QUANTUM POINT CHARGES INTERACTING WITH QUASI-CLASSICAL ELECTROMAGNETIC FIELDS

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Abstract. We study effective models describing systems of quantum particles interacting with quantized (electromagnetic) fields in the quasi-classical regime, i.e., when the field's state shows a large average number of excitations. Once the field's degrees of freedom are traced out on factorized states, the reduced dynamics of the particles' system is described by an effective Schrödinger operator keeping track of the field's state. We prove that, under suitable assumptions on the latter, such effective models are well-posed even if the particles are point-like, that is no ultraviolet cut-off is imposed on the interaction with quantum fields.

1. Introduction

It is widely known that models describing quantum particles in interaction with quantized fields are ill-posed if the former are assumed to be point-like or, equivalently, no ultraviolet cut-off is imposed on the interaction with the fields [Spo04]. A typical and paradigmatic example is provided by the Nelson model [Nel64], where nucleons are linearly coupled to a scalar quantized field: the ultraviolet regularization can in this case be removed up to the extraction of an infinite self-energy and a suitable renormalization procedure [Nel64]. A more relevant model is the Pauli-Fierz (PF) Hamiltonian [PF38], describing quantum particles interacting with the electromagnetic radiation, which is well-posed only if the large frequencies of the radiation are suitably cut off. The removal of such ultraviolet cut-off is one of the major open problems in non-relativistic Quantum ElectroDynamics (QED) (see, *e.g.*, [Spo04, §19.3]). In this work, we aim at tackling this problem in the *quasi-classical regime* recently introduced in [CF18, CCFO21, CFO19, CFO23b, CFO23a].

The quasi-classical regime consists of an average number of field's excitations that is much larger than 1 (we use natural units in which $\hbar = 1$): more precisely, a quasi-classical field state Ψ_ε satisfies

$$\langle \mathcal{N} \rangle_{\Psi_\varepsilon} \simeq \tfrac{1}{\varepsilon}, \qquad \text{for } 0 < \varepsilon \ll 1,$$

where

$$\mathcal{N} = \int a^\dagger(k) a(k) \mathrm{d}k$$

is the number of the field's excitations. When this is the case, we can consider the commutator between the canonical variables $[a(k), a^\dagger(k')] = \delta(k - k')$ to be negligible and introduce rescaled variables $a_\varepsilon^\sharp := \sqrt{\varepsilon} a^\sharp$, so that

$$\left[a_\varepsilon(k), a_\varepsilon^\dagger(k')\right] = \varepsilon \delta(k - k'). \tag{1.1}$$

Such variables are the ones we are going to use throughout the paper. Their semiclassical nature is made apparent in the vanishing of the commutator as $\varepsilon \to 0$, so that the field can be well approximated by its classical counterpart.

Let us specify now the setting in more details. We consider a bipartite quantum system whose space of states is

$$\mathscr{H} := L^2(\mathbb{R}^3; \mathbb{C}^{2s+1}) \otimes \mathscr{F}_\varepsilon, \tag{1.2}$$

where $\mathscr{F}_\varepsilon$ is a suitable Fock space describing the field's degrees of freedom and $s \in \frac{1}{2}\mathbb{N}$ stands for the spin of the particle. We are assuming for simplicity that there is a single quantum particle interacting with the field but the model can be easily generalized to many-body systems. The Hamiltonian for the full system is denoted by $\mathbb{H}_\varepsilon$ and contains a non-trivial interaction term, i.e., not factorized: we aim to address models of non-relativistic QED and therefore the PF Hamiltonian, but for the sake of providing a simpler and pedagogical example we will also discuss the Nelson Hamiltonian. However, the general scheme is independent of the specific details of the Hamiltonian $\mathbb{H}_\varepsilon$. Our main goal is indeed to study the reduced operators obtained by tracing out the field's degrees of freedom on a product state of the form

$$\psi \otimes \Psi_\varepsilon \in \mathscr{H},$$

where $\Psi_\varepsilon \in \mathscr{F}_\varepsilon$ is a quasi-classical state in the sense specified above. We thus consider the quadratic form

$$\mathcal{Q}_\varepsilon[\psi] := \langle \psi \otimes \Psi_\varepsilon | \mathbb{H}_\varepsilon | \psi \otimes \Psi_\varepsilon \rangle_{\mathscr{H}} - \langle \psi \otimes \Psi_\varepsilon | \mathrm{d}\Gamma_\varepsilon(\omega) | \psi \otimes \Psi_\varepsilon \rangle_{\mathscr{H}}, \tag{1.3}$$

and study its limit as $\varepsilon \to 0$, where $\mathrm{d}\Gamma_\varepsilon(\omega)$ stands for the second quantization of the dispersion relation ω.

More precisely, we are going to show that, while the operator $\mathbb{H}_\varepsilon$ is in general well defined only in presence of a suitable ultraviolet regularization, the form $\mathcal{Q}_\varepsilon[\psi]$ is well posed even without such an ultraviolet cut-off, provided the field's state Ψ_ε is regular enough. Then, under the same assumptions on Ψ_ε, we prove that the quadratic form $\mathcal{Q}_\varepsilon$ converges as $\varepsilon \to 0$ to a quadratic form $\mathcal{Q}_\mu$ depending on a classical Wigner measure μ on the one-excitation space of the field:

$$\mathcal{Q}_\varepsilon \xrightarrow[\varepsilon \to 0]{\Gamma} \mathcal{Q}_\mu,$$

where the convergence is in the sense of $\Gamma-$convergence of functionals [DM93]. Furthermore, $\mathcal{Q}_\mu$ uniquely identifies a self-adjoint Schrödinger operator H_μ characterizing the particle's reduced dynamics. Since (weak and strong) $\Gamma-$convergence of quadratic forms is equivalent [DM93, §13] to strong resolvent convergence of the associated operators, we deduce that the reduced particle's dynamics converges as $\varepsilon \to 0$ to the one generated by H_μ. In addition, if the particle is trapped, i.e., a confining potential is present, then the convergence of the generators is lifted to norm resolvent sense. We already point out that in the case of the PF model, in order to prove the above convergence, we will have to perform a vacuum renormalization and remove some energy diverging as $\varepsilon \to 0$, or, equivalently, consider the normal ordered version of the PF Hamiltonian $\mathbb{H}_\varepsilon$. One of the tools we use to handle quadratic forms without ultraviolet cutoff is the use of suitable Lorentz spaces, in a similar fashion as in the works [BFP23, BFP24].

Organisation of the paper. We present the statements of the main results in § 2. § 3 is devoted to some functional inequalities in Lorentz spaces along with the semiclassical analysis framework used along the paper. § 4 presents the proofs of our results for the Nelson model while § 5 contains the proofs of our results for the Pauli-Fierz model. Finally, in Appendix A, we prove a Γ-convergence result which we need in the core of the article.

2. Main Results

We present here our main results. In order to provide a precise statement we have first to address the notion of convergence in the quasi-classical limit and provide a definition of the Wigner measures associated to (families of) field states Ψ_ε. Then, we first state the results concerning the Nelson model and next discuss non-relativistic QED.

We recall that the Hilbert space on which $\mathbb{H}_\varepsilon$ acts is $L^2(\mathbb{R}^3;\mathbb{C}^{2s+1})\otimes\mathscr{F}_\varepsilon$, where $\mathscr{F}_\varepsilon$ is the symmetric Fock space constructed over the one-excitation space $\mathfrak{h}^\sharp$, i.e.,

$$\mathscr{F}_\varepsilon = \Gamma_\varepsilon(\mathfrak{h}^\sharp) = \bigoplus_{n\in\mathbb{N}} \mathfrak{h}^{\sharp^{\otimes_s n}}, \tag{2.1}$$

where

$$\mathfrak{h}^{\mathrm{Nel}} := L^2(\mathbb{R}^3), \qquad \mathfrak{h}^{\mathrm{PF}} := L^2(\mathbb{R}^3;\mathbb{C}^2). \tag{2.2}$$

The Hamiltonian of the complete system is assumed to have the formal structure

$$\mathbb{H}_\varepsilon^\sharp = H_0^\sharp + \mathrm{d}\Gamma_\varepsilon(\omega) + \mathbb{H}_{I,\varepsilon}^\sharp, \tag{2.3}$$

where $H_0^\sharp$ is the particle's Hamiltonian acting non-trivially only on $L^2(\mathbb{R}^3;\mathbb{C}^{2s+1})$, Γ_ε the second quantization map with canonical variables satisfying (1.1), ω the field's dispersion relation and $\mathbb{H}_{I,\varepsilon}^\sharp$ a non-factorized operators describing the particle-field interaction. Here and in the sequel, in the case where $\mathfrak{h}^\sharp = \mathfrak{h}^{\mathrm{Nel}}$, we identify a map $\omega^\alpha : \mathbb{R}^3 \to \mathbb{R}$, for $\alpha\in\mathbb{R}$, with the operator of multiplication by ω^α on $L^2(\mathbb{R}^3)$. Likewise, in the case where $\mathfrak{h}^\sharp = \mathfrak{h}^{\mathrm{PF}}$, we identify ω^α with the operator $\omega^\alpha\,\mathbb{1}_{\mathbb{C}^2}$ on $L^2(\mathbb{R}^3;\mathbb{C}^2)$. We then recall that the operator $\mathrm{d}\Gamma_\varepsilon(\omega^\alpha)$ in (2.3) is defined by

$$\mathrm{d}\Gamma_\varepsilon(\omega^\alpha)\big|_{\mathfrak{h}^{\sharp\otimes_s n}} = \varepsilon\sum_{k=1}^n \mathbb{1}_{\mathfrak{h}^{\sharp\otimes k-1}}\otimes\omega^\alpha\otimes\mathbb{1}_{\mathfrak{h}^{\sharp\otimes n-k}}.$$

Concerning the dispersion relation ω, we are going to assume the following properties, which are satisfied by the typical choices $\omega(k)=|k|$ or $\omega(k)=\sqrt{k^2+m^2}$.

Assumption (A_ω). *The map $\omega:\mathbb{R}^3\to\mathbb{R}_+$ is measurable, it grows at least linearly, i.e.,*

$$\liminf_{|k|\to+\infty}\frac{\omega(k)}{|k|} > 0, \tag{2.4}$$

and it admits an (unbounded) inverse ω^{-1} with dense domain $\mathscr{D}(\omega^{-1})\subset\mathfrak{h}^\sharp$.

2.1. **Quasi-classical limit.** We recall some facts on semiclassical measures (see, *e.g.*, [AN08, AN09, AN11, AN15, Fal18b]). Analogously to the notation introduced above, $\mathfrak{h}^\sharp_\omega$ denote the one-excitation spaces

$$\mathfrak{h}^{\mathrm{Nel}}_\omega := L^2(\mathbb{R}^3, \omega(k)\mathrm{d}k), \qquad \mathfrak{h}^{\mathrm{PF}}_\omega := L^2(\mathbb{R}^3, \omega(k)\mathrm{d}k; \mathbb{C}^2). \tag{2.5}$$

More generally, we define the spaces $\mathfrak{h}^\sharp_{\omega^\alpha}$ as the weighted L^2-spaces with weight ω^α, $\alpha \in \mathbb{R}$. Observe that for all $\alpha \in \mathbb{R}$, $\mathfrak{h}^\sharp \cap \mathfrak{h}^\sharp_{\omega^\alpha}$ is dense in both $\mathfrak{h}^\sharp$ and $\mathfrak{h}^\sharp_{\omega^\alpha}$ thanks to Assumption (A_ω). The Weyl operator associated to Nelson and Pauli-Fierz fields reads

$$W_\varepsilon(z) := e^{i(a^\dagger_\varepsilon(z)+a_\varepsilon(z))} \tag{2.6}$$

for $z \in \mathfrak{h}^\sharp$. To shorten the notation, in the following we will omit the label Nel/PF distinguishing between the Nelson and Pauli-Fierz models, when the statement applies to both cases. We denote by $\mathscr{P}(\mathfrak{h})$ the set of Borel probability measures on $\mathfrak{h}$.

Definition 2.1 (Semiclassical convergence)**.**
Given a family of normalized microscopic states $\{\Psi_\varepsilon\}_{\varepsilon\in(0,1)} \subset \mathscr{F}_\varepsilon$, let us define the associated set of quasi-classical $\mathfrak{h}_{\omega^\alpha}$-Wigner measures $\mathscr{M}_{\omega^\alpha}(\Psi_\varepsilon) \subset \mathscr{P}(\mathfrak{h}_{\omega^\alpha})$, $\alpha \in \mathbb{R}$, as the subset of all probability measures μ, such that $\exists\, \{\varepsilon_n\}_{n\in\mathbb{N}}$, $\varepsilon_n \xrightarrow[n\to+\infty]{} 0$, so that

$$\lim_{n\to+\infty} \left\langle \Psi_{\varepsilon_n} \middle| W_{\varepsilon_n}(\eta)\Psi_{\varepsilon_n} \right\rangle_{\mathscr{F}_{\varepsilon_n}} = \widehat{\mu}(\eta) := \int_{\mathfrak{h}_{\omega^\alpha}} \mathrm{d}\mu(z)\, e^{2i\mathrm{Re}\left\langle \omega^{-\alpha/2}\eta \middle| \omega^{\alpha/2} z \right\rangle_{\mathfrak{h}}} \tag{2.7}$$

for all $\eta \in \mathfrak{h}_{\omega^{-\alpha}} \cap \mathfrak{h}$, which we also denote for short as $\Psi_{\varepsilon_n} \xrightarrow[n\to+\infty]{\omega^\alpha-\mathrm{sc}} \mu$.

To ensure that the sequence of field states we are considering admits at least one limit point, i.e., the set of associated Wigner measures $\mathscr{M}_\omega(\Psi_\varepsilon)$ is non-empty, we assume a uniform control on these states in ε. The precise statement of such a control depends on the model and therefore we make two different assumptions for the Nelson and PF models, respectively. We preliminary recall that

$$\mathrm{d}\Gamma^{(2)}_\varepsilon(\omega\otimes\omega) = \mathrm{d}\Gamma_\varepsilon(\omega)^2 - \varepsilon\,\mathrm{d}\Gamma_\varepsilon(\omega^2).$$

Assumption ($\mathrm{A}^{\mathrm{Nel}}_\Psi$)**.** *The family $\{\Psi_\varepsilon\}_{\varepsilon\in(0,1)} \subset \Gamma_\varepsilon(L^2(\mathbb{R}^3))$ is such that*

$$\left\langle \Psi_\varepsilon \middle| 1 + \mathrm{d}\Gamma_\varepsilon(\omega) \middle| \Psi_\varepsilon \right\rangle_{\mathscr{F}_\varepsilon} \leqslant C \tag{2.8}$$

uniformly in ε.

Assumption ($\mathrm{A}^{\mathrm{PF}}_\Psi$)**.** *The family $\{\Psi_\varepsilon\}_{\varepsilon\in(0,1)} \subset \Gamma_\varepsilon(L^2(\mathbb{R}^3;\mathbb{C}^2))$ is such that*

$$\left\langle \Psi_\varepsilon \middle| 1 + \mathrm{d}\Gamma_\varepsilon(\omega) + \mathrm{d}\Gamma^{(2)}_\varepsilon(\omega\otimes\omega)) \middle| \Psi_\varepsilon \right\rangle_{\mathscr{F}_\varepsilon} \leqslant C \tag{2.9}$$

uniformly in ε.

We anticipate that for any family of states $\{\Psi_\varepsilon\}_\varepsilon \subset \mathscr{F}_\varepsilon$ as in Definition 2.1, such that Assumption ($\mathrm{A}^{\mathrm{Nel}}_\Psi$) or Assumption ($\mathrm{A}^{\mathrm{PF}}_\Psi$) holds, we have that $\mathscr{M}_\omega(\Psi_\varepsilon) \neq \varnothing$ [AN08, Fal18a], i.e., there exists at least one $\mu \in \mathscr{M}_\omega(\Psi_\varepsilon)$, such that

$$\Psi_{\varepsilon_n} \xrightarrow[n\to+\infty]{\omega-\mathrm{sc}} \mu.$$

In addition, for any such converging sequence $\{\Psi_{\varepsilon_n}\}_{n\in\mathbb{N}}$ and μ,

$$\left\langle \Psi_{\varepsilon_n} \left| a^*_{\varepsilon_n}(g)\Psi_{\varepsilon_n} \right.\right\rangle_{\mathscr{F}_{\varepsilon_n}} \xrightarrow[n\to\infty]{} \int_{\mathfrak{h}_\omega} \left\langle \omega^{1/2} z \left| \omega^{-1/2} g \right.\right\rangle_{\mathfrak{h}} \,\mathrm{d}\mu(z) \,, \tag{2.10}$$

for all $g \in \mathfrak{h}_{\omega^{-1}}$ (see Proposition 3.4 below).

2.2. **Nelson model.** We consider a spinless non-relativistic particle linearly coupled to a quantized scalar field. The space of states (1.2) takes in this case the following form

$$\mathscr{H}^{\mathrm{Nel}} = L^2(\mathbb{R}^3) \otimes \mathscr{F}^{\mathrm{Nel}}_\varepsilon = L^2(\mathbb{R}^3) \otimes \Gamma_\varepsilon(\mathfrak{h}^{\mathrm{Nel}}), \qquad \mathfrak{h}^{\mathrm{Nel}} = L^2(\mathbb{R}^3). \tag{2.11}$$

The energy of the total quantum system is described by a Nelson-type Hamiltonian, given by

$$\mathbb{H}^{\mathrm{Nel}}_\varepsilon = \mathbf{P}^2 + U(x) + \mathrm{d}\Gamma_\varepsilon(\omega) + \Phi_\varepsilon(e^{ix\cdot k}\omega^{-\frac{1}{2}}\chi). \tag{2.12}$$

Here $\mathbf{P} := -i\nabla_x$ is the momentum[1] of the electron, $U : \mathbb{R}^3 \to \mathbb{R}$ is a real external potential, and

$$\Phi_\varepsilon(e^{ix\cdot k}\omega^{-\frac{1}{2}}\chi) := a^*_\varepsilon(e^{ix\cdot k}\omega^{-\frac{1}{2}}\chi) + a_\varepsilon(e^{ix\cdot k}\omega^{-\frac{1}{2}}\chi) = \int_{\mathbb{R}^3} \frac{\chi(k)}{\omega^{\frac{1}{2}}(k)} \big(e^{ix\cdot k}a^*_\varepsilon(k) + e^{-ix\cdot k}a_\varepsilon(k)\big)\mathrm{d}k,$$

is the field operator corresponding to the interaction between the non-relativistic particle and the field. The function $\chi : \mathbb{R}^3 \to \mathbb{R}$ is an ultraviolet cut-off, which might be required for the Hamiltonian to identify a self-adjoint operator in the Hilbert space $\mathscr{H}^{\mathrm{Nel}}$, but which, in our paper, can subsequently be put equal to 1.

Before stating the assumptions on U and χ, let us denote by $U_+ := \max(U, 0)$ and $U_- := \max(-U, 0)$ the positive and negative parts of U, respectively.

Assumption (A_U)**.** *The potential $U : \mathbb{R}^3 \to \mathbb{R}$ is such that $U_+ \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ and U_- is KLMN form-bounded w.r.t. $-\Delta$, i.e., there exists $a \in (0, 1)$ and $b \in \mathbb{R}$ such that*

$$\langle \psi \,| U_- |\, \psi\rangle \leqslant a \,\langle \psi \,| -\Delta |\, \psi\rangle + b \,\|\psi\|^2 \,, \qquad \forall \psi \in H^1(\mathbb{R}^3). \tag{2.13}$$

Concerning the ultraviolet cut-off function, we preliminary recall the definition of weak L^p spaces: for $1 \leqslant p < \infty$, we denote by $L^{p,\infty}(\mathbb{R}^3)$ the set of (equivalence classes of) measurable functions $f : \mathbb{R}^3 \to \mathbb{C}$ such that the quasi-norm

$$\|f\|_{L^{p,\infty}} := p \left\| \big|\{|f| > t\}\big|^{1/p}\, t \right\|_{L^\infty((0,\infty),\mathrm{d}t/t)} \tag{2.14}$$

is finite, where, for any measurable set S, $|S|$ stands for its Lebesgue measure.

Assumption (A_χ)**.** *The function $\chi : \mathbb{R}^3 \to \mathbb{R}$ is such that $\chi/\omega \in L^{3,\infty}(\mathbb{R}^3)$.*

We remark that the function $\chi \equiv 1$ can be easily seen to satisfy the above Assumption (A_χ), at least in the physically relevant cases $\omega(k) = \sqrt{k^2 + m^2}$ and $\omega(k) = |k|$, since

$$\|1/\omega\|_{L^{3,\infty}(\mathbb{R}^3)} \leqslant 3 \left\| \big|\{1/|k| > t\}\big|^{1/3}\, t \right\|_{L^\infty((0,\infty),\mathrm{d}t/t)} = 3^{2/3}(4\pi)^{1/3}.$$

More generally, it can be readily seen that $1/\omega \in L^{3,\infty}(\mathbb{R}^3)$ for any ω satisfying Assumption (A_ω). Hence, Assumption (A_χ) does not require a decay for large $|k|$ of the function χ, as it occurs for ultraviolet cut-offs.

[1]We use boldface letters to denote vectors in $\mathbb{R}^3$, whenever we need to stress the vector nature of the object.

We will see in § 4 that the following quadratic form on the Schwartz space $\mathcal{S}(\mathbb{R}^3)$ indeed defines a function $V_\varepsilon(x)$ belonging to some Lorentz space

$$\int_{\mathbb{R}^3} V_\varepsilon|\psi|^2 := \int_{\mathbb{R}^3} 2\mathrm{Re}\Big((2\pi)^{3/2}\overline{\mathcal{F}(|\psi|^2)(k)}\frac{\chi(k)}{\sqrt{\omega(k)}}\langle\Psi_\varepsilon|a^*_\varepsilon(k)\Psi_\varepsilon\rangle_{\mathscr{F}_\varepsilon}\Big)\,\mathrm{d}k\,, \quad \forall\psi\in\mathcal{S}(\mathbb{R}^3)\,, \tag{2.15}$$

where $\mathcal{F}$ denotes the Fourier transform on $L^2(\mathbb{R}^d)$, i.e.,

$$\mathcal{F}(f)(k) := \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} e^{-ik\cdot x}f(x)\,\mathrm{d}x\,. \tag{2.16}$$

If both χ/ω and $\chi/\sqrt{\omega}$ are in $L^2_k(\mathbb{R}^3)$ then the following explicit expression holds

$$V_\varepsilon(x) = \Big\langle\Psi_\varepsilon\,\Big|2\mathrm{Re}\,a^*_\varepsilon(e^{ix\cdot k}\tfrac{\chi}{\sqrt{\omega}})\Big|\,\Psi_\varepsilon\Big\rangle_{\mathscr{F}_\varepsilon}\,, \tag{2.17}$$

so that the expectation of $\mathbb{H}^{\mathrm{Nel}}_\varepsilon - \mathrm{d}\Gamma_\varepsilon(\omega)$ on the factorized state $\psi\otimes\Psi_\varepsilon$ as in (1.3) reads

$$Q_\varepsilon[\psi] := \langle\psi\,|H^{\mathrm{Nel}}_\varepsilon|\,\psi\rangle_{L^2_x}\,, \qquad H^{\mathrm{Nel}}_\varepsilon = -\Delta + U + V_\varepsilon. \tag{2.18}$$

Finally, given $\mu\in\mathscr{P}(\mathfrak{h}^{\mathrm{Nel}}_\omega)$, we will also see in § 4 that the following quadratic form on the Schwartz space $\mathcal{S}(\mathbb{R}^3)$ defines a function V_μ belonging to some Lorentz space

$$\int_{\mathbb{R}^3} V_\mu|\psi|^2 := \int_{\mathfrak{h}_\omega} 2\mathrm{Re}\Big\langle\sqrt{\omega}\,z\Big|(2\pi)^{3/2}\overline{\mathcal{F}(|\psi|^2)}\frac{\chi}{\omega}\Big\rangle_{L^2_k}\mathrm{d}\mu(z)\,, \quad \forall\psi\in\mathcal{S}(\mathbb{R}^3)\,. \tag{2.19}$$

If χ/ω is in $L^2_k(\mathbb{R}^3)$ then the following explicit expression holds

$$V_\mu(x) = 2\mathrm{Re}\int_{\mathfrak{h}_\omega}\big\langle\omega^{1/2}z\big|\,\tfrac{\chi}{\omega}e^{ik\cdot x}\big\rangle_{L^2_k}\,\mathrm{d}\mu(z)\,. \tag{2.20}$$

Our main result for the Nelson model is the following. Let us preliminarily introduce a slightly modified notion of Γ-convergence, adapted to our needs.

Definition 2.2. *Given $\mathcal{E}_n$, with $n\in\mathbb{N}\cup\{\infty\}$, functionals from $\mathfrak{h}^\sharp$ to $\mathbb{R}$ defined on a suitable common dense domain $\mathscr{Q}\subset\mathfrak{h}^\sharp$ we write*

$$\mathcal{E}_n[\cdot]\xrightarrow[n\to\infty]{\Gamma}\mathcal{E}_\infty[\cdot]\,,$$

if and only if the following two statements hold:

- $[\Gamma-\limsup\,]$ *For any $\psi\in\mathscr{Q}$, there exists one sequence $\{\psi_n\}_{n\in\mathbb{N}}\subset\mathscr{Q}$ such that $\psi_n\xrightarrow[n\to\infty]{\mathfrak{h}^\sharp}\psi$, and*
$$\limsup_{n\to\infty}\mathcal{E}_n[\psi_n]\leqslant\mathcal{E}_\infty[\psi]\,;$$
- $[\Gamma-\liminf\,]$ *For any sequence $\{\psi_n\}_{n\in\mathbb{N}}\subset\mathscr{Q}$, such that $\psi_n\xrightarrow[n\to\infty]{\mathrm{w}-\mathfrak{h}^\sharp}\psi$, $\psi\in\mathscr{Q}$,*
$$\liminf_{n\to\infty}\mathcal{E}_n[\psi_n]\geqslant\mathcal{E}_\infty[\psi]\,.$$

The convergence $\mathcal{E}_n[\cdot]\xrightarrow[n\to\infty]{\Gamma}\mathcal{E}_\infty[\cdot]$ holds if and only if $\mathcal{E}_n$ Γ-converges to $\mathcal{E}_\infty$ both in the weak and strong topologies of $\mathfrak{h}^\sharp$.

Let us recall that with this definition, the Γ-convergence of quadratic forms bounded from below is equivalent to strong resolvent convergence of the associated self-adjoint operators

[DM93, Theorem 13.6]. If, in addition, the operators have compact resolvent, then the Γ-convergence is equivalent to norm resolvent convergence.

Theorem 2.3 (Convergence of $H_\varepsilon^{\mathrm{Nel}}$).
Suppose that Assumptions (A_ω), $(\mathrm{A}_\Psi^{\mathrm{Nel}})$, (A_U) *and* (A_χ) *hold. For any* $\mu \in \mathscr{M}^{\mathrm{Nel}}(\Psi_\varepsilon)$ *and for any sequence* $\{\varepsilon_n\}_{n\in\mathbb{N}}$, $\varepsilon_n \to 0$, *such that* $\Psi_{\varepsilon_n} \xrightarrow[n\to+\infty]{\mathrm{sc}} \mu$, *then,*

$$H_{\varepsilon_n}^{\mathrm{Nel}} = -\Delta + U + V_{\varepsilon_n} \qquad \text{and} \qquad H_\mu^{\mathrm{Nel}} = -\Delta + U + V_\mu \tag{2.21}$$

define symmetric closed quadratic forms with form domain

$$\mathcal{Q} := \mathcal{Q}(H_{\varepsilon_n}^{\mathrm{Nel}}) = \mathcal{Q}(H_\mu^{\mathrm{Nel}}) = H^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3, U_+\,\mathrm{d}x)$$

and hence define self-adjoint operators on domains $\mathcal{D}(H_{\varepsilon_n}^{\mathrm{Nel}})$, $\mathcal{D}(H_\mu^{\mathrm{Nel}}) \subseteq \mathcal{Q}$, *respectively. Moreover, as quadratic forms,*

$$\left\langle \varphi \left| H_{\varepsilon_n}^{\mathrm{Nel}} \right| \varphi \right\rangle \xrightarrow[n\to\infty]{\Gamma} \left\langle \varphi \left| H_\mu^{\mathrm{Nel}} \right| \varphi \right\rangle . \tag{2.22}$$

Consequently, $H_{\varepsilon_n}^{\mathrm{Nel}}$ *converges to* H_μ^{Nel} *in strong resolvent sense. If in addition* U *is confining, i.e.* $U_+(x) \to \infty$ *as* $|x| \to \infty$, *then* $H_{\varepsilon_n}^{\mathrm{Nel}}$ *converges to* H_μ^{Nel} *in norm resolvent sense.*

Remark 2.4 (Ultraviolet renormalization).
In the work [CFO24] it is shown that the ultraviolet renormalization of the Nelson model commutes with the quasi-classical limit. However, the former calls for the extraction of an infinite particle self-energy, and the introduction of a suitable dressing transformation that modifies substantially the properties of the microscopic Hamiltonian as well as of its quasi-classical limit. In this framework, the above result shows that, on product states and at the level of the quadratic form, such a renormalization procedure is actually not needed. Indeed, one can easily figure out that Theorem 2.3 entails that (see Assumption (A_χ) and discussion thereafter), if one imposes an ultraviolet cut-off χ_Λ such that $\chi_\Lambda \to 1$, as $\Lambda \to +\infty$, then, at the level of the reduced quadratic form, the limits $\varepsilon \to 0$ and $\Lambda \to +\infty$ yield the same result irrespective of the order in which they are taken. Furthermore, if the cut-off parameter $\Lambda = \Lambda(\varepsilon)$ depends on ε and $\Lambda(\varepsilon) \to +\infty$ as $\varepsilon \to 0$, i.e., the ultraviolet renormalization is performed at the same time as the quasi-classical limit, the rate of the former does not matter.

Remark 2.5 (Wigner measures).
As discussed above, the set of Wigner measures of a generic family of states $\{\Psi_\varepsilon\}_{\varepsilon\in(0,1)}$ might contain more than a single point. In that case, the family of operators $H_{\varepsilon_n}^{\mathrm{Nel}}$ depends on the choice of sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ determining the limit point μ. However, if the set of Wigner measures consists of a single point, then the Γ-convergence holds as $\varepsilon \to 0$.

Remark 2.6 (Pseudo-relativistic kinetic energy).
A close inspection of the proof shows that the result easily extends to Nelson-type Hamiltonians with pseudo-relativistic kinetic energy, i.e., for $H_0 = \sqrt{-\Delta + \nu^2} + U$, $\nu \geqslant 0$, up to a straightforward modification of Assumption (A_U).

2.3. **Pauli-Fierz model.** We next consider a second, physically more relevant setting, where the spin of the non-relativistic particle is taken into account, the radiation is described by the quantized electromagnetic field in the Coulomb gauge and the particle-field interaction is given by Pauli coupling. For the sake of simplicity we set $s = \frac{1}{2}$ but the extension to different values of the spin is straightforward. The space states is then

$$\mathscr{H}^{\mathrm{PF}} = L^2(\mathbb{R}^3;\mathbb{C}^2) \otimes \Gamma_\varepsilon(\mathfrak{h}), \qquad \mathfrak{h}^{\mathrm{PF}} = L^2(\mathbb{R}^3;\mathbb{C}^2), \tag{2.23}$$

and the Hamiltonian is given by

$$\mathbb{H}_\varepsilon^{\mathrm{PF}} = \big(\boldsymbol{\sigma}\cdot(\mathbf{P} - \mathbb{A}_\varepsilon(x))\big)^2 + U(x) + \mathrm{d}\Gamma_\varepsilon(\omega), \tag{2.24}$$

where the vector of Pauli matrices $\boldsymbol{\sigma} = (\sigma_1,\sigma_2,\sigma_3)$ is

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The vector potential of the quantized electromagnetic field is of the form

$$\mathbb{A}_\varepsilon(x) = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \frac{\chi(k)}{\omega^{\frac{1}{2}}(k)} \mathbf{e}_\lambda(k) \left(e^{ix\cdot k} a^\dagger_{\lambda,\varepsilon}(k) + e^{-ix\cdot k} a_{\lambda,\varepsilon}(k) \right) \mathrm{d}k,$$

with $(\mathbf{e}_1(k), \mathbf{e}_2(k))$ polarization vectors, such that $(\mathbf{e}_1(k), \mathbf{e}_2(k), k/|k|)$ forms an orthonormal basis of $\mathbb{R}^3$ for all $k \neq 0$. The creation and annihilation operators $a^\dagger_{\lambda,\varepsilon}$, $a_{\lambda,\varepsilon}$ are now labelled by the polarization directions $\lambda, \lambda' \in \{1,2\}$ and satisfy the canonical commutation relations

$$\left[a^\sharp_{\lambda,\varepsilon}(k), a^\sharp_{\lambda',\varepsilon}(k')\right] = 0, \qquad \left[a_{\lambda,\varepsilon}(k), a^\dagger_{\lambda',\varepsilon}(k')\right] = \varepsilon\delta_{\lambda\lambda'}\delta(k-k'),$$

for $k, k' \in \mathbb{R}^3$ and $\lambda, \lambda' \in \{1,2\}$. To shorten notations, we will sometimes write

$$\mathbb{A}_\varepsilon(x) = a^*_\varepsilon(\mathbf{w}_x) + a_\varepsilon(\mathbf{w}_x),$$

with

$$\mathbf{w}_x(k,\lambda) := \frac{\chi(k)}{\omega^{\frac{1}{2}}(k)} \mathbf{e}_\lambda(k) e^{ix\cdot k}. \tag{2.25}$$

The Hamiltonian (2.24) is not normal ordered, which implies that its vacuum energy may diverge in absence of an ultraviolet cut-off. For this reason, we actually work with its Wick-ordered counterpart

$$\begin{aligned} :\mathbb{H}_\varepsilon: = (\boldsymbol{\sigma}\cdot\mathbf{P})^2 - (\boldsymbol{\sigma}\cdot\mathbf{P})(\boldsymbol{\sigma}\cdot\mathbb{A}_\varepsilon(x)) - (\boldsymbol{\sigma}\cdot\mathbb{A}_\varepsilon(x))(\boldsymbol{\sigma}\cdot\mathbf{P}) \\ + a^*_\varepsilon(\mathbf{w}_x)a^*_\varepsilon(\mathbf{w}_x) + a_\varepsilon(\mathbf{w}_x)a_\varepsilon(\mathbf{w}_x) + 2a^*_\varepsilon(\mathbf{w}_x)a_\varepsilon(\mathbf{w}_x) + U(x). \end{aligned} \tag{2.26}$$

Note that $:\mathbb{H}_\varepsilon:$ and $\mathbb{H}_\varepsilon$ only differ by an ε-dependent constant,

$$:\mathbb{H}_\varepsilon: = \mathbb{H}_\varepsilon - [a_\varepsilon(\mathbf{w}_x), a^*_\varepsilon(\mathbf{w}_x)] = \mathbb{H}_\varepsilon - 2\varepsilon \left\|\omega^{-1/2}\chi\right\|^2_{L^2_k},$$

which formally vanishes in the limit $\varepsilon \to 0$.

The counterparts of the potential V_ε defined in (2.15) for the PF model are a vector potential $\mathbf{A}_\varepsilon$ along with a potential W_ε defined through the following quadratic form on the Schwartz space $\mathcal{S}(\mathbb{R}^3)$. For $\mathbf{A}_\varepsilon$, with $\mathbf{w}_x$ is defined in (2.25):

$$\int_{\mathbb{R}^3} |\psi|^2 \mathbf{A}_\varepsilon := \int_{\mathbb{R}^6} 2\mathrm{Re}\Big(|\psi(x)|^2 \ \mathbf{w}_x(k) \ \langle \Psi_\varepsilon | a^*_\varepsilon(k)\Psi_\varepsilon\rangle_{\mathscr{F}_\varepsilon} \Big) \,\mathrm{d}x\,\mathrm{d}k, \quad \forall \psi \in \mathcal{S}(\mathbb{R}^3). \tag{2.27}$$

If both χ/ω and $\chi/\sqrt{\omega}$ are in $L^2_k(\mathbb{R}^3)$ then the following explicit expression holds

$$\mathbf{A}_\varepsilon(x) := \langle \Psi_\varepsilon \,| \mathbb{A}_\varepsilon(x)\, \Psi_\varepsilon \rangle_{\mathscr{F}_\varepsilon} = \left\langle \Psi_\varepsilon \,\middle|\, \big(a^\dagger_\varepsilon(\mathbf{w}_x) + a_\varepsilon(\mathbf{w}_x)\big)\ \Psi_\varepsilon \right\rangle_{\mathscr{F}_\varepsilon} . \tag{2.28}$$

For W_ε:

$$\int_{\mathbb{R}^3} |\psi|^2\, W_\varepsilon := \int_{\mathbb{R}^9} |\psi(x)|^2\, 2\mathrm{Re}\Big(\mathbf{w}_x(k)\cdot\mathbf{w}_x(k')\ \langle a_\varepsilon(k)a_\varepsilon(k')\Psi_\varepsilon|\Psi_\varepsilon\rangle_{\mathscr{F}_\varepsilon} \\ + \mathbf{w}_x(k)\cdot\overline{\mathbf{w}_x(k')}\langle a_\varepsilon(k)\Psi_\varepsilon|a_\varepsilon(k')\Psi_\varepsilon\rangle_{\mathscr{F}_\varepsilon}\Big)\,\mathrm{d}x\,\mathrm{d}k\,\mathrm{d}k' , \quad \forall\psi\in\mathcal{S}(\mathbb{R}^3) . \tag{2.29}$$

If both χ/ω and $\chi/\sqrt{\omega}$ are in $L^2_k(\mathbb{R}^3)$ then the following explicit expression holds

$$W_\varepsilon(x) := \left\langle \Psi_\varepsilon \,\middle|\!: \big(\mathbb{A}_\varepsilon(x)\big)^2 : \Psi_\varepsilon \right\rangle_{\mathscr{F}_\varepsilon} \\ = \left\langle \Psi_\varepsilon \,\middle|\, \big(a^\dagger_\varepsilon(\mathbf{w}_x)a^\dagger_\varepsilon(\mathbf{w}_x) + a_\varepsilon(\mathbf{w}_x)a_\varepsilon(\mathbf{w}_x) + 2a^\dagger_\varepsilon(\mathbf{w}_x)a_\varepsilon(\mathbf{w}_x)\big)\,\Psi_\varepsilon \right\rangle_{\mathscr{F}_\varepsilon} . \tag{2.30}$$

The reduced Hamiltonian has then the following expression:

$$H^{\mathrm{PF}}_\varepsilon = (\boldsymbol{\sigma}\cdot\mathbf{P})^2 - (\boldsymbol{\sigma}\cdot\mathbf{P})(\boldsymbol{\sigma}\cdot\mathbf{A}_\varepsilon(x)) - (\boldsymbol{\sigma}\cdot\mathbf{A}_\varepsilon(x))(\boldsymbol{\sigma}\cdot\mathbf{P}) + W_\varepsilon(x) + U(x). \tag{2.31}$$

Its quasi-classical counterpart is identified by a measure $\mu\in\mathscr{M}^{\mathrm{PF}}(\Psi_\varepsilon)$ in the set of Wigner measures associated to $\{\Psi_\varepsilon\}_{\varepsilon\in(0,1)}$ and such that, up to the extraction of a subsequence,

$$\Psi_{\varepsilon_n} \xrightarrow[n\to+\infty]{\mathrm{sc}} \mu,$$

and reads

$$H^{\mathrm{PF}}_\mu := (\boldsymbol{\sigma}\cdot\mathbf{P})^2 - (\boldsymbol{\sigma}\cdot\mathbf{P})(\boldsymbol{\sigma}\cdot\mathbf{A}_\mu(x)) - (\boldsymbol{\sigma}\cdot\mathbf{A}_\mu(x))(\boldsymbol{\sigma}\cdot\mathbf{P}) + W_\mu(x) + U(x), \tag{2.32}$$

where the multiplication operators by $\mathbf{A}_\mu(x)$ and $W_\mu(x)$ are defined through the following quadratic forms on $\mathcal{S}(\mathbb{R}^3)$ (see § 5):

$$\int_{\mathbb{R}^3} \mathbf{A}_\mu\, |\psi|^2 := \int_{\mathfrak{h}^{\mathrm{PF}}} 2\mathrm{Re}\Big(\int_{\mathbb{R}^6} \overline{z(k)}\ \mathbf{w}_x(k)\ |\psi(x)|^2\,\mathrm{d}x\,\mathrm{d}k\Big)\mathrm{d}\mu(z) \tag{2.33}$$

and

$$\int_{\mathbb{R}^3} W_\mu\, |\psi|^2 := \int_{\mathfrak{h}^{\mathrm{PF}}_\omega}\int_{\mathbb{R}^9} 4\mathrm{Re}(\overline{z(k_1)}\ \mathbf{w}_x(k_1))\mathrm{Re}(\overline{z(k_2)}\ \mathbf{w}_x(k_2))|\psi(x)|^2\,\mathrm{d}x\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}\mu(z) . \tag{2.34}$$

with $\psi\in\mathcal{S}(\mathbb{R}^3)$. In the simple case that $\chi/\sqrt{\omega}$ is in $L^2_k(\mathbb{R}^3)$ then the following explicit expressions hold

$$\mathbf{A}_\mu(x) := \int_{\mathfrak{h}^{\mathrm{PF}}_\omega} 2\mathrm{Re}\ \left\langle \omega^{1/2}z \,\middle|\, \omega^{-1/2}\mathbf{w}_x \right\rangle_{L^2_k} \mathrm{d}\mu(z) , \tag{2.35}$$

$$W_\mu(x) := \int_{\mathfrak{h}^{\mathrm{PF}}_\omega} \Big(2\mathrm{Re}\ \left\langle \omega^{1/2}z \,\middle|\, \omega^{-1/2}\mathbf{w}_x \right\rangle_{L^2_k}\Big)^2 \mathrm{d}\mu(z) . \tag{2.36}$$

To prove our result on the Pauli-Fierz model, we need to slightly strengthen Assumption (A_χ):

Assumption (A'_χ)**.** *The function* $\chi:\mathbb{R}^3\to\mathbb{R}$ *is such that* $|k|\chi/\omega\in L^\infty(\mathbb{R}^3)$.

Similarly as for Assumption (A_χ), the function $\chi \equiv 1$ satisfies Assumption (A'_χ) in the physically relevant cases $\omega(k) = \sqrt{k^2+m^2}$ and $\omega(k) = |k|$.

Our main result is then:

Theorem 2.7 (Convergence of $H_\varepsilon^{\mathrm{PF}}$)**.**
Suppose that Assumptions (A_ω), ($\mathrm{A}_\Psi^{\mathrm{PF}}$), ($\mathrm{A}_U$) and ($\mathrm{A}'_\chi$) hold. Then, for any $\mu \in \mathscr{M}^{\mathrm{PF}}(\Psi_\varepsilon)$ and for any sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$, $\varepsilon_n \to 0$, such that $\Psi_{\varepsilon_n} \xrightarrow[n\to+\infty]{\mathrm{sc}} \mu$, $H_{\varepsilon_n}^{\mathrm{PF}}$ and H_μ^{PF} define symmetric closed quadratic forms with form domain

$$\mathscr{Q} = H^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3, U_+\,\mathrm{d}x; \mathbb{C}^2), \tag{2.37}$$

and hence define self-adjoint operators on domains $\mathscr{D}(H_n^{\mathrm{PF}})$, $\mathscr{D}(H_\mu^{\mathrm{PF}}) \subseteq \mathscr{Q}$, respectively. Moreover, as quadratic forms,

$$\left\langle \varphi \middle| H_{\varepsilon_n}^{\mathrm{PF}} \middle| \varphi \right\rangle \xrightarrow[n\to\infty]{\Gamma} \left\langle \varphi \middle| H_\mu^{\mathrm{PF}} \middle| \varphi \right\rangle . \tag{2.38}$$

Consequently, $H_{\varepsilon_n}^{\mathrm{PF}}$ converges to H_μ^{PF} in strong resolvent sense. If in addition U is confining, i.e. $U_+(x) \to \infty$ as $|x| \to \infty$, then $H_{\varepsilon_n}^{\mathrm{PF}}$ converges to H_μ^{PF} in norm resolvent sense.

Remark 2.8 (Ultraviolet renormalization)**.**
Analogously to Remark 2.4, the above Theorem 2.7 implies (see Assumption (A_χ) and discussion thereafter) that one can remove the ultraviolet cut-off at the level of the energy form and such an operation commutes with the quasi-classical limit $\varepsilon \to 0$. However, unlike for the Nelson model, the rate at which the cut-off is removed does matter: recalling that the result above applies to the normal ordered form of the PF Hamiltonian, the vacuum energy we are implicitly extracting is

$$-2\varepsilon \left\| \omega^{-1/2}\chi_\Lambda \right\|_{L_k^2}^2 = -C\varepsilon\Lambda^2(1 + o_\Lambda(1))$$

in the simple case of a sharp cut-off $\chi_\Lambda = \mathbb{1}_{|k|\leqslant\Lambda}$. Hence, such an energy remains bounded as $\varepsilon \to 0$, if $\Lambda \lesssim \frac{1}{\sqrt{\varepsilon}}$, suggesting that the ultraviolet renormalization is actually trivial if the cut-off is removed slowly enough.

Remark 2.9 (Quantum and classical divergences in non-relativistic electrodynamics)**.**
The choice $\chi \equiv 1$ corresponds to a point distribution of the electric charge of the quantum Schrödinger particle. On the one hand, at the quantum level, a point charge yields an energy unbounded from below due to the interaction with the *quantized* field. On the other hand, in classical electrodynamics, a point charge yields an energy unbounded from below due to the Coulomb singularity of the electrostatic potential. The model in between, namely with a quantum point charge and a classical electromagnetic field has an energy bounded from below, at least for regular enough vector potentials of the electromagnetic field. Theorem 2.7 agrees with such a physical picture, since we can derive the Hamiltonian of a point charge in a classical electromagnetic field from a fully quantized non-relativistic electrodynamics, under suitable assumptions.

3. Preliminaries

3.1. **Functional inequalities in Lorentz spaces.** Our proofs rely on suitable functional inequalities in Lorentz spaces. For the convenience of the reader we recall basic facts about

Lorentz spaces (see, *e.g.*, [O'N63, Yap69, LR02, BLNS17] or [Gra08, 1.4.19] for more details). For $1 \leqslant p < \infty$ and $1 \leqslant q \leqslant \infty$, the Lorentz spaces $L^{p,q} := L^{p,q}(\mathbb{R}^d)$ are defined as the set of (equivalence classes of) measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ such that the quasi-norm

$$\|f\|_{L^{p,q}} := p^{1/q} \left\| |\{|f| > t\}|^{1/p}\ t \right\|_{L^q((0,\infty),\mathrm{d}t/t)} \tag{3.1}$$

is finite. For $1 \leqslant p < \infty$ and $1 \leqslant q_1 \leqslant q_2 \leqslant \infty$, the continuous embedding $L^{p,q_1} \subseteq L^{p,q_2}$ holds. Moreover, $L^{p,p}$ identifies with the Lebesgue space L^p. In the sequel, the notation $\lesssim$ stands for the inequality $\leqslant$ up to a multiplicative constant which is independent of the chosen function.

We use the following generalizations of Hölder and Young's inequality in Lorentz spaces. For $1 \leqslant p_1, p_2 < \infty$, $1 \leqslant q_1, q_2 \leqslant \infty$, Hölder's inequality states that

$$\|f_1 f_2\|_{L^{p,q}} \lesssim \|f_1\|_{L^{p_1,q_1}} \|f_2\|_{L^{p_2,q_2}}\ , \qquad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}\,, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}\,,$$

whenever the r.h.s. is finite. Young's inequality states that, for $1 < p, p_1, p_2 < \infty$, $1 \leqslant q_1, q_2 \leqslant \infty$,

$$\|f_1 * f_2\|_{L^{p,q}} \lesssim \|f_1\|_{L^{p_1,q_1}} \|f_2\|_{L^{p_2,q_2}}\ , \qquad 1 + \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}\,, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}\,.$$

Then, we have the following property of the Lorentz norms:

Lemma 3.1. *If $\alpha_j \geqslant 0$, $j \in \{1,2\}$, $(\alpha_1+\alpha_2)/d = 1/p \in (0,1]$ and $1 \leqslant q \leqslant \infty$, then, for all $\psi_j \in \dot{H}^{\alpha_j}$,*

$$\|\mathcal{F}(\psi_1\psi_2)\|_{L^{p,q}} \lesssim \|\psi_1\|_{\dot{H}^{\alpha_1}} \|\psi_2\|_{\dot{H}^{\alpha_2}}\ . \tag{3.2}$$

Proof. Using the Young inequality in Lorentz spaces with $\frac{1}{p_j} = \frac{\alpha_j}{d} + \frac{1}{2}$ yields

$$\|\mathcal{F}(\psi_1\psi_2)\|_{L^{p,q}} \lesssim \|\psi_1\|_{L^{p_1,2q}} \|\psi_2\|_{L^{p_2,2q}} \tag{3.3}$$

we can then estimate the two terms on the right hand side in the same way, using the Hölder inequality in Lorentz spaces, and the continuity of the embedding $L^{2,2} \subseteq L^{2,2q}$,

$$\|\psi_j\|_{L^{p_1,2q}} \lesssim \left\| |k|^{-\alpha_j} \right\|_{L^{d/\alpha_j,\infty}} \left\| |k|^{-\alpha_j}\psi_j \right\|_{L^{2,2q}} \lesssim \left\| |k|^{-\alpha_j}\psi_j \right\|_{L^{2,2}} = \|\psi_j\|_{\dot{H}^{\alpha_j}} \tag{3.4}$$

which achieves the proof. □

From now on we set the space dimension d equal to 3. We state a simple Lorentz space estimate, which is useful both for the Nelson model and the Pauli-Fierz model:

Lemma 3.2. *If Assumptions* (A_ω) *and* (A_χ) *hold, and ψ, φ are Lebesgue measurable functions such that $\psi, \varphi, \bar{\psi}\varphi \in \mathcal{S}'(\mathbb{R}^3)$, and $\mathcal{F}(\bar{\psi}\varphi) \in L^{6,2}$, then*

$$\left\| \overline{\mathcal{F}}(\overline{\psi}\varphi)\frac{\chi}{\omega} \right\|_{L^2} \lesssim \left\| \mathcal{F}(\bar{\psi}\varphi) \right\|_{L^{6,2}}\ .$$

Proof. Hölder's inequality in Lorentz spaces and Assumption (A_χ) yield

$$\left\| \overline{\mathcal{F}}(\overline{\psi}\varphi)\tfrac{\chi}{\omega} \right\|_{L^2} \leqslant \left\| \overline{\mathcal{F}}(\overline{\psi}\varphi) \right\|_{L^{6,2}} \left\| \tfrac{\chi}{\omega} \right\|_{L^{3,\infty}}\ ,$$

which is the claimed result. □

The Lorentz spaces estimates we use for the Pauli-Fierz model are given below.

Lemma 3.3. *If Assumption* (A_ω) *and* ((A_χ)) *hold, and* ψ, φ *are Lebesgue measurable functions such* $\psi, \varphi, \bar{\psi}\varphi \in \mathcal{S}'(\mathbb{R}^3)$, *and* $\mathcal{F}(\bar{\psi}\varphi) \in L^{3,2}$, *then*

$$\left\| \overline{\mathcal{F}}(\overline{\psi}\varphi)(k-k') \frac{\chi(k)}{\omega(k)} \frac{\chi(k')}{\omega(k')} \right\|_{L^2_{k,k'}} \lesssim \left\| \mathcal{F}(\bar{\psi}\varphi) \right\|_{L^{3,2}}$$

and $\left\| \mathcal{F}(\bar{\psi}\varphi) \right\|_{L^{3,2}}$ *can be controlled either by* $\|\psi\|_{\dot{H}^{1/2}} \|\varphi\|_{\dot{H}^{1/2}}$ *or* $\|\psi\|_{L^2} \|\varphi\|_{\dot{H}^1}$.

Proof. By Hölder and Young's inequalities in Lorentz spaces

$$\begin{aligned} \left\| \left(|\overline{\mathcal{F}}(\overline{\psi}\varphi)|^2 * \tfrac{\chi^2}{\omega^2} \right) \tfrac{\chi^2}{\omega^2} \right\|_{L^1} &\lesssim \left\| \tfrac{\chi^2}{\omega^2} \right\|_{L^{3/2,\infty}} \left\| |\overline{\mathcal{F}}(\overline{\psi}\varphi)|^2 * \tfrac{\chi^2}{\omega^2} \right\|_{L^{3,1}} \\ &\lesssim \left\| \tfrac{\chi}{\omega} \right\|^2_{L^{3,\infty}} \left\| |\overline{\mathcal{F}}(\overline{\psi}\varphi)|^2 \right\|_{L^{3/2,1}} \left\| \tfrac{\chi^2}{\omega^2} \right\|_{L^{3/2,\infty}} \lesssim \left\| \tfrac{\chi}{\omega} \right\|^4_{L^{3,\infty}} \left\| \overline{\mathcal{F}}(\overline{\psi}\varphi) \right\|_{L^{3,2}} \end{aligned}$$

and Lemma 3.1 yields the bounds on $\left\| \mathcal{F}(\bar{\psi}\varphi) \right\|_{L^{3,2}}$. □

3.2. **Semiclassical Analysis.** We present here the main tool of semiclassical analysis ensuring the existence of at least one Wigner measure (see Definition 2.1) associated to the family of field's states $\{\Psi_\varepsilon\}_{\varepsilon \in (0,1)}$, as well as the convergence of expectations of suitable observables (monomials of creations and annihilation operators). In order to properly state the latter, let us introduce a class[2] $\mathscr{S}_{\ell,m}$, $\ell, m \in \mathbb{N}$, of (cylindrical) classical symbols of the form

$$\mathcal{S}(z) = \prod_{i=1}^{\ell} \prod_{j=\ell+1}^{\ell+m} \left\langle \omega^{1/2} z | \omega^{-1/2} g_i \right\rangle_{\mathfrak{h}} \left\langle \omega^{-1/2} g_j | \omega^{1/2} z \right\rangle_{\mathfrak{h}}, \tag{3.5}$$

where $g_i \in \mathfrak{h}_{\omega^{-1}}$ for any $i \in \{1, \ldots, \ell+m\}$ and $z \in \mathfrak{h}_\omega$. The Wick quantization of such symbols is simply given by the normal ordered monomial

$$\mathrm{Op}_\varepsilon^{\mathrm{Wick}}\left(\mathcal{S}(z)\right) = \prod_{i=1}^{\ell} \prod_{j=\ell+1}^{\ell+m} a_\varepsilon^\dagger(g_i) a_\varepsilon(g_j). \tag{3.6}$$

Proposition 3.4 (Existence of Wigner measures)**.**
Suppose that Assumption (A_ω) *holds. If there exists* $\delta > 0$ *such that the family of states* $\{\Psi_\varepsilon\}_{\varepsilon \in (0,1)} \subset \mathscr{F}_\varepsilon$ *satisfies*

$$\left| \left\langle \Psi_\varepsilon \middle| (1 + \mathrm{d}\Gamma_\varepsilon(\omega))^\delta \middle| \Psi_\varepsilon \right\rangle_{\mathscr{F}_\varepsilon} \right| \leqslant C\,, \tag{3.7}$$

uniformly in ε, *then* $\mathscr{M}_\omega(\Psi_\varepsilon) \neq \varnothing$ *and*

$$\int_{\mathfrak{h}_\omega} \left(1 + \left\| \omega^{1/2} z \right\|_{\mathfrak{h}}^2 \right)^\eta \mathrm{d}\mu(z) < +\infty, \qquad \forall \eta \leqslant \delta. \tag{3.8}$$

Furthermore, if $\Psi_{\varepsilon_n} \xrightarrow[n\to+\infty]{\omega-\mathrm{sc}} \mu$ *and* $\mathcal{S}(z) \in \mathscr{S}_{\ell,m}$ *is a symbol of the form* (3.5) *with* $\frac{\ell+m}{2} < \delta$, *then*

$$\left\langle \Psi_{\varepsilon_n} | \mathrm{Op}_{\varepsilon_n}^{\mathrm{Wick}}\left(\mathcal{S}(z)\right) \Psi_{\varepsilon_n} \right\rangle_{\mathscr{F}_{\varepsilon_n}} \xrightarrow[n\to\infty]{} \int_{\mathfrak{h}_\omega} \mathcal{S}(z)\, \mathrm{d}\mu(z). \tag{3.9}$$

[2]We use the following convention: if either ℓ or $m = 0$, no factor of the corresponding kind is present in the symbol.

Proof. The result concerning the existence of Wigner measures for $\omega = 1$ is already stated in [AN08, Thm. 6.2]. The extension to a generic $\omega \geqslant 0$ and the result about the convergence of cylindrical symbols can be found in [Fal18a, Thm. 3.3]. □

Remark 3.5 (Nelson model)**.**
Combining Proposition 3.4 above with Assumption $(\mathrm{A}_\Psi^{\mathrm{Nel}})$, we immediately see that, if the latter holds, then not only the set of Wigner measures is non-empty, but also for any $\mu \in \mathscr{M}_\omega(\Psi_\varepsilon)$ and any subsequence such that $\Psi_{\varepsilon_n} \xrightarrow[n\to+\infty]{\omega-\mathrm{sc}} \mu$,

$$\left\langle \Psi_{\varepsilon_n} \left| a^\dagger_{\varepsilon_n}(g) \Psi_{\varepsilon_n} \right.\right\rangle_{\mathscr{F}_{\varepsilon_n}} \xrightarrow[n\to\infty]{} \int_{\mathfrak{h}^{\mathrm{Nel}}_\omega} \left\langle \omega^{1/2} z \left| \omega^{-1/2} g \right.\right\rangle_{\mathfrak{h}^{\mathrm{Nel}}} \mathrm{d}\mu(z), \tag{3.10}$$

for all $g \in \mathfrak{h}^{\mathrm{Nel}}_{\omega^{-1}}$. Obviously, the analogous result for the expectation of $a_{\varepsilon_n}(g)$ holds true.

Remark 3.6 (PF model)**.**
Analogously to Remark 3.5, the combination of Proposition 3.4 with Assumption $(\mathrm{A}_\Psi^{\mathrm{PF}})$ guarantees the non-emptiness of the set of Wigner measures $\mathscr{M}_\omega(\Psi_\varepsilon)$, as well as the convergence (over subsequences) of the expectation values of monomials of degree up to 2 of creation and annihilation operators, *i.e.*, in addition to the analogue of (3.10), one also has that

$$\left\langle \Psi_{\varepsilon_n} \left| \left(a^\dagger_{\varepsilon_n}(g)\right)^2 \Psi_{\varepsilon_n} \right.\right\rangle_{\mathscr{F}_{\varepsilon_n}} \xrightarrow[n\to\infty]{} \int_{\mathfrak{h}^{\mathrm{PF}}_\omega} \left(\left\langle \omega^{1/2} z \left| \omega^{-1/2} g \right.\right\rangle_{\mathfrak{h}^{\mathrm{PF}}} \right)^2 \mathrm{d}\mu(z), \tag{3.11}$$

$$\left\langle \Psi_{\varepsilon_n} \left| a^\dagger_{\varepsilon_n}(g) a_{\varepsilon_n}(g) \Psi_{\varepsilon_n} \right.\right\rangle_{\mathscr{F}_{\varepsilon_n}} \xrightarrow[n\to\infty]{} \int_{\mathfrak{h}^{\mathrm{PF}}_\omega} \left| \left\langle \omega^{-1/2} g \left| \omega^{1/2} z \right.\right\rangle_{\mathfrak{h}^{\mathrm{PF}}} \right|^2 \mathrm{d}\mu(z), \tag{3.12}$$

for all $g \in \mathfrak{h}^{\mathrm{PF}}_{\omega^{-1}}$.

4. The Nelson model: Proof of Theorem 2.3

We present first the method on the Nelson model, where most ideas can already be understood. Throughout this section, the spaces L^p, $L^{p,q}$ and H^s have their variables in $\mathbb{R}^3$.

Lemma 3.2 allows us to make sense of the potential V_ε (recall its definition in (2.15)) once the fields are traced out, and also yields a useful estimate:

Proposition 4.1 (Estimate of V_ε)**.**
Suppose that Assumptions (A_ω), $(\mathrm{A}_\Psi^{\mathrm{Nel}})$ and (A_χ) hold. If $\mathcal{F}(\bar{\psi}\varphi) \in L^{6,2}$, then,

$$\left| \langle \psi | V_\varepsilon \varphi \rangle_{L^2_x} \right| \lesssim \left\| \mathcal{F}(\bar{\psi}\varphi) \right\|_{L^{6,2}} \langle \Psi_\varepsilon \left| \mathrm{d}\Gamma_\varepsilon(\omega) \right| \Psi_\varepsilon \rangle^{1/2} \left\| \Psi_\varepsilon \right\|_{\mathscr{F}_\varepsilon} . \tag{4.1}$$

Moreover, $V_\varepsilon \left| \mathbf{P} \right|^{-1/2} \in \mathscr{B}(L^2)$ uniformly in ε.

Proof. Taking the modulus of both sides of the identity

$$\left\langle \psi \otimes \Psi_\varepsilon \left| a^*_\varepsilon \left(e^{ix\cdot k} \tfrac{\chi}{\sqrt{\omega}} \right) \varphi \otimes \Psi_\varepsilon \right.\right\rangle = \int \overline{\psi}(x) \varphi(x) e^{ix\cdot k} \frac{\chi(k)}{\sqrt{\omega(k)}} \left\langle \Psi_\varepsilon \left| a^*_\varepsilon(k) \Psi_\varepsilon \right.\right\rangle_{\mathscr{F}_\varepsilon} \mathrm{d}x\mathrm{d}k$$

yields

$$\left| \left\langle \psi \otimes \Psi_\varepsilon \left| a^*_\varepsilon \left(e^{ix\cdot k} \tfrac{\chi}{\sqrt{\omega}} \right) \varphi \otimes \Psi_\varepsilon \right.\right\rangle \right| \leqslant \left\| \overline{\mathcal{F}}(\overline{\psi}\varphi) \tfrac{\chi}{\omega} \right\|_{L^2_k} \left\| \sqrt{\omega(\cdot)} \left\| a_\varepsilon(\cdot) \Psi_\varepsilon \right\|_{\mathscr{F}_\varepsilon} \right\|_{L^2_k} \left\| \Psi_\varepsilon \right\|_{\mathscr{F}_\varepsilon} .$$

We have that $\left\| \sqrt{\omega(\cdot)}\, \|a_\varepsilon(\cdot)\Psi_\varepsilon\|_{\mathscr{F}_\varepsilon} \right\|_{L^2_k} = \langle \Psi_\varepsilon \,|\mathrm{d}\Gamma_\varepsilon(\omega)|\, \Psi_\varepsilon\rangle^{1/2}$ and the term $\left\|\overline{\mathcal{F}}(\overline{\psi}\varphi)\frac{\chi}{\omega}\right\|_{L^2_k}$ is bounded by Lemma 3.2. This shows (5.1). The facts that the operator $V_\varepsilon\,|\mathbf{P}|^{-1/2}$ is bounded on L^2 uniformly in ε follows from Lemma 3.1. □

We can then make sense out of the effective potential V_μ (recall its definition in (2.19)) obtained as the quasi-classical limit of V_ε.

Proposition 4.2 (Estimates of V_μ)**.**
Suppose that Assumptions (A_ω), $(\mathrm{A}_\Psi^{\mathrm{Nel}})$ *and* (A_χ) *hold. Let* $\mu \in \mathscr{M}_\omega(\Psi_\varepsilon)$*. Then,*

$$|\langle \psi \,|V_\mu|\, \varphi\rangle| \lesssim \left\|\mathcal{F}(\bar{\psi}\varphi)\right\|_{L^{6,2}} \left(\int_{\mathfrak{h}_\omega} \left\|\omega^{1/2} z\right\|_{L^2}^2 \,\mathrm{d}\mu(z)\right)^{1/2} . \tag{4.2}$$

Moreover, $V_\mu\,|\mathbf{P}|^{-1/2} \in \mathscr{B}(L^2)$*.*

Proof. The Cauchy-Schwarz inequality followed by Lemma 3.2 yield

$$\begin{aligned}|\langle \psi \,|V_\mu|\, \varphi\rangle| &\lesssim \|\mathcal{F}(\psi\overline{\varphi})\chi/\omega\|_{L^2} \left(\int_{\mathfrak{h}_\omega} \left\|\omega^{1/2} z\right\|_{L^2}^2 \,\mathrm{d}\mu(z)\right)^{1/2} \\ &\lesssim \left\|\mathcal{F}(\bar{\psi}\varphi)\right\|_{L^{6,2}} \left\|\tfrac{\chi}{\omega}\right\|_{L^{3,\infty}} \left(\int_{\mathfrak{h}_\omega} \left\|\omega^{1/2} z\right\|_{L^2}^2 \,\mathrm{d}\mu(z)\right)^{1/2} .\end{aligned}$$

As in the previous proof, Lemma 3.1 shows that the operator $V_\mu|\mathbf{P}|^{-1/2}$ is bounded on L^2. □

We are now ready to prove a convergence result of V_ε towards V_μ:

Proposition 4.3 (Convergence of V_ε)**.**
Suppose that Assumptions (A_ω), $(\mathrm{A}_\Psi^{\mathrm{Nel}})$ *and* (A_χ) *hold. Let* $\mu \in \mathscr{M}_\omega(\Psi_\varepsilon)$ *and* $\{\varepsilon_n\}_{n\in\mathbb{N}}$ *be such that* $\Psi_{\varepsilon_n} \xrightarrow[n\to+\infty]{\omega-\mathrm{sc}} \mu$*. If* $\mathcal{F}(\bar{\psi}\varphi) \in L^{6,2}$*, then*

$$\int_{\mathbb{R}^3} (V_{\varepsilon_n} - V_\mu)\,\bar{\psi}\varphi \xrightarrow[n\to+\infty]{} 0. \tag{4.3}$$

Moreover, the family of operators $(V_{\varepsilon_n} - V_\mu)\,|\mathbf{P}|^{-1/2}$ *is bounded on* L^2 *uniformly in* n *and converges weakly to* 0*.*

Proof. If $\mathcal{F}(\bar{\psi}\varphi) \in L^{6,2}$, we have

$$\begin{aligned}\langle \psi \,|V_{\varepsilon_n}|\, \varphi\rangle_{L^2_x} &= \left\langle \psi \otimes \Psi_{\varepsilon_n} \left| \mathrm{Re}\, a^\dagger_{\varepsilon_n}\left(e^{ix\cdot k}\tfrac{\chi}{\sqrt{\omega}}\right)\right| \varphi \otimes \Psi_{\varepsilon_n}\right\rangle \\ &= \left\langle \Psi_{\varepsilon_n} \left| a^\dagger_{\varepsilon_n}\left(\left\langle \psi \left| e^{ix\cdot k} \varphi\right.\right\rangle_{L^2_x} \tfrac{\chi}{\sqrt{\omega}}\right) \Psi_{\varepsilon_n}\right.\right\rangle_{\mathscr{F}_{\varepsilon_n}} + \left\langle \Psi_{\varepsilon_n} \left| a_{\varepsilon_n}\left(\left\langle \varphi \left| e^{ix\cdot k} \psi\right.\right\rangle_{L^2_x} \tfrac{\chi}{\sqrt{\omega}}\right) \Psi_{\varepsilon_n}\right.\right\rangle_{\mathscr{F}_{\varepsilon_n}} \\ &\xrightarrow[n\to+\infty]{} \int_{\mathfrak{h}_\omega} \left\langle \psi \left| \mathrm{Re}\left(\left\langle \omega^{1/2} z \left| e^{ix\cdot k}\tfrac{\chi}{\omega}\right.\right\rangle_{L^2_k}\right)\right| \varphi\right\rangle_{L^2_x} \mathrm{d}\mu(z)\,,\end{aligned}$$

where we used that $\mu \in \mathscr{M}_\omega(\Psi_\varepsilon)$ (see Definition 2.1) and that $\overline{\mathcal{F}}(\overline{\psi}\varphi)\chi/\omega \in L^2$ by Lemma 3.2.

The uniform boundedness in n of the family of operators $(V_{\varepsilon_n} - V_\mu)\,|\mathbf{P}|^{-1/2}$ in L^2 follows from Propositions 4.1 and 4.2. The fact that $(V_{\varepsilon_n} - V_\mu)\,|\mathbf{P}|^{-1/2}$ converges weakly to 0 follows from Lemma 3.1 with $\alpha_1 = 0$, $\alpha_2 = 1/2$ and $d = 3$: it shows that if $\varphi \in \dot{H}^{1/2}$ and $\psi \in L^2$, then $\mathcal{F}(\bar{\psi}\varphi) \in L^{6,2}$ and we can then conclude thanks to (4.3). □

Remark 4.4.
The previous proof also shows that $\mathcal{F}(V_{\varepsilon_n}) \xrightarrow[n\to+\infty]{} \mathcal{F}(V_\mu)$ weakly in $L^{6/5,2}$.

We are now ready to prove the main result of this section on the Nelson model.

Proof of Theorem 2.3. Let us fix $\mu \in \mathscr{M}_\omega(\Psi_\varepsilon)$, and the sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ such that $\Psi_{\varepsilon_n} \to \mu$.

To justify that $H^{\mathrm{Nel}}_{\varepsilon_n} = -\Delta + U + V_{\varepsilon_n}$ and $H^{\mathrm{Nel}}_\mu = -\Delta + U + V_\mu$ define symmetric closed quadratic forms with form domain $\mathcal{Q} := \mathcal{Q}(H^{\mathrm{Nel}}_{\varepsilon_n}) = \mathcal{Q}(H^{\mathrm{Nel}}_\mu) = H^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3, U_+\,\mathrm{d}x)$, it suffices to argue as follows: first, since U_+ is non-negative and belongs to L^1_{loc}, $-\Delta + U_+$ identifies with a self-adjoint operator with form domain $\mathcal{Q}$. Next, Assumption (A_U) together with Proposition 4.1 show that $U_- + V_{\varepsilon_n}$ is relatively form bounded w.r.t. $-\Delta$ (and hence w.r.t. $-\Delta + U_+$) with a relative bound < 1. The KLMN Theorem (see e.g. [RS75, Theorem X.17]) then shows that $H^{\mathrm{Nel}}_{\varepsilon_n}$ identifies with a self-adjoint operator with form domain $\mathcal{Q}$. The same holds for H^{Nel}_μ, using Proposition 4.2 instead of Proposition 4.1.

Now the goal is to prove Γ-convergence of the quadratic form $\left\langle \varphi \left| H^{\mathrm{Nel}}_n \right| \varphi \right\rangle$ to $\langle \varphi | H_\mu | \varphi \rangle$ as $n \to \infty$, in the common domain of definition $\mathcal{Q}$.

Let us start with the Γ-lim sup. For any $\varphi \in \mathcal{Q}$, we have to construct a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{Q}$ such that

$$\varphi_n \xrightarrow[n\to+\infty]{\mathcal{Q}} \varphi\,, \qquad \limsup_{n\to+\infty} \left\langle \varphi_n \left| H^{\mathrm{Nel}}_n \right| \varphi_n \right\rangle \leqslant \langle \varphi | H_\mu | \varphi \rangle\ . \tag{4.4}$$

In view of Proposition 4.3, it is sufficient to choose $\varphi_n \equiv \varphi$ the constant sequence, to get

$$\lim_{n\to+\infty} \left\langle \varphi_n \left| H^{\mathrm{Nel}}_n \right| \varphi_n \right\rangle = \langle \varphi | H_\mu | \varphi \rangle\ .$$

For the Γ-lim inf, we apply Proposition A.2: the fact that the assumptions of Proposition A.2 are satisfied follows from Propositions 4.1 to 4.3 (note in particular that (A.2) holds with $\delta = 1/4$). This concludes the proof.

□

5. The Pauli-Fierz Model: Proof of Theorem 2.7

In this section, we set $L^p := L^p(\mathbb{R}^3 \otimes \mathbb{C}^2)$, $L^{p,q} := L^{p,q}(\mathbb{R}^3 \otimes \mathbb{C}^2)$ and $H^s := H^s(\mathbb{R}^3 \otimes \mathbb{C}^2)$. Recall the definition of the vector potential $\mathbf{A}_\varepsilon$ in (2.27) with form domain $H^1 \cap L^2(U_+\,\mathrm{d}x)$. With the same proof as Proposition 4.1 for linearly coupled models, one gets:

Proposition 5.1 (Estimate of $\mathbf{A}_\varepsilon$)**.**
Suppose that Assumptions (A_ω), ($\mathrm{A}^{\mathrm{PF}}_\Psi$) and ($\mathrm{A}_\chi$) hold. If $\mathcal{F}(\bar{\psi}\varphi) \in L^{6,2}$, then,

$$\left| \langle \psi | \mathbf{A}_\varepsilon \varphi \rangle_{L^2_x} \right| \lesssim \left\| \mathcal{F}(\bar{\psi}\varphi) \right\|_{L^{6,2}} \langle \Psi_\varepsilon | \mathrm{d}\Gamma_\varepsilon(\omega) | \Psi_\varepsilon \rangle^{1/2} \left\| \Psi_\varepsilon \right\|_{\mathscr{F}_\varepsilon}\ . \tag{5.1}$$

Moreover, $\mathbf{A}_\varepsilon \, |\mathbf{P}|^{-1/2} \in \mathscr{B}(L^2)$ uniformly in ε.

Next we decompose the potential W_ε defined in (2.29) as

$$W_\varepsilon = W_{a^*a^*,\varepsilon} + W_{aa,\varepsilon} + 2W_{a^*a,\varepsilon}$$

with

$$W_{a^*a,\varepsilon}(x) = \left\| a_\varepsilon \left(e^{ix\cdot k} \tfrac{\chi}{\sqrt{\omega}} \right) \Psi_\varepsilon \right\|^2_{\mathscr{F}_\varepsilon}, \tag{5.2}$$

$$W_{aa,\varepsilon}(x) = \left\langle \Psi_\varepsilon \left| a_\varepsilon^2 \left(e^{ix\cdot k} \tfrac{\chi}{\sqrt{\omega}} \right) \Psi_\varepsilon \right.\right\rangle_{\mathscr{F}_\varepsilon}, \tag{5.3}$$

and likewise for $W_{a^*a^*,\varepsilon}(x) = \overline{W_{aa,\varepsilon}(x)}$.

Proposition 5.2 (Estimate of W_ε)**.**
Suppose that Assumptions (A_ω), $(\mathrm{A}_\Psi^{\mathrm{PF}})$ *and* (A_χ) *hold. If* $\mathcal{F}(\bar{\psi}\varphi) \in L^{3,2}$, *then*

$$\left| \langle \psi \left| W_{a^*a,\varepsilon} \right| \varphi \rangle_{L^2_x} \right| \lesssim \left\| \mathcal{F}(\bar{\psi}\varphi) \right\|_{L^{3,\infty}} \langle \Psi_\varepsilon \left| \mathrm{d}\Gamma_\varepsilon(\omega) \right| \Psi_\varepsilon \rangle , \tag{5.4}$$

and

$$\left| \langle \psi \left| W_{aa,\varepsilon} \varphi \right\rangle_{L^2_x} \right| \lesssim \left\| \mathcal{F}(\bar{\psi}\varphi) \right\|_{L^{3,2}} \left\| \Psi_\varepsilon \right\|_{\mathscr{F}_\varepsilon} \left\langle \Psi_\varepsilon \left| \mathrm{d}\Gamma_\varepsilon^{(2)}(\omega\otimes\omega) \right| \Psi_\varepsilon \right\rangle^{1/2} . \tag{5.5}$$

As a consequence,

$$\left| \langle \psi \left| W_\varepsilon \right| \varphi \rangle_{L^2_x} \right| \lesssim \left\| \mathcal{F}(\bar{\psi}\varphi) \right\|_{L^{3,2}} , \tag{5.6}$$

and $W_\varepsilon \left| \mathbf{P} \right|^{-1} \in \mathscr{B}(L^2)$ *uniformly in* ε.

Proof. By the definition of $W_{a^*a,\varepsilon}$, we have

$$\begin{aligned}
\langle \psi \left| W_{a^*a,\varepsilon} \right| \varphi \rangle_{L^2_x} &= \left\langle a_\varepsilon \left(e^{ix\cdot k} \tfrac{\chi}{\sqrt{\omega}} \right) \psi \otimes \Psi_\varepsilon \left| a_\varepsilon \left(e^{ix\cdot k} \tfrac{\chi}{\sqrt{\omega}} \right) \varphi \otimes \Psi_\varepsilon \right.\right\rangle \\
&= \iiint \bar{\psi}(x)\varphi(x) e^{-ix\cdot(k-k')} \frac{\chi(k)}{\sqrt{\omega(k)}} \frac{\chi(k')}{\sqrt{\omega(k')}} \left\langle a_\varepsilon(k)\Psi_\varepsilon \left| a_\varepsilon(k')\Psi_\varepsilon \right.\right\rangle_{\mathscr{F}_\varepsilon} \mathrm{d}x\mathrm{d}k\mathrm{d}k' \\
&= \iint \mathcal{F}(\bar{\psi}\varphi)(k-k') \frac{\chi(k)}{\sqrt{\omega(k)}} \frac{\chi(k')}{\sqrt{\omega(k')}} \left\langle a_\varepsilon(k)\Psi_\varepsilon \left| a_\varepsilon(k')\Psi_\varepsilon \right.\right\rangle_{\mathscr{F}_\varepsilon} \mathrm{d}k\mathrm{d}k',
\end{aligned}$$

and hence, taking the modulus of the previous equalities provides the bound

$$\begin{aligned}
\left| \langle \psi \left| W_{a^*a,\varepsilon} \right| \varphi \rangle_{L^2_x} \right| &\leqslant \iint \left| \mathcal{F}(\bar{\psi}\varphi)(k-k') \right| \frac{\chi(k)}{\sqrt{\omega(k)}} \frac{\chi(k')}{\sqrt{\omega(k')}} \left\| a_\varepsilon(k)\Psi_\varepsilon \right\|_{\mathscr{F}_\varepsilon} \left\| a_\varepsilon(k')\Psi_\varepsilon \right\|_{\mathscr{F}_\varepsilon} \mathrm{d}k\mathrm{d}k' \\
&= \int \frac{\chi(k)}{\sqrt{\omega(k)}} \left\| a_\varepsilon(k)\Psi_\varepsilon \right\|_{\mathscr{F}_\varepsilon} \left[\left| \mathcal{F}(\bar{\psi}\varphi) \right| * \frac{\chi(\cdot)}{\sqrt{\omega(\cdot)}} \left\| a_\varepsilon(\cdot)\Psi_\varepsilon \right\|_{\mathscr{F}_\varepsilon} \right](k)\mathrm{d}k \,.
\end{aligned}$$

Hölder, Young and again Hölder's inequalities in Lorentz spaces yield

$$\begin{aligned}
\left| \langle \psi \left| W_{a^*a,\varepsilon} \right| \varphi \rangle_{L^2_x} \right| &\lesssim \left\| \frac{\chi(k)}{\sqrt{\omega(k)}} \left\| a_\varepsilon(k)\Psi_\varepsilon \right\|_{\mathscr{F}_\varepsilon} \right\|_{L_k^{6/5,2}} \left\| \left| \mathcal{F}(\bar{\psi}\varphi) \right| * \frac{\chi(\cdot)}{\sqrt{\omega(\cdot)}} \left\| a_\varepsilon(\cdot)\Psi_\varepsilon \right\|_{\mathscr{F}_\varepsilon} \right\|_{L^{6,2}} \\
&\lesssim \left\| \frac{\chi(k)}{\sqrt{\omega(k)}} \left\| a_\varepsilon(k)\Psi_\varepsilon \right\|_{\mathscr{F}_\varepsilon} \right\|^2_{L_k^{6/5,2}} \left\| \mathcal{F}(\bar{\psi}\varphi) \right\|_{L^{3,\infty}} \\
&\lesssim \left\| \tfrac{\chi}{\omega} \right\|^2_{L^{3,\infty}} \left\| \left\| \omega(\cdot)^{1/2} a_\varepsilon(\cdot)\Psi_\varepsilon \right\|_{\mathscr{F}_\varepsilon} \right\|^2_{L^2} \left\| \mathcal{F}(\bar{\psi}\varphi) \right\|_{L^{3,\infty}} \\
&= \left\| \tfrac{\chi}{\omega} \right\|^2_{L^{3,\infty}} \left\| \mathcal{F}(\bar{\psi}\varphi) \right\|_{L^{3,\infty}} \langle \Psi_\varepsilon \left| \mathrm{d}\Gamma_\varepsilon(\omega) \right| \Psi_\varepsilon \rangle \,.
\end{aligned}$$

Next, by the definition of $W_{aa,\varepsilon}$,

$$\begin{aligned}\langle \psi \,|W_{aa,\varepsilon}\varphi\rangle_{L^2_x} &= \Big\langle \psi\otimes\Psi_\varepsilon \,\Big|a^2_\varepsilon\Big(e^{ix\cdot k}\tfrac{\chi}{\sqrt{\omega}}\Big)\ \varphi\otimes\Psi_\varepsilon\Big\rangle\\ &= \iint \mathcal{F}(\bar\psi\varphi)(-k-k')\tfrac{\chi(k)}{\omega(k)}\tfrac{\chi(k')}{\omega(k')}\sqrt{\omega(k)\omega(k')}\,\langle\Psi_\varepsilon\,|a_\varepsilon(k)a_\varepsilon(k')\Psi_\varepsilon\rangle_{\mathscr{F}_\varepsilon}\,\mathrm{d}k\mathrm{d}k'\,.\end{aligned}$$

The Cauchy-Schwarz inequality in $L^2_{k,k'}$ along with Lemma 3.3 yield

$$\begin{aligned}&\Big|\langle \psi\,|W_{aa,\varepsilon}\varphi\rangle_{L^2_x}\Big|\\ &\leqslant \Big\|\mathcal{F}(\bar\psi\varphi)(-k-k')\tfrac{\chi(k)}{\omega(k)}\tfrac{\chi(k')}{\omega(k')}\Big\|_{L^2_{k,k'}}\Big\|\sqrt{\omega(k)\omega(k')}\,\langle\Psi_\varepsilon\,|a_\varepsilon(k)a_\varepsilon(k')\Psi_\varepsilon\rangle_{\mathscr{F}_\varepsilon}\Big\|_{L^2_{k,k'}}\\ &\lesssim \big\|\tfrac{\chi}{\omega}\big\|^2_{L^{3,\infty}}\big\|\mathcal{F}(\bar\psi\varphi)\big\|_{L^{3,2}}\|\Psi_\varepsilon\|_{\mathscr{F}_\varepsilon}\Big\|\sqrt{\omega(k)\omega(k')}\,\|a_\varepsilon(k)a_\varepsilon(k')\Psi_\varepsilon\|_{\mathscr{F}_\varepsilon}\Big\|_{L^2_{k,k'}}\\ &= \big\|\tfrac{\chi}{\omega}\big\|^2_{L^{3,\infty}}\big\|\mathcal{F}(\bar\psi\varphi)\big\|_{L^{3,2}}\|\Psi_\varepsilon\|_{\mathscr{F}_\varepsilon}\Big\langle\Psi_\varepsilon\,\Big|\mathrm{d}\Gamma^{(2)}_\varepsilon(\omega\otimes\omega)\Big|\,\Psi_\varepsilon\Big\rangle^{1/2}\end{aligned}$$

where we recall that $\mathrm{d}\Gamma^{(2)}_\varepsilon(\omega\otimes\omega)=\mathrm{d}\Gamma_\varepsilon(\omega)^2-\varepsilon\mathrm{d}\Gamma_\varepsilon(\omega^2)$.

We deduce the uniform bound on the norm of $W_\varepsilon\,|\mathbf{P}|^{-1}$ from Lemma 3.1 with $\alpha_1=0$, $\alpha_2=1$ and $d=3$, since, if $\varphi\in\dot H^1$ and $\psi\in L^2$, then $\mathcal{F}(\bar\psi\varphi)\in L^{3,2}$. □

In our proof of the Γ-convergence below, we will need a related estimate on

$$\mathbf{B}_\varepsilon := \nabla\wedge\mathbf{A}_\varepsilon.$$

In this respect, it turns out that Assumption (A_χ) is not sufficient for our purpose. The next result holds under the slightly stronger Assumption (A'_χ).

Proposition 5.3 (Estimate of $\mathbf{B}_\varepsilon$)**.**
Suppose that Assumptions (A_ω), (A^{PF}_Ψ) and (A'_χ) hold. If $\bar\psi\varphi\in L^2$, then

$$|\langle\psi\,|\mathbf{B}_\varepsilon|\,\varphi\rangle|\lesssim\big\|\bar\psi\varphi\big\|_{L^2}\,.$$

As a consequence, $|\mathbf{P}|^{-3/4}\mathbf{B}_\varepsilon\,|\mathbf{P}|^{-3/4}\in\mathscr{B}(L^2)$ uniformly in ε.

Proof. Note that

$$\langle\psi\,|\mathbf{B}_\varepsilon(x)|\,\varphi\rangle=\langle\psi\otimes\Psi_\varepsilon\,|\mathbb{B}_\varepsilon(x)|\,\varphi\otimes\Psi_\varepsilon\rangle\,,$$

where

$$\mathbb{B}_\varepsilon(x)=\sqrt{\varepsilon}\sum_{\lambda=1}^{2}\int\frac{\chi(k)}{\sqrt{\omega(k)}}k\wedge\mathbf{e}_\lambda(k)\Big(e^{ik\cdot x}a_\lambda(k)+e^{-ik\cdot x}a^\dagger_\lambda(k)\Big)\,\mathrm{d}k\,.$$

Proceeding in the same way as for $\mathbf{A}_\varepsilon$ or V_ε, we can thus estimate

$$|\langle\psi\,|\mathbf{B}_\varepsilon(x)|\,\varphi\rangle|\lesssim\Big\|\tfrac{|k|\chi}{\omega}\mathcal{F}(\bar\psi\varphi)\Big\|_{L^2}\lesssim\big\|\bar\psi\varphi\big\|_{L^2}\,,$$

since $|k|\chi/\omega\in L^\infty$ thanks to Assumption (A'_χ).

The uniform boundedness of $|\mathbf{P}|^{-3/4}\mathbf{B}_\varepsilon\,|\mathbf{P}|^{-3/4}$ then follows from Lemma 3.1 with $\alpha_1=\alpha_2=3/4$ and $d=3$. □

Recall that $\mathbf{A}_\mu$ has been defined in (2.33). Similarly as for $\mathbf{B}_\varepsilon$, we set

$$\mathbf{B}_\mu := \nabla\wedge\mathbf{A}_\mu.$$

Proposition 5.4 (Estimate of $\mathbf{B}_\mu$)**.**
Suppose that Assumptions (A_ω) *and* (A_χ) *hold. Let* $\mu \in \mathscr{M}_\omega(\Psi_\varepsilon)$*. If* $\mathcal{F}(\bar{\psi}\varphi) \in L^{6,2}$*, then*

$$|\langle \psi \,| \mathbf{A}_\mu |\, \varphi \rangle| \lesssim \left\| \mathcal{F}(\bar{\psi}\varphi) \right\|_{L^{6,2}} \left(\int_{\mathfrak{h}_\omega} \left\| \omega^{1/2} z \right\|_{L^2}^2 \mathrm{d}\mu(z) \right)^{1/2} \tag{5.7}$$

and $\mathbf{A}_\mu \,|\mathbf{P}|^{-1/2} \in \mathscr{B}(L^2)$*. Likewise, if* $\mathcal{F}(\bar{\psi}\varphi) \in L^{3,2}$*, then*

$$|\langle \psi \,| W_\mu |\, \varphi \rangle| \lesssim \left\| \mathcal{F}(\bar{\psi}\varphi) \right\|_{L^{3,2}} \tag{5.8}$$

and $W_\mu \,|\mathbf{P}|^{-1} \in \mathscr{B}(L^2)$*.*

Moreover, if Assumption (A_χ) *is replaced by the stronger Assumption* (A'_χ) *and if* $\bar{\psi}\varphi \in L^2$*, then*

$$|\langle \psi \,| \mathbf{B}_\mu |\, \varphi \rangle| \lesssim \left\| \bar{\psi}\varphi \right\|_{L^2} \tag{5.9}$$

and $|\mathbf{P}|^{-3/4} \mathbf{B}_\mu \,|\mathbf{P}|^{-3/4} \in \mathscr{B}(L^2)$*.*

Proof. To prove the first estimate, it suffices to write

$$|\langle \psi \,| \mathbf{A}_\mu |\, \varphi \rangle| \lesssim \| \mathcal{F}(\psi \overline{\varphi}) \chi / \omega \|_{L^2} \left(\int_{\mathfrak{h}_\omega} \left\| \omega^{1/2} z \right\|_{L^2}^2 \mathrm{d}\mu(z) \right)^{1/2}$$
$$\lesssim \left\| \mathcal{F}(\bar{\psi}\varphi) \right\|_{L^{6,2}} \left\| \tfrac{\chi}{\omega} \right\|_{L^{3,\infty}} \left(\int_{\mathfrak{h}_\omega} \left\| \omega^{1/2} z \right\|_{L^2}^2 \mathrm{d}\mu(z) \right)^{1/2}$$

where we used Lemma 3.3 as before. The statements concerning W_μ and $\mathbf{B}_\mu$ can be proven in the same way. □

Proposition 5.5 (Convergence of $\mathbf{A}_\varepsilon$, W_ε, $\mathbf{B}_\varepsilon$)**.**
Suppose that Assumptions (A_ω), (A_Ψ^{PF}) *and* (A_χ) *hold. Let* $\mu \in \mathscr{M}_\omega(\Psi_\varepsilon)$ *and* $\{\varepsilon_n\}_{n\in\mathbb{N}}$ *be such that* $\Psi_{\varepsilon_n} \xrightarrow[n\to+\infty]{\omega-\mathrm{sc}} \mu$*. If* $\mathcal{F}(\bar{\psi}\varphi) \in L^{6,2}$*, then*

$$\int_{\mathbb{R}^3} \left(\mathbf{A}_{\varepsilon_n} - \mathbf{A}_\mu \right) \bar{\psi}\varphi \xrightarrow[n\to+\infty]{} 0 \tag{5.10}$$

and the family of operators $(\mathbf{A}_{\varepsilon_n} - \mathbf{A}_\mu)\,|\mathbf{P}|^{-1/2}$ *is bounded on* L^2 *uniformly in* n *and converges weakly to* 0*. Likewise, if* $\mathcal{F}(\bar{\psi}\varphi) \in L^{3,2}$*, then*

$$\int_{\mathbb{R}^3} \left(W_{\varepsilon_n} - W_\mu \right) \bar{\psi}\varphi \xrightarrow[n\to+\infty]{} 0 \tag{5.11}$$

and the family of operators $(W_{\varepsilon_n} - W_\mu)\,|\mathbf{P}|^{-1}$ *is bounded on* L^2 *uniformly in* n *and converges weakly to* 0*.*

Moreover, if Assumption (A_χ) *is replaced by the stronger Assumption* (A'_χ) *and if* $\bar{\psi}\varphi \in L^2$*, then*

$$\int_{\mathbb{R}^3} \left(\mathbf{B}_{\varepsilon_n} - \mathbf{B}_\mu \right) \bar{\psi}\varphi \xrightarrow[n\to+\infty]{} 0, \tag{5.12}$$

and the family of operators $|\mathbf{P}|^{-3/4}(\mathbf{B}_{\varepsilon_n} - \mathbf{B}_\mu)\,|\mathbf{P}|^{-3/4}$ *is bounded on* L^2 *uniformly in* n *and converges weakly to* 0*.*

Proof. The proof of the convergence of $\mathbf{A}_{\varepsilon_n}$ is the same as the proof of Proposition 4.3.

Likewise, using that $\mu \in \mathscr{M}_\omega(\Psi_\varepsilon)$ (see Definition 2.1), we have that

$$\begin{aligned}\langle \psi \,| W_{\varepsilon_n} |\, \varphi\rangle &= \left\langle \psi(x) \otimes \Psi_{\varepsilon_n} \left| \left(a^\dagger_{\varepsilon_n}(\mathbf{w}_x)\right)^2 + (a_{\varepsilon_n}(\mathbf{w}_x))^2 + 2a^\dagger_{\varepsilon_n}(\mathbf{w}_x) a_{\varepsilon_n}(\mathbf{w}_x) \right| \varphi(x) \otimes \Psi_{\varepsilon_n} \right\rangle \\ &= \left\langle \Psi_{\varepsilon_n} \left| \left\langle \psi(x) \left| \left(a^\dagger_{\varepsilon_n}(\mathbf{w}_x)\right)^2 + (a_\varepsilon(\mathbf{w}_x))^2 + 2a^\dagger_{\varepsilon_n}(\mathbf{w}_x) a_{\varepsilon_n}(\mathbf{w}_x) \right| \varphi(x) \right\rangle_{L^2_x} \Psi_{\varepsilon_n} \right\rangle_{\mathscr{F}_{\varepsilon_n}} \\ &\xrightarrow[n\to+\infty]{} \int_{\mathfrak{h}_\omega} \left\langle \psi(x) \left| (2\mathrm{Re}\, \langle z | w_x\rangle)^2 \right| \varphi(x) \right\rangle \mathrm{d}\mu(z) = \langle \psi \mid W_\mu \varphi\rangle,\end{aligned}$$

since $\overline{\mathcal{F}}(\bar{\psi}\varphi)(k+k')\frac{\chi(k)}{\omega(k)}\frac{\chi(k')}{\omega(k')} \in L^2_{k,k'}$ and μ is supported on functions z such that $\sqrt{\omega} z \in L^2$.

The weak convergence results follow similarly and the uniform boundedness results follow from Propositions 5.1 to 5.4. □

Remark 5.6.

The previous proof also implies that

- $\mathcal{F}(\mathbf{A}_{\varepsilon_n}) \xrightarrow[n\to+\infty]{} \mathcal{F}(\mathbf{A}_\mu)$ weakly in $L^{6/5,2}$,
- $\mathcal{F}(W_{\varepsilon_n}) \xrightarrow[n\to+\infty]{} \mathcal{F}(W_\mu)$ weakly in $L^{3/2,2}$,
- $\mathbf{B}_{\varepsilon_n} \xrightarrow[n\to+\infty]{} \mathbf{B}_\mu$ weakly in L^2.

We are now ready to prove our main result about the Pauli-Fierz model.

Proof of Theorem 2.7. We proceed as in the proof of Theorem 2.3. Let us fix $\mu \in \mathscr{M}_\omega(\Psi_\varepsilon)$, and the sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ such that $\Psi_{\varepsilon_n} \to \mu$.

To justify that $H^{\mathrm{PF}}_{\varepsilon_n}$ and H^{PF}_μ define symmetric closed quadratic forms with form domains $\mathcal{Q} := H^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3, U_+\,\mathrm{d}x)$, it suffices to argue exactly as in the proof of Theorem 2.3, using that $H^{\mathrm{PF}}_{\varepsilon_n} - (-\Delta + U_+)$ and $H^{\mathrm{PF}}_{\varepsilon_n} - (-\Delta + U_+)$ are relatively form bounded w.r.t. $-\Delta$ with a relative bound < 1, which follows from Assumption (A_U) and Propositions 5.1 to 5.4.

Next, we prove Γ-convergence of the quadratic forms $\left\langle \varphi \left| H^{\mathrm{PF}}_{\varepsilon_n} \right| \varphi \right\rangle$ to $\left\langle \varphi \left| H^{\mathrm{PF}}_\mu \right| \varphi \right\rangle$.

For the Γ- lim sup we take again the constant sequence and use Proposition 5.5: for any $\varphi \in \mathcal{Q}$,

$$\left\langle \varphi \left| H^{\mathrm{PF}}_{\varepsilon_n} - H^{\mathrm{PF}}_\mu \right| \varphi \right\rangle = 2\mathrm{Re}\, \langle \boldsymbol{\sigma} \cdot \mathbf{P}\varphi | \boldsymbol{\sigma} \cdot (\mathbf{A}_\mu - \mathbf{A}_{\varepsilon_n})\, \varphi\rangle + \langle \varphi \,| W_{\varepsilon_n} - W_\mu |\, \varphi\rangle \xrightarrow[n\to+\infty]{} 0\,.$$

For the Γ- lim inf, we apply Proposition A.2: the fact that the assumptions of Proposition A.2 are satisfied follows from Propositions 5.1 to 5.5 (in particular, (A.2) holds with $\delta = 3/8$). This concludes the proof.

□

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Appendix A. Weak Γ-convergence for Schrödinger and Pauli operators

In this short appendix, we will prove some results concerning the limes inferior of a sequence of operators in the weak topology that are useful for proving Γ-convergence of quasi-classical operators in the main text. We consider Schrödinger or Pauli type operators, with perturbations that are uniformly KLMN-relatively-small and converge weakly.

We first prove a preliminary useful lemma.

Lemma A.1. *Let $Q : \mathcal{Q} \to \mathbb{R}_+$ be a non-negative, densely defined quadratic form, and $Q[\,\cdot\,,\,\cdot\,]$ the associated sesquilinear form. Then,*

$$Q[\psi] = \sup_{\phi \in \mathcal{Q}} \operatorname{Re} Q[\phi, 2\psi - \phi] \ .$$

Proof. By the polarization identity, we can write

$$\operatorname{Re} Q[\phi, 2\psi - \phi] = \tfrac{1}{4} \operatorname{Re} \Big\{ Q[2\psi - \phi + \phi] - Q[2\psi - \phi - \phi] + iQ[2\psi - \phi - i\phi] - iQ[2\psi - \phi - i\phi] \Big\}$$
$$= \tfrac{1}{4} \Big(Q[2\psi] - Q[2(\psi - \phi)] \Big) = Q[\psi] - Q[\psi - \phi] \ .$$

Therefore,

$$\sup_{\phi \in \mathcal{Q}} \operatorname{Re} Q[\phi, 2\psi - \phi] = Q[\psi] - \inf_{\phi \in \mathcal{Q}} Q[\psi - \phi] = Q[\psi] \ ,$$

since $Q[\,\cdot\,] \geqslant 0$ (choose $\phi = \psi$). □

Let $\mathfrak{h} = L^2(\mathbb{R}^d; \mathbb{C}^\nu)$ with $\nu = 1$ in the case of Schrödinger operators and $\nu = 2$ for Pauli operators, and consider

$$H_n = -\Delta + U + V_n \ , \tag{A.1}$$

where U satisfies Assumption (A_U), $V_n = W_n$ for Schrödinger, $V_n = W_n + \nabla \cdot \mathbf{A}_n + \sigma \cdot \mathbf{B}_n$ for Pauli, with W_n measurable from $\mathbb{R}^d$ to $\mathbb{R}$, $\nabla \cdot \mathbf{A}_n = \mathbf{A}_n \cdot \nabla$, and $(\mathbf{A}_n)_j$, $(\mathbf{B}_n)_j$, for $1 \leqslant j \leqslant d$, measurable from $\mathbb{R}^d$ to $\mathbb{R}$. Let us suppose that V_n converges to some $V_\infty = W_\infty + \nabla \cdot \mathbf{A}_\infty + \sigma \cdot \mathbf{B}_\infty$ weakly on $\mathcal{Q}(-\Delta + U)$: for all $\psi, \phi \in \mathcal{Q}(-\Delta + U)$,

$$\lim_{n \to \infty} \langle \phi \,| V_n |\, \psi \rangle = \langle \phi \,| V_\infty |\, \psi \rangle \ .$$

We set $H_\infty := -\Delta + U + V_\infty$. Furthermore, we suppose that for some $0 < \delta < \frac{1}{2}$ and $\lambda_0 > 0$,

$$\begin{aligned} \Big\| (-\Delta + \lambda_0)^{-\delta} W_n (-\Delta + \lambda_0)^{-\delta} \Big\| &\leqslant C \ , \\ \Big\| \mathbf{A}_n (-\Delta + \lambda_0)^{-\delta} \Big\| &\leqslant C \ , \\ \Big\| (-\Delta + \lambda_0)^{-\delta} \mathbf{B}_n (-\Delta + \lambda_0)^{-\delta} \Big\| &\leqslant C \ , \end{aligned} \tag{A.2}$$

all uniformly with respect to $n \in \mathbb{N} \cup \{+\infty\}$. Let us remark that this implies that V_n (for $n \in \mathbb{N} \cup \{+\infty\}$) is a $-\Delta$-relatively small perturbation in the sense of quadratic forms: there exist $a < 1$ and $b \geqslant 0$ such that for all $\psi \in H^1(\mathbb{R}^d)$ and $n \in \mathbb{N} \cup \{+\infty\}$,

$$|\langle \psi \,| V_n |\, \psi \rangle| \leqslant a \, \langle \psi \,| -\Delta |\, \psi \rangle + b \|\psi\|^2 \ . \tag{A.3}$$

In turn, this implies that there exists $m \in \mathbb{R}$ bounding from below the spectrum of all H_n and H_∞, and that any non-uniformly bounded sequence in $\mathcal{Q}(-\Delta + U)$ makes $\langle \psi_n \,| H_n |\, \psi_n \rangle$ diverge as $n \to \infty$.

Proposition A.2 (weak Γ-lower bound)**.**
Let $H_n = -\Delta + U + V_n$, and $H_\infty = -\Delta + U + V_\infty$ be defined as above, with U satisfying Assumption (A_U) and such that: V_n converges weakly on $\mathcal{Q} := \mathcal{Q}(-\Delta+U)$ to V_∞, and (A.2) holds uniformy in $\mathbb{N}$ for some $\delta > 0$ and $\lambda_0 > 0$. Then, for any $\{\psi_n\}_{n\in\mathbb{N}} \subset \mathcal{Q}$ such that $\psi_n \xrightarrow[n\to\infty]{\mathrm{w}-L^2} \psi$ and $\psi \in \mathcal{Q}$, it holds that for all $\lambda > \max(-m, \lambda_0)$,

$$\liminf_{n\to\infty} \langle \psi_n | H_n + \lambda | \psi_n \rangle \geqslant \langle \psi | H_\infty + \lambda | \psi \rangle \ . \tag{A.4}$$

Proof. First of all, since a non-uniformly-bounded sequence ψ_n on $\mathcal{Q} := \mathcal{Q}(-\Delta + U)$ makes the l.h.s. of the inequality to prove diverge, we can restrict to uniformly bounded weakly convergent sequences, hence to sequences $\psi_n \xrightarrow[n\to+\infty]{\mathrm{w}-\mathcal{Q}} \psi$. Let $\psi_\kappa \in C_0^\infty(\mathbb{R}^d)$ be such that

$$\|\psi_\kappa - \psi\|_{\mathcal{Q}} < \kappa \ ,$$

and $\mathrm{supp}(\psi_\kappa) \subset B_0(R_\kappa)$, with $R_\kappa \to \infty$ as $\kappa \to 0$.

Using Lemma A.1, we can write

$$\begin{aligned} \langle \psi_n | H_n + \lambda | \psi_n \rangle = \sup_{\phi\in\mathcal{Q}} \mathrm{Re} \langle \phi | H_n + \lambda | 2\psi_n - \phi \rangle \geqslant \mathrm{Re} \langle \psi_\kappa | H_n + \lambda | 2\psi_n - \psi_\kappa \rangle \\ = - \langle \psi_\kappa | H_n + \lambda | \psi_\kappa \rangle + 2\mathrm{Re} \langle \psi_\kappa | H_n + \lambda | \psi_n \rangle \ . \end{aligned}$$

Now, the first term on the rightmost hand side converges by weak convergence of V_n:

$$\lim_{n\to\infty} \langle \psi_\kappa | H_n + \lambda | \psi_\kappa \rangle = \langle \psi_\kappa | H_\infty + \lambda | \psi_\kappa \rangle \ .$$

We rewrite the remaining term as

$$\mathrm{Re} \langle \psi_\kappa | H_n + \lambda | \psi_n \rangle = \mathrm{Re} \langle \psi_\kappa | -\Delta + U + \lambda | \psi_n \rangle + \mathrm{Re} \langle \psi_\kappa | V_n | \psi_n \rangle \ .$$

Since $\psi_n \xrightarrow[n\to+\infty]{\mathrm{w}-\mathcal{Q}} \psi$,

$$\mathrm{Re} \langle \psi_\kappa | -\Delta + U + \lambda | \psi_n \rangle \xrightarrow[n\to+\infty]{} \mathrm{Re} \langle \psi_\kappa | -\Delta + U + \lambda | \psi \rangle \ .$$

Finally, it remains to consider $\mathrm{Re} \langle \psi_\kappa | V_n | \psi_n \rangle = \mathrm{Re} \langle \psi_\kappa | V_n | \psi \rangle + \mathrm{Re} \langle \psi_\kappa | V_n | \psi_n - \psi \rangle$: since V_n converges weakly on $\mathcal{Q}$ to V_∞, we have

$$\lim_{n\to+\infty} \mathrm{Re} \langle \psi_\kappa | V_n | \psi \rangle = \mathrm{Re} \langle \psi_\kappa | V_\infty | \psi \rangle \ .$$

Next, we write

$$\begin{aligned} \mathrm{Re} \langle \psi_\kappa | V_n | \psi_n - \psi \rangle &= \mathrm{Re} \left\langle \chi_{B_0(R_\kappa)} \psi_\kappa | V_n | \psi_n - \psi \right\rangle \\ &= \mathrm{Re} \left\langle \psi_\kappa | V_n | \chi_{B_0(R_\kappa)} (\psi_n - \psi) \right\rangle + \mathrm{Re} \left\langle \psi_\kappa \left| [\chi_{B_0(R_\kappa)}, V_n] (\psi_n - \psi) \right. \right\rangle \ , \end{aligned}$$

where $\chi_{B_0(R_\kappa)}$ is a smooth characteristic function that is $\equiv 1$ inside $B_0(R_\kappa)$, and is supported on $B_0(2R_\kappa)$. For the first term on the r.h.s., observing that $(-\Delta + \lambda_0)^{-1/2} V_n (-\Delta + \lambda_0)^{-\delta}$ is uniformly bounded in n by (A.2) and that $\|\psi_\kappa\|_{\mathcal{Q}}$ is uniformly bounded in $0 < \kappa < 1$, we can estimate

$$\begin{aligned}\left|\left\langle\psi_\kappa\left|V_n\right|\chi_{B_0(R_\kappa)}(\psi_n-\psi)\right\rangle\right| & \\ \leqslant \|\psi_\kappa\|_{\mathcal{Q}}\left\|(-\Delta+\lambda_0)^{-1/2}V_n(-\Delta+\lambda_0)^{-\delta}\right\|&\left\|(-\Delta+\lambda_0)^{\delta}\chi_{B_0(R_\kappa)}(\psi_n-\psi)\right\| \\ &\leqslant C\left\|(-\Delta+\lambda_0)^{\delta}\chi_{B_0(R_\kappa)}(\psi_n-\psi)\right\|.\end{aligned}$$

Now we can write

$$\begin{aligned}(-\Delta+\lambda_0)^{\delta}\chi_{B_0(R_\kappa)}(\psi_n-\psi) &= (-\Delta+\lambda_0)^{-1/2+\delta}\chi_{B_0(R_\kappa)}(-\Delta+\lambda_0)^{1/2}(\psi_n-\psi) \\ &\quad - (-\Delta+\lambda_0)^{\delta}\left[\chi_{B_0(R_\kappa)},(-\Delta+\lambda_0)^{-1/2}\right](-\Delta+\lambda_0)^{1/2}(\psi_n-\psi).\end{aligned}$$

For any fixed $0<\kappa<1$, the first term goes to 0 in norm as $n\to+\infty$ since $(-\Delta+\lambda_0)^{-1/2+\delta}\chi_{B_0(R_\kappa)}$ is compact and $(-\Delta+\lambda_0)^{1/2}(\psi_n-\psi)\to 0$ as $n\to+\infty$ weakly in L^2. As for the second term, we use the representation formula

$$(-\Delta+\lambda_0)^{-1/2}=\pi^{-1}\int_0^\infty s^{-1/2}\big(-\Delta+\lambda_0+s\big)^{-1}\mathrm{d}s,$$

which gives

$$\left[\chi_{B_0(R_\kappa)},(-\Delta+\lambda_0)^{-1/2}\right]=\pi^{-1}\int_0^\infty s^{-1/2}\big(-\Delta+\lambda_0+s\big)^{-1}\left[-\Delta,\chi_{B_0(R_\kappa)}\right]\big(-\Delta+\lambda_0+s\big)^{-1}\mathrm{d}s.$$

Since $\delta<\frac{1}{2}$, standard estimates exploiting the scaling properties of the Laplacian resolvent then show that

$$\left\|(-\Delta+\lambda_0)^{\delta}\left[\chi_{B_0(R_\kappa)},(-\Delta+\lambda_0)^{-1/2}\right]\right\|\leqslant CR_\kappa^{-1},$$

and hence, since in addition $\|\psi_n\|_{H^1}$ is uniformly bounded in n,

$$\left\|(-\Delta+\lambda_0)^{\delta}\left[\chi_{B_0(R_\kappa)},(-\Delta+\lambda_0)^{-1/2}\right](-\Delta+\lambda_0)^{1/2}(\psi_n-\psi)\right\|\leqslant CR_\kappa^{-1}.$$

The previous estimates yield

$$\limsup_{n\to+\infty}\left|\left\langle\psi_\kappa\left|V_n\chi_{B_0(R_\kappa)}\right|\psi_n-\psi\right\rangle\right|\leqslant CR_\kappa^{-1}.$$

It remains to consider the term $\mathrm{Re}\left\langle\psi_\kappa\left|[\chi_{B_0(R_\kappa)},V_n](\psi_n-\psi)\right.\right\rangle$. We compute

$$\mathrm{Re}\left\langle\psi_\kappa\left|\left[V_n,\chi_{B_0(R_\kappa)}\right](\psi_n-\psi)\right.\right\rangle=\mathrm{Re}\left\langle\psi_\kappa\left|\mathbf{A}_n\cdot(\nabla\chi_{B_0(R_\kappa)})(\psi_n-\psi)\right.\right\rangle.$$

This term converges to zero by the Rellich-Kondrachov theorem, since $\mathbf{A}_n\psi_\kappa$ is uniformly bounded in n by (A.2), $\nabla\chi_{B_0(R_\kappa)}$ is a compactly supported function and ψ_n converges weakly to ψ in H^1.

Putting all together, we obtain that

$$\liminf_{n\to+\infty}\langle\psi_n|H_n+\lambda|\psi_n\rangle\geqslant-\langle\psi_\kappa|H_\infty+\lambda|\psi_\kappa\rangle+2\mathrm{Re}\,\langle\psi_\kappa|H_\infty+\lambda|\psi\rangle-CR_\kappa^{-1}.$$

Now, the right hand side converges, as $\kappa\to 0$, to $\langle\psi|H_\infty+\lambda|\psi\rangle$, thus completing the proof. □

References

[AN08] Z. Ammari, F. Nier. Mean field limit for bosons and infinite dimensional phase-space analysis. *Ann. Henri Poincaré* **9** (8), pp. 1503–1574 (2008). arXiv:0711.4128.

[AN09] Z. Ammari, F. Nier. Mean field limit for bosons and propagation of Wigner measures. *J. Math. Phys.* **50** (4), pp. 042,107, 16 (2009).

[AN11] Z. Ammari, F. Nier. Mean field propagation of Wigner measures and BBGKY hierarchies for general bosonic states. *J. Math. Pures Appl. (9)* **95** (6), pp. 585–626 (2011).

[AN15] Z. Ammari, F. Nier. Mean field propagation of infinite-dimensional Wigner measures with a singular two-body interaction potential. *Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5)* **14** (1), pp. 155–220 (2015).

[BFP23] S. Breteaux, J. Faupin, J. Payet. Quasi-classical ground states. I. Linearly coupled Pauli-Fierz Hamiltonians. *Doc. Math.* **28** (5), pp. 1191–1233 (2023).

[BFP24] S. Breteaux, J. Faupin, J. Payet. Quasi-classical Ground States. II. Standard Model of Non-relativistic QED. *Ann. Inst. Fourier* (2024). Online first.

[BLNS17] N. Bez, S. Lee, S. Nakamura, Y. Sawano. Sharpness of the Brascamp-Lieb inequality in Lorentz spaces. *Electron. Res. Announc. Math. Sci.* **24**, pp. 53–63 (2017).

[CCFO21] R. Carlone, M. Correggi, M. Falconi, M. Olivieri. Emergence of Time-Dependent Point Interactions in Polaron Models. *SIAM J. Math. Anal.* **53** (4), pp. 4657–4691 (2021). arXiv:1904.11012.

[CF18] M. Correggi, M. Falconi. Effective Potentials Generated by Field Interaction in the Quasi-Classical Limit. *Ann. Henri Poincaré* **19** (1), pp. 189–235 (2018). arXiv:1701.01317.

[CFO19] M. Correggi, M. Falconi, M. Olivieri. Magnetic Schrödinger Operators as the Quasi-Classical Limit of Pauli-Fierz-type Models. *J. Spectr. Theory* **9** (4), pp. 1287–1325 (2019). arXiv:1711.07413.

[CFO23a] M. Correggi, M. Falconi, M. Olivieri. Ground State Properties in the Quasi-Classical Regime. *Anal. PDE* **16** (8), pp. 1745–1798 (2023). arXiv:2007.09442.

[CFO23b] M. Correggi, M. Falconi, M. Olivieri. Quasi-Classical Dynamics. *J. Eur. Math. Soc. (JEMS)* **25** (2), pp. 731–783 (2023). arXiv:1909.13313.

[CFO24] M. Correggi, M. Falconi, M. Olivieri. Quasi-classics of the renormalized nelson model. *In preparation* (2024).

[DM93] G. Dal Maso. *An introduction to* Γ*-convergence*, *Progress in Nonlinear Differential Equations and their Applications*, vol. 8 (Birkhäuser Boston, Inc., Boston, MA, 1993). ISBN 0-8176-3679-X, xiv+340 pp.

[Fal18a] M. Falconi. Concentration of cylindrical Wigner measures. *Commun. Contemp. Math.* **20** (5), p. 1750,055 (2018). arXiv:1704.07676.

[Fal18b] M. Falconi. Cylindrical Wigner measures. *Doc. Math.* **23**, pp. 1677–1756 (2018). arXiv:1605.04778.

[Gra08] L. Grafakos. *Classical Fourier analysis*, *Grad. Texts Math.*, vol. 249 (New York, NY: Springer, 2008), 2nd ed. edn. ISBN 978-0-387-09431-1.

[LR02] P. G. Lemarié-Rieusset. *Recent developments in the Navier-Stokes problem*, *Chapman Hall/CRC Res. Notes Math.*, vol. 431 (Boca Raton, FL: Chapman & Hall/CRC, 2002). ISBN 1-58488-220-4.

[Nel64] E. Nelson. Interaction of nonrelativistic particles with a quantized scalar field. *J. Math. Phys.* **5**, pp. 1190–1197 (1964).

[O'N63] R. O'Neil. Convolution operators and L(p, q) spaces. *Duke Math. J.* **30**, pp. 129–142 (1963).

[PF38] W. Pauli, M. Fierz. Zur Theorie der Emission langwelliger Lichtquanten. *Il Nuovo Cimento* **15** (3), pp. 167–188 (1938).

[RS75] M. Reed, B. Simon. *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness* (Academic Press, New York, 1975), xv+361 pp.

[Spo04] H. Spohn. *Dynamics of charged particles and their radiation field* (Cambridge University Press, Cambridge, 2004). ISBN 0-521-83697-2, xvi+360 pp.

[Yap69] L. Y. H. Yap. Some remarks on convolution operators and L(p,q) spaces. *Duke Math. J.* **36**, pp. 647–658 (1969).

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