle^{-2})\langle\nabla\rangle^{-s}\,du\Big\|_{\mathcal{B}(L^p)}\\ &\qquad\lesssim\int_0^\infty\frac{u^{s/2}}{(1+u)^2}\,\mathrm{d}u\lesssim 1\end{aligned}$$

  where we applied Lemma A.2. Since $\Delta(x^2)=2d$ the corresponding term in (A.4) is treated applying directly Lemma A.2. This and the previous inequality, together with expression (A.4) prove the statement for $s\in[0,2)$ and $\gamma=2$.

- $s,\gamma\in[0,2)$.

  We reason analogously, applying Lemma A.1 to obtain the expression

$$\begin{aligned}[\langle x\rangle^\gamma,\langle\nabla\rangle^s]&=\int_0^\infty t^{\gamma/2}\,u^{s/2}[(x^2+1+t)^{-1},(-\Delta+1+u)^{-1}]\,\mathrm{d}u\,\mathrm{d}t\\ &=\int_0^\infty t^{\gamma/2}\,u^{s/2}R_{x^2}(t)R_{-\Delta}(u)(\Delta(x^2)+2\nabla(x^2)\cdot\nabla)R_{-\Delta}(u)R_{x^2}(t)\,\mathrm{d}u\,\mathrm{d}t.\end{aligned} \tag{A.5}$$

  If $0\le s<1$ we rewrite the contribution of $\nabla(x^2)\cdot\nabla=2x\cdot\nabla$ as

$$\begin{aligned}&R_{x^2}(t)R_{-\Delta}(u)\,x\cdot\nabla R_{-\Delta}(u)R_{x^2}(t)\\ &\qquad=R_{x^2}(t)R_{-\Delta}(u)^2\Big(\sum_{k=1}^d\partial_k R_{x^2}(t)x_k+dR_{x^2}(t)-2R_{-\Delta}(u)\Delta R_{x^2}(t)\Big).\end{aligned} \tag{A.6}$$

Using the inequality

$$\|R_{x^2}(t)x_k\langle x\rangle^{-\gamma}\|_{\mathcal{B}(L^p)} \lesssim \begin{cases}(1+t)^{-1} & 1\le\gamma<2\\ (1+t)^{-1/2} & 0\le\gamma<1\end{cases}$$

as well as Lemma A.2 to estimate (A.6) we obtain

$$\begin{aligned}\Big\|\Big(\int_0^\infty t^{\gamma/2}\,u^{s/2}R_{x^2}(t)R_{-\Delta}(u)\,\nabla(x^2)\cdot\nabla\,R_{-\Delta}(u)R_{x^2}(t)\,\mathrm{d}u\,\mathrm{d}t\Big)\langle x\rangle^{-\gamma}\langle\nabla\rangle^{-s}\Big\|_{\mathcal{B}(L^p)}&\\ \lesssim\begin{cases}\int_0^\infty\frac{t^{\gamma/2}}{(1+t)^2}\mathrm{d}t\int_0^\infty\frac{u^{s/2}}{(1+u)^{3/2}}\mathrm{d}u & 1\le\gamma<2\\ \int_0^\infty\frac{t^{\gamma/2}}{(1+t)^{3/2}}\mathrm{d}t\int_0^\infty\frac{u^{s/2}}{(1+u)^{3/2}}\mathrm{d}u & 0\le\gamma<1\end{cases}&\\ \lesssim 1 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (A.7)&\end{aligned}$$

since we assumed $s<1$. If $1\le s<2$ we commute ∂_k to the right in (A.6) obtaining

$$\begin{aligned}&R_{x^2}(t)R_{-\Delta}(u)\,x\cdot\nabla R_{-\Delta}(u)R_{x^2}(t)\langle x\rangle^{-\gamma}\langle\nabla\rangle^{-s}\\ &=R_{x^2}(t)R_{-\Delta}(u)^2\Big(\sum_{k=1}^d R_{x^2}(t)x_k\langle x\rangle^{-\gamma}\partial_k\langle\nabla\rangle^{-s}+\partial_k(R_{x^2}(t)x_k\langle x\rangle^{-\gamma})\langle\nabla\rangle^{-s}\\ &\qquad\qquad+dR_{x^2}(t)\langle x\rangle^{-\gamma}\langle\nabla\rangle^{-s}-2R_{-\Delta}(u)\Delta R_{x^2}(t)\langle x\rangle^{-\gamma}\langle\nabla\rangle^{-s}\Big).\end{aligned}$$

Since $\|\partial_k(R_{x^2}(t)x_k\langle x\rangle^{-\gamma})\|_{\mathcal{B}(L^p)}\lesssim(1+t)^{-1}$ and applying again Lemma A.2 as before we obtain

$$\begin{aligned}\Big\|\Big(\int_0^\infty u^{\gamma/2}\,t^{s/2}R_{x^2}(u)R_{-\Delta}(t)\,\nabla(x^2)\cdot\nabla\,R_{-\Delta}(t)R_{x^2}(u)\,\mathrm{d}u\,\mathrm{d}t\Big)\langle x\rangle^{-\gamma}\langle\nabla\rangle^{-s}\Big\|_{\mathcal{B}(L^p)}&\\ \lesssim\begin{cases}\int_0^\infty\frac{t^{\gamma/2}}{(1+t)^2}\mathrm{d}t\int_0^\infty\frac{u^{s/2}}{(1+u)^2}\mathrm{d}u & 1\le\gamma<2\\ \int_0^\infty\frac{t^{\gamma/2}}{(1+t)^{3/2}}\mathrm{d}t\int_0^\infty\frac{u^{s/2}}{(1+u)^2}\mathrm{d}u & 0\le\gamma<1\end{cases}&\\ \lesssim 1.&\end{aligned}$$

The term in (A.5) involving $\Delta(x^2)$ can be bounded in an analogous way.

We have therefore proved the statement holds for $\gamma,s\in[0,2]$, we now use this fact to prove the case $s\in(2,4]$. Indeed, we need to prove that

$$[\langle x\rangle^\gamma,\langle\nabla\rangle^s]\langle x\rangle^{-\gamma}\langle\nabla\rangle^{-s}=\langle x\rangle^\gamma\langle\nabla\rangle^s\langle x\rangle^{-\gamma}\langle\nabla\rangle^{-s}-\mathrm{Id}$$

is bounded for $\gamma\in[0,2]$ and $s\in(2,4]$. We rewrite the first operator in the sum as

$$\langle x\rangle^\gamma\langle\nabla\rangle^s\langle x\rangle^{-\gamma}\langle\nabla\rangle^{-s}=\langle x\rangle^\gamma\langle\nabla\rangle^{s-2}\langle x\rangle^{-\gamma}\langle\nabla\rangle^{-s+2}+\langle x\rangle^\gamma\langle\nabla\rangle^{s-2}[-\Delta,\langle x\rangle^{-\gamma}]\langle\nabla\rangle^{-s} \tag{A.8}$$

where the first operator is bounded, since $\gamma,s-2\in[0,2]$. For the second operator, let us first consider $\Delta(\langle x\rangle^{-\gamma})=c\langle x\rangle^{-\gamma-2}+c'\langle x\rangle^{-\gamma-4}$. Let $j=2,4$, then we have

$$\langle x\rangle^\gamma\langle\nabla\rangle^{s-2}\langle x\rangle^{-\gamma-j}\langle\nabla\rangle^{-s}=\big(\langle x\rangle^\gamma\langle\nabla\rangle^{s-2}\langle x\rangle^{-\gamma}\langle\nabla\rangle^{-s+2}\big)\big(\langle\nabla\rangle^{s-2}\langle x\rangle^{-j}\langle\nabla\rangle^{-s}\big)$$

where again $\langle x\rangle^\gamma\langle\nabla\rangle^{s-2}\langle x\rangle^{-\gamma}\langle\nabla\rangle^{-s+2}$ is bounded from the previous step and it holds

$$\langle\nabla\rangle^{s-2}\langle x\rangle^{-2}\langle\nabla\rangle^{-s}=\langle x\rangle^{-2}(\langle x\rangle^2\langle\nabla\rangle^{s-2}\langle x\rangle^{-2}\langle\nabla\rangle^{-s+2})\langle\nabla\rangle^{-2},$$

which is again a composition of bounded operators on L^p, and similarly for

$$\langle\nabla\rangle^{s-2}\langle x\rangle^{-4}\langle\nabla\rangle^{-s} = [\langle x\rangle^{-2}(\langle x\rangle^{2}\langle\nabla\rangle^{s-2}\langle x\rangle^{-2}\langle\nabla\rangle^{-s+2})]^2\langle\nabla\rangle^{-2}.$$

Hence $\langle x\rangle^{\gamma}\langle\nabla\rangle^{s-2}\Delta(\langle x\rangle^{-\gamma})\langle\nabla\rangle^{-s}$ is a bounded operator on L^p and the same can be proved for $\langle x\rangle^{\gamma}\langle\nabla\rangle^{s-2}\nabla(\langle x\rangle^{-\gamma})\cdot\nabla\langle\nabla\rangle^{-s}$ with similar computations. These facts, combined with the expression given in (A.8) prove the statement for $s\in(2,4]$. For larger s we reason analogously. □

B Time decay for the free propagator

In this appendix we consider the linear flow $e^{-it\langle\nabla\rangle}$. First, in Subsection B.1, we recall the pointwise time-decay estimates in L^p-spaces that were used in Section 6. We also formulate pointwise time-decay estimates in Besov-Lorentz spaces. In Subsection B.2, we deduce from the latter the Strichartz estimates in Lorentz spaces that were used in Section 5. Finally, in Subsection B.3, we derive the pointwise estimates in weighted L^p spaces that were used in Section 8.

B.1 Pointwise estimates in L^p spaces

We will use the following lemma due to Hörmander [37]. In what follows we denote by $\mathcal{F}$ the unitary Fourier transform on L^2.

Lemma B.1. *Let $\phi:\mathbb{R}^d\to\mathbb{C}$ be such that $\mathcal{F}\phi\in C^\infty(\mathbb{R}^d)$ and*

$$|\nabla^\alpha\mathcal{F}\phi(\xi)|\le C_\alpha\langle\xi\rangle^{-\frac{d}{2}-1-|\alpha|}\quad\forall\,\xi\in\mathbb{R}^d,\alpha\in\mathbb{N}^d,$$

for some $C_\alpha>0$. Then

$$\left|e^{-it\langle\nabla\rangle}\phi(x)\right|\le c_m(|t|+|x|)^{-\frac{d}{2}}\left(1+(|x|^2-t^2)\mathbf{1}_{|x|\ge t}\right)^{-m},\tag{B.1}$$

for all $t\in\mathbb{R}$, $x\in\mathbb{R}^d$, $m\in\mathbb{N}$ and for some $c_m>0$. In particular, for all $2\le p\le\infty$, $\gamma\ge0$ and $t\neq0$,

$$\|e^{-it\langle\nabla\rangle}\phi\|_{L^p_\gamma}\lesssim\langle t\rangle^{\gamma-\frac{d}{2r}},\tag{B.2}$$

with $p=\frac{2r}{r-1}$.

Proof. The estimate (B.1) is proven in [37, Theorem 7.2.1]. (B.2) directly follows from (B.1). □

We introduce the notations

$$\phi_d:=\mathcal{F}^{-1}\big(\langle\xi\rangle^{-s_d}\big),\qquad s_d:=\frac{d}{2}+1.$$

The following provides a useful representation of $e^{-it\langle\nabla\rangle}f$ for f regular enough. The proof is straightforward.

Lemma B.2. *Let $f\in H^{s_d,p'}$ for some $1\le p'\le2$. Then, for all $t\in\mathbb{R}$,*

$$e^{-it\langle\nabla\rangle}f=\big(e^{-it\langle\nabla\rangle}\phi_d\big)*\big(\langle\nabla\rangle^{s_d}f\big).\tag{B.3}$$

Proof. We rewrite f as

$$f = \langle\nabla\rangle^{-s_d}\langle\nabla\rangle^{s_d} f = \mathcal{F}^{-1}[\langle\xi\rangle^{-s_d}\mathcal{F}(\langle\nabla\rangle^{s} f)] = \phi_d * (\langle\nabla\rangle^{s_d} f).$$

Hence, observing that the convolution product is well-defined by Lemma B.1, we obtain (B.3). □

Using the previous two lemmas, we have the following estimate.

Lemma B.3. *Let $s \geq s_d = d/2+1$ and $1 \leq r \leq \infty$. Then, for all $f \in H^{s,p'} \cap H^s$ and $t \geq 0$,*

$$\|e^{-it\langle\nabla\rangle} f\|_{L^p\cap L^\infty} \lesssim \langle t\rangle^{-\frac{d}{2r}} \|f\|_{H^{s,p'}\cap H^s},$$

with $p = \frac{2r}{r-1}$.

Proof. Since $H^{s,p'} \cap H^s \subset H^{s_d,p'} \cap H^{s_d}$, it suffices to prove the lemma for $s = s_d$. For $t \leq 1$, using Sobolev's embedding $H^s \hookrightarrow L^p \cap L^\infty$ (since $p \geq 2$ and $s > \frac{d}{2}$) and the unitarity of $e^{-it\langle\nabla\rangle}$ in H^s, we have

$$\|e^{-it\langle\nabla\rangle} f\|_{L^p\cap L^\infty} \lesssim \|f\|_{H^s}. \tag{B.4}$$

For $t > 1$, using Lemma B.2 and the fact that ϕ_d satisfies the conditions of Lemma B.1, we obtain

$$\begin{aligned}\|e^{-it\langle\nabla\rangle} f\|_{L^\infty} &= \left\|\left(e^{-it\langle\nabla\rangle}\phi_d\right) * \left(\langle\nabla\rangle^s f\right)\right\|_{L^\infty} \\ &\lesssim \left\|e^{-it\langle\nabla\rangle}\phi_d\right\|_{L^p}\left\|\langle\nabla\rangle^s f\right\|_{L^{p'}} \lesssim |t|^{-\frac{d}{2r}}\|f\|_{H^{s,p'}}.\end{aligned} \tag{B.5}$$

Interpolating the inequality

$$\|e^{-it\langle\nabla\rangle} f\|_{L^\infty} \lesssim |t|^{-d/2}\|\langle\nabla\rangle^s f\|_{L^1},$$

(obtained from the previous bound for $p = \infty$), with

$$\|e^{-it\langle\nabla\rangle} f\|_{L^2} = \|f\|_{L^2} \leq \|\langle\nabla\rangle^s f\|_{L^2},$$

we obtain, using e.g. [33, Theorem 1.3.4],

$$\|e^{-it\langle\nabla\rangle} f\|_{L^a} \lesssim |t|^{-d/2(1-\theta)}\|\langle\nabla\rangle^s f\|_{L^b}, \quad \frac{1}{a} = \frac{\theta}{2}, \quad \frac{1}{b} = \frac{1-\theta}{1} + \frac{\theta}{2}, \quad 0 \leq \theta \leq 1.$$

Setting $\theta = 1 - \frac{1}{r}$ yields

$$\|e^{-it\langle\nabla\rangle} f\|_{L^p} \lesssim |t|^{-\frac{d}{2r}}\|\langle\nabla\rangle^s f\|_{L^{p'}} \tag{B.6}$$

Combining (B.4), (B.5) and (B.6) leads to the statement of the lemma. □

One can formulate a more precise time-decay estimate in Besov-Lorentz spaces as follows. For $s \in \mathbb{R}$, $1 \leq p, q < \infty$ and $1 \leq r \leq \infty$, we denote by $B^s_{(p,r),q}$ the Besov-Lorentz space associated to the norm

$$\|f\|_{B^s_{(p,r),q}} = \Big(\sum_{k\in\mathbb{N}_0} 2^{ksq}\|\Lambda_k f\|^q_{L^{p,r}}\Big)^{\frac{1}{q}},$$

where $(\Lambda_k)_{k\in\mathbb{N}_0}$ stands for a Littlewood-Paley decomposition, see [54] for more details.

Lemma B.4. *Let $2 \leq b \leq \infty$ and $0 \leq \beta \leq 1$. Then*

$$\|e^{-it\langle\nabla\rangle} f\|_{B'} \lesssim |t|^{-(d-1+\beta)(\frac{1}{2}-\frac{1}{b})}\|f\|_B,$$

where B stands for the Besov-Lorentz space $B = B^s_{(b',2),2}$, and $B' = B^{-s}_{(b,2),2}$, with $s = \frac{1}{2}(d + 1 + \beta)(\frac{1}{2} - \frac{1}{b})$.

Proof. It suffices to modify the end of the proof of [30, Lemma 2.1] (see also [5]) in a straightforward way, using real interpolation $L^{b,2} = [L^1, L^\infty]_{1/b',2}$. □

B.2 Strichartz estimates

Lemma B.4 implies the following Strichartz estimates. Compared to the estimates previously used in the literature, see e.g. [14, 49], a difference here is that we need a Strichartz estimate in Lorentz space. On the other hand, we do not consider the endpoint case $a = 2$, $b = 2(d-1+\beta)(d-3+\beta)^{-1}$ since we do not need it in our application.

In the following statement, $\beta = 0$ corresponds to the usual wave admissible pairs and $\beta = 1$ to Schrödinger admissible pairs. We only consider here the estimates that were used in Section 5, we do not state more general estimates that could be deduced from Lemma B.4.

Proposition B.5. *Let $a < 2 \leq \infty$, $0 \leq \beta \leq 1$ and $2 \leq b < 2(d-1+\beta)(d-3+\beta)^{-1}$ be such that*

$$\frac{2}{a} + \frac{d-1+\beta}{b} = \frac{d-1+\beta}{2}.$$

Then

$$\|e^{-it\langle\nabla\rangle} f\|_{L^a_t L^{b,2}_x} \lesssim \|f\|_{H^s}, \tag{B.7}$$

for any $s \geq \frac{1}{2} + \frac{1}{a} - \frac{1}{b}$ and

$$\Big\| \int_0^t e^{-i(t-\tau)\langle\nabla\rangle} f(\tau, x) \mathrm{d}\tau \Big\|_{L^a_t L^{b,2}_x} \lesssim \|f\|_{L^1_t H^s_x}. \tag{B.8}$$

Proof. Recall that B stands for the Besov-Lorentz space $B^s_{(b',2),2}$. Applying [39, Theorem 10.1] (with $B_0 = H = L^2$ and $B_1 = B$), we obtain from Lemma B.4 that

$$\|e^{-it\langle\nabla\rangle} f\|_{L^a_t B'} \lesssim \|f\|_{L^2}, \qquad a = 2\Big[(d-1+\beta)(\frac{1}{2} - \frac{1}{b})\Big]^{-1},$$

provided that $a > 2$. Equivalently

$$\|e^{-it\langle\nabla\rangle} f\|_{L^a_t B^0_{(b,2),2}} \lesssim \|f\|_{H^s},$$

with $s = \frac{1}{2}(d+1+\beta)(\frac{1}{2} - \frac{1}{b})$. Since, by [54, Theorem 1.1], we have that $B^0_{(b,2),2} \hookrightarrow L^{b,2}$, we deduce from the previous inequality that (B.7) holds. The estimate (B.8) follows in the same way. □

B.3 Pointwise estimates in weighted L^p spaces

The next lemma was used in Section 8.2. Recall the notation $\Theta := -i\nabla\langle\nabla\rangle^{-1}$.

Lemma B.6. *Let $0 \leq \gamma \leq 2$. Then for all $\varphi \in L^2_\gamma$,*

$$\|e^{-it\langle\nabla\rangle}\varphi\|_{L^2_\gamma} \lesssim \|\varphi\|_{L^2_\gamma} + t^\gamma \|\varphi\|_{L^2}. \tag{B.9}$$

Proof. From an explicit computation we obtain the expression

$$e^{it\langle\nabla\rangle} x e^{-it\langle\nabla\rangle} = x + t\Theta,$$

which gives the statement when $\gamma = 1$. Moreover, this relation further implies $x^2 e^{-it\langle\nabla\rangle} = e^{-it\langle\nabla\rangle}(x+t\Theta)^2$. Since $\langle x\rangle^2 = 1 + x^2$, we have

$$\|e^{-it\langle\nabla\rangle}\varphi\|_{L^2_2} \leq \|\varphi\|_{L^2} + \|(x+t\Theta)^2\varphi\|_{L^2}, \tag{B.10}$$

with $(x+t\Theta)^2 = x^2 + t^2\Theta^2 + t(\Theta\cdot x + x\cdot\Theta) = x^2 + t^2\Theta^2 + 2t\Theta\cdot x + t[x,\Theta]$, where we use the notation $[x,\Theta] = \sum_{k=1}^d [x_k,\Theta_k]$. Since Θ_k and $[x_k,\Theta_k]$ are bounded operators in L^2, we deduce that

$$\big\|e^{-it\langle\nabla\rangle}\varphi\|_{L^2_2} \lesssim \|\langle x\rangle^2\varphi\|_{L^2} + t^2\|\varphi\|_{L^2} \lesssim \|(\langle x\rangle^2 + t^2)\varphi\|_{L^2}. \tag{B.11}$$

Hence we have the bounded operators

$$e^{-it\langle\nabla\rangle} : L^2(\mathrm{d}x) \to L^2(\mathrm{d}x) \qquad e^{-it\langle\nabla\rangle} : L^2((\langle x\rangle^2 + t^2)^2\mathrm{d}x) \to L^2(\langle x\rangle^4\mathrm{d}x)$$

By an application of the Riesz-Thorin theorem (see [60, Theorem 2.11]) we obtain that $e^{-it\langle\nabla\rangle} : L^2((\langle x\rangle^2 + t^2)^{2\theta}\mathrm{d}x) \to L^2(\langle x\rangle^{4\theta}\mathrm{d}x)$ is a bounded operator for all $\theta \in (0,1)$, from which the statement follows since $(\langle x\rangle^2 + t^2)^\theta \lesssim \langle x\rangle^{2\theta} + t^{2\theta}$. □

Lemma B.7. *Let $s \geq d/2+1$, $1 \leq r \leq \infty$ and $0 \leq \gamma \leq 2$. Then, for all $f \in H^s_\gamma \cap H^{s,p'}$ and $t \geq 0$,*

$$\|e^{-it\langle\nabla\rangle} f\|_{L^p_\gamma \cap L^\infty_\gamma} \lesssim \|f\|_{H^s_\gamma} + \langle t\rangle^{\gamma - \frac{d}{2r}}\|f\|_{H^{s,p'}\cap H^s},$$

with $p = \frac{2r}{r-1}$.

Proof. For $t \leq 1$, it suffices to use Sobolev's embedding $H^s \hookrightarrow L^p \cap L^\infty$ together with Corollary A.4 and Lemma B.6. Suppose that $t > 1$. Using Lemma B.2, we write

$$\begin{aligned}\big|\langle x\rangle^\gamma e^{-it\langle\nabla\rangle} f\big| &= \langle x\rangle^\gamma\big|\big(e^{-it\langle\nabla\rangle}\phi_d\big) * \big(\langle\nabla\rangle^s f\big)\big| \\ &\lesssim \big(\langle x\rangle^\gamma\big|e^{-it\langle\nabla\rangle}\phi_d\big|\big) * \big|\langle\nabla\rangle^s f\big| + \big|e^{-it\langle\nabla\rangle}\phi_d\big| * \big(\langle x\rangle^\gamma\big|\langle\nabla\rangle^s f\big|\big).\end{aligned}$$

This implies, for the L^∞-norm,

$$\begin{aligned}\big\|e^{-it\langle\nabla\rangle} f\big\|_{L^\infty_\gamma} &\lesssim \big\|e^{-it\langle\nabla\rangle}\phi_d\big\|_{L^p_\gamma}\big\|f\big\|_{H^{s,p'}} + \big\|e^{-it\langle\nabla\rangle}\phi_d\big\|_{L^2}\big\|f\big\|_{H^s_\gamma} \\ &\lesssim \langle t\rangle^{\gamma-\frac{d}{2r}}\big\|f\big\|_{H^{s,p'}} + \big\|f\big\|_{H^s_\gamma},\end{aligned}$$

where we used Lemma B.1 in the second inequality.

Now we consider the $\|\cdot\|_{L^p}$ norm for $2 \leq p < \infty$. First, for $\gamma = 2$, we have

$$\begin{aligned}\|x^2 f\|_{L^p} &\leq \big\|(x^2 + t^2\Theta^2)f\|_{L^p} + t^2\|\Theta^2 f\|_{L^p} \\ &\leq \big\|(x - t\Theta)^2 f\big\|_{L^p} + 2t\|(x\cdot\Theta + \Theta\cdot x)f\|_{L^p} + t^2\|\Theta^2 f\|_{L^p} \\ &\leq \big\|(x - t\Theta)^2 f\big\|_{L^p} + 2t\|(\Theta\cdot x)f\|_{L^p} + t\|[x,\Theta]f\|_{L^p} + t^2\|\Theta^2 f\|_{L^p} \\ &\lesssim \big\|(x - t\Theta)^2 f\big\|_{L^p} + t^2\|f\|_{L^p},\end{aligned}$$

where in the last inequality we used that Θ, $[x,\Theta]$ and $\Theta^2 = \mathrm{Id} - (-\Delta+1)^{-1}$ are bounded operators from L^p to L^p by Lemma A.2 (using that $2 \leq p < \infty$ for Θ). Using, similarly as before, that $\langle x - t\Theta\rangle^2 e^{-it\langle\nabla\rangle} = e^{-it\langle\nabla\rangle}\langle x\rangle^2$, this yields

$$\begin{aligned}\big\|e^{-it\langle\nabla\rangle} f\big\|_{L^p_2} &\lesssim \big\|\langle x - t\Theta\rangle^2 e^{-it\langle\nabla\rangle} f\big\|_{L^p} + t^2\big\|e^{-it\langle\nabla\rangle} f\big\|_{L^p} \\ &= \big\|e^{-it\langle\nabla\rangle}\langle x\rangle^2 f\big\|_{L^p} + t^2\big\|e^{-it\langle\nabla\rangle} f\big\|_{L^p} \\ &\lesssim \big\|f\big\|_{H^s_2} + |t|^{2-\frac{d}{2r}}\big\|f\big\|_{H^{s,p'}},\end{aligned}$$

where we used Sobolev's embedding $H^s \hookrightarrow L^p$, Lemma 3.3 and (B.6) in the last inequality. In order to interpolate later on, we rewrite this as

$$\left\| \langle x \rangle^2 e^{-it\langle\nabla\rangle} \langle\nabla\rangle^{-s} f \right\|_{L^p} \lesssim \|f\|_{B_0\cap B_1}, \tag{B.12}$$

where we have set

$$B_0 := L^2_2 = L^2(\langle x\rangle^4 \mathrm{d}x), \qquad B_1 := L^{p'}(|t|^{p'(2-\frac{d}{2r})}\mathrm{d}x).$$

Next we interpolate (B.12) with the inequality obtained for $\gamma = 0$. The argument may be well-known, we provide some details for the convenience of the reader. From the proof of Lemma B.3 and Sobolev's embedding $H^s \hookrightarrow L^p$, we deduce that

$$\|e^{-it\langle\nabla\rangle} f\|_{L^p} \lesssim \min\left(|t|^{-\frac{d}{2r}}\|f\|_{H^{s,p'}}, \|f\|_{H^s}\right),$$

which in turn implies that

$$\|e^{-it\langle\nabla\rangle}\langle\nabla\rangle^{-s} f\|_{L^p} \lesssim \|f\|_{A_0+A_1}, \tag{B.13}$$

where we have set

$$A_0 := L^2(\mathrm{d}x), \qquad A_1 := L^{p'}(|t|^{-\frac{d}{2r}p'}\mathrm{d}x).$$

By real interpolation, we deduce from (B.12), (B.13) and [3, Theorem 5.4.1] that, for all $0 \le \gamma \le 2$,

$$\left\| \langle x \rangle^\gamma e^{-it\langle\nabla\rangle} \langle\nabla\rangle^{-s} f \right\|_{L^p} \lesssim \|f\|_{[B_0\cap B_1; A_0+A_1]_{\gamma,p}}. \tag{B.14}$$

Now since A_0, A_1, B_0, B_1 are all Banach function lattices, we have

$$\begin{aligned}[B_0\cap B_1; A_0+A_1]_{\gamma,p} &= [B_0; A_0+A_1]_{\gamma,p} \cap [B_1; A_0+A_1]_{\gamma,p} \\ &= \left([B_0;A_0]_{\gamma,p} + [B_0;A_1]_{\gamma,p}\right) \cap \left([B_1;A_0]_{\gamma,p} + [B_1;A_1]_{\gamma,p}\right),\end{aligned}$$

(see [50] for the commutativity between interpolation and intersection, and use duality [3, Theorem 2.7.1] for the commutativity between interpolation and sum). Hence

$$\|f\|_{[B_0\cap B_1;A_0+A_1]_{\gamma,p}} \lesssim \|f\|_{[B_0;A_0]_{\gamma,p}} + \|f\|_{[B_1;A_1]_{\gamma,p}}. \tag{B.15}$$

Since $p \ge 2$, using again [3, Theorem 5.4.1],

$$\|f\|_{[B_0;A_0]_{\gamma,p}} \lesssim \|f\|_{[B_0;A_0]_{\gamma,2}} \lesssim \|\langle x\rangle^\gamma f\|_{L^2}, \tag{B.16}$$

and likewise, since $p \ge p'$,

$$\|f\|_{[B_1;A_1]_{\gamma,p}} \lesssim \|f\|_{[B_1;A_1]_{\gamma,p'}} \lesssim |t|^{\gamma-\frac{d}{2r}}\|f\|_{L^{p'}}. \tag{B.17}$$

Putting together (B.14) and (B.15)–(B.17) concludes the proof. □

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