ON THE NUMBER OF BOUND STATES FOR FRACTIONAL SCHRÖDINGER OPERATORS WITH CRITICAL AND SUPER-CRITICAL EXPONENT

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Abstract. We study the number $N_{<0}(H_s)$ of negative eigenvalues, counting multiplicities, of the fractional Schrödinger operator $H_s = (-\Delta)^s - V(x)$ on $L^2(\mathbb{R}^d)$, for any $d \geq 1$ and $s \geq d/2$. We prove a bound on $N_{<0}(H_s)$ which depends on $s - d/2$ being either an integer or not, the critical case $s = d/2$ requiring a further analysis. Our proof relies on a splitting of the Birman-Schwinger operator associated to this spectral problem into low- and high-energies parts, a projection of the low-energies part onto a suitable subspace, and, in the critical case $s = d/2$, a Cwikel-type estimate in the weak trace ideal $\mathcal{L}^{2,\infty}$ to handle the high-energies part.

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1. Introduction

Estimating the number of bound states of the two-body Schrödinger operator

$$H := -\Delta - V(x)$$

on $L^2(\mathbb{R}^d)$ constitutes a rich problem that has attracted lots of attention in the mathematical literature. Classical textbook references include [27, Chapter XIII.3], [34, Chapter 7], [8, Chapter XI], [22, Chapter 4], see also [16, 33] for review articles and [13, Chapter 4] for a more recent exposition.

Roughly speaking, the question raised is as follows. Consider a real-valued measurable function $V : \mathbb{R}^d \to \mathbb{R}$ such that H identifies with a self-adjoint operator on $L^2(\mathbb{R}^d)$, with essential spectrum $[0, \infty)$ (see e.g. [26] or [13] for sufficient conditions on V implying these properties, see also Hypothesis B.1 and Remark B.2 below for the conditions considered in this paper, in the setting of the fractional Schrödinger operator). The bound states of H are defined as the normalized eigenvectors corresponding to negative eigenvalues. One then aims at estimating $N_{<0}(H)$, the number of negative eigenvalues of H counting multiplicities.

Note that, decomposing $V = V_+ - V_-$ with $V_\pm \geq 0$, we have $H \geq -\Delta - V_+(x)$ in the sense of quadratic forms, which implies that

$$N_{<0}(H) \leq N_{<0}\big(-\Delta - V_+(x)\big).$$

Therefore, to obtain a bound on the number of negative eigenvalues of H, it suffices to consider the case where $V = V_+ \geq 0$. Throughout the paper, to simplify the exposition, we thus assume that

$$V \geq 0.$$

Among the various bounds obtained in the literature, we mention the following ones. The celebrated Cwikel-Lieb-Rozenblum (CLR) bounds state that

$$N_{<0}(H) \lesssim_d \int_{\mathbb{R}^d} V^{\frac{d}{2}}, \quad d \geq 3, \tag{1.1}$$

for any V in $L^{\frac{d}{2}}$. Throughout this paper, $a \lesssim_{y_1,\dots,y_n} b$ means that there exists a constant $C_{y_1,\dots,y_n} > 0$ depending only on the parameters $y_1, \dots, y_n$ such that $a \leq C_{y_1,\dots,y_n} b$, and this constant may change from one line to the other.

The estimates (1.1) enjoy the important property that they are consistent with Weyl's semi-classical asymptotics. Namely, for sufficiently regular and fast-decaying V,

$$\lambda^{-\frac{d}{2}} N_{<0}(-\Delta - \lambda V) \to L_d \int_{\mathbb{R}^d} V^{\frac{d}{2}}, \quad \lambda \to \infty,$$

for some positive constant L_d (see e.g. [22, Section 4.1.1] or [13, Theorem 4.28]). The CLR bounds were proven independently by Cwikel [7], Lieb [21] and Rozenblum [28]. They are the crucial endpoint case of a more general family of bounds on the moments of the negative eigenvalues of H, the Lieb-Thirring inequalities [24], that in turn have important consequences for the stability of matter [22, 23]. Estimating the best constant in the CLR bound (1.1) therefore remains a well-studied open problem. We refer to [17] for important recent progress regarding this question and to [12, 13, 17, 29] for detailed discussions concerning the history, applications, recent developments and open problems related to the Lieb-Thirring inequalities.

Note that the CLR bound (1.1) implies in particular that if $\|V\|_{L^{d/2}}$ is small enough, in dimension ≥ 3, then H has no bound states. The situation is different in dimension one or two. In these cases, it is well-known that H has at least one bound state for any V in C_0^∞ which is not identically zero (see e.g. [27, Theorem XIII.11], see also the recent work [15] for similar results for Schrödinger operators with general kinetic energies). In one-dimension, the estimate

$$N_{<0}(H) - 1 \leq \int_{\mathbb{R}} |x| V(x) \mathrm{d}x, \quad d = 1, \tag{1.2}$$

was obtained in [5, 18], as a consequence of Bargmann's bound [1]. See also [27, Theorem XIII.9] for other related bounds for central potentials in 3-dimension.

The two-dimensional case is the most subtle one. In this case it is known that no estimate of the form

$$N_{<0}(H) \lesssim 1 + \int_{\mathbb{R}} w(x)V(x)\mathrm{d}x,$$

can hold, provided that w is bounded in a neighborhood of at least one point [14]. Several papers have been devoted to estimating the number of bound states of 2-dimensional Schrödinger operators in the recent years [3, 6, 14, 19, 20, 31, 35]. In particular, conditions on V ensuring the semi-classical growth $N_{<0}(-\Delta - \lambda V) = \mathcal{O}(\lambda)$ as $\lambda \to \infty$ are derived in [19, 20]. Among the various bounds obtained in 2-dimension, we mention the following ones:

$$N_{<0}(H) - 1 \lesssim \int_{\mathbb{R}^2} (1 + \ln\langle x\rangle)V(x)\mathrm{d}x - \int_{|x|\le 1} (\ln|x|)V^*(|x|)\mathrm{d}x, \quad d = 2, \tag{1.3}$$

and

$$N_{<0}(H) - 1 \lesssim \int_{\mathbb{R}^2} (1 + \ln\langle x\rangle)V(x)\mathrm{d}x + \|V\|_{L\log L}, \quad d = 2. \tag{1.4}$$

In (1.3), V^* stands for the decreasing rearrangement of V defined, for all $t \in [0,\infty)$, by

$$V^*(t) := \inf\{s \in [0,\infty) \mid \mu_V(s) \le t\},$$

where $\mu_V(s) := |\{x \in \mathbb{R}^2 \,|\, |V(x)| > s\}|$. In (1.4), $\|\cdot\|_{L\log L}$ stands for the norm in the Orlicz space $L\log L$ defined by

$$\|f\|_{L\log L} := \inf\Big\{\kappa > 0 \mid \int_{\mathbb{R}^2} \Phi(|f|/\kappa) \le 1\Big\},$$

with $\Phi(s) = s\ln(2+s)$ for all $s \in [0,\infty)$. Estimates (1.3) and (1.4) are proven in [31]; previously, estimate (1.3) was proven in the case where V is radial, and conjectured in the general case, in [6]; estimate (1.4) relies on previous important results obtained in [35]. We refer to [31] for further (and stronger) inequalities obtained in the two-dimensional case.

For $0 < s < \frac{d}{2}$, one can similarly study the fractional Schrödinger operator

$$H_s := (-\Delta)^s - V(x) \tag{1.5}$$

on $L^2(\mathbb{R}^d)$. The proof of the CLR bounds (1.1) extends to this case, leading to

$$N_{<0}(H_s) \lesssim_{d,s} \int_{\mathbb{R}^d} V^{\frac{d}{2s}}, \quad d \ge 1, \quad 0 < s < \frac{d}{2}. \tag{1.6}$$

We refer to the review [11] and references therein for bounds on the number of negative eigenvalues and Lieb-Thirring inequalities for H_s with $s < \frac{d}{2}$.

In this paper we consider the fractional Schrödinger operator (1.5) in the case $s \ge \frac{d}{2}$. This includes in particular the critical case $s = \frac{d}{2}$, as well as "polyharmonic Schrödinger operators", namely the fractional Schrödinger operators H_s with integer exponent $s \in \mathbb{N}$. For polyharmonic Schrödinger operators with $\mathbb{N} \ni s \ge \frac{d}{2}$, it was proven in [9, 10] that

$$N_{<0}(H_s) - s \lesssim_{s,q} \int_{\mathbb{R}} |x|^{2sq-1}V(x)^q\mathrm{d}x, \quad d = 1, \quad s \in \mathbb{N}, \quad q \ge 1, \tag{1.7}$$

in one-dimension, and

$$N_{<0}(H_s) - \binom{d+n}{d} \lesssim_{d,s,q} \begin{cases} \displaystyle\int_{\mathbb{R}^d} |x|^{2sq-d} V(x)^q \mathrm{d}x, & d \text{ odd}, \quad s \in \mathbb{N}, \quad q > 1, \\ \displaystyle\int_{\mathbb{R}^d} (1 + |\ln|x||)^{2q-1} |x|^{2sq-d} V(x)^q \mathrm{d}x, & d \text{ even}, \quad s \in \mathbb{N}, \quad q > 1, \end{cases} \tag{1.8}$$

in any dimension, where $n = \lfloor s - \frac{d}{2} \rfloor$. Still for $s \geq d/2$, s an integer, the Lieb-Thirring inequalities for moments of the negative eigenvalues of H_s of order $\mu > 1 - \frac{d}{2s}$ have been obtained in [25]; moreover the asymptotics of $N_{<0}((-\Delta)^s - \lambda V)$ as $\lambda \to \infty$ has been studied in [3, 4], giving in particular sufficient conditions on V, for d odd, to ensure the usual semi-classical behavior at large coupling.

Here we aim at proving a bound on $N_{<0}(H_s)$ in any dimension d and for any real $s \geq \frac{d}{2}$, comparable to the bounds of the form (1.2) or (1.7) (with $q = 1$) in dimension one, or (1.3)–(1.4) in dimension two.

1.1. **Statement of the main result.** Before stating our main results, Theorems 1.1 and 1.5, we recall and introduce some notations.

The symbol $\mathbb{N}$ denotes the set of integers larger than or equal to 1, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We use the japanese bracket notation $\langle x \rangle := \sqrt{1 + |x|^2}$ for $x \in \mathbb{R}^d$. We recall that for $1 \leq p < \infty$, the Schatten ideals $\mathcal{L}^p$ (or trace ideals) and the weak trace ideals $\mathcal{L}^{p,\infty}$ are defined, respectively, as the spaces of compact operators A such that the following quantities are finite:

$$\|A\|_{\mathcal{L}^p} := \Big(\sum_{j \geq 0} \lambda_j (A^*A)^{p/2} \Big)^{1/p}, \quad \|A\|^*_{\mathcal{L}^{p,\infty}} := \sup_{j \geq 0} (j+1)^{1/p} \sqrt{\lambda_j(A^*A)}, \tag{1.9}$$

where $\lambda_j(A^*A)$ is the sequence of the eigenvalues of A^*A sorted in decreasing order. The star in the notation $\| \cdot \|^*_{\mathcal{L}^{p,\infty}}$ is a reminder that it is a quasinorm but not necessarily a norm. (See e.g. [32] for more information on the weak trace ideals $\mathcal{L}^{p,\infty}$.) Similarly, the space of bounded operators on L^2 is denoted by $\mathcal{L}^\infty$.

Our main results are the following.

Theorem 1.1 ("Super-critical case", $s > \frac{d}{2}$)**.** *Let $d \geq 1$, $s > \frac{d}{2}$, $n = \lfloor s - \frac{d}{2} \rfloor$ and set $v := V^{\frac{1}{2}}$. Then*

$$N_{<0}(H_s) - \binom{d+n}{d} \lesssim_{d,s} \begin{cases} \big\| |x|^{s-\frac{d}{2}} v \big\|^2_{L^2} & \text{if } s - \frac{d}{2} \notin \mathbb{N}_0, \\ \big\| \langle x \rangle^{s-\frac{d}{2}} \sqrt{1 + \ln\langle x \rangle}\, v \big\|^2_{L^2} & \text{if } s - \frac{d}{2} \in \mathbb{N}, \end{cases}$$

for all v such that the right hand side is finite.

We have the following accompanying remarks. As usual, for $\alpha \in \mathbb{N}_0^d$ and $x \in \mathbb{R}^d$, we use the notations $x^\alpha = \prod_{j=1}^d x_j^{\alpha_j}$ and $|\alpha| = \sum_{j=1}^d \alpha_j$.

Remark 1.2. *The constant $\binom{d+n}{d}$ can be replaced by the possibly smaller constant*

$$\mathrm{c}_{d,n}(v) := \dim \operatorname{span}\big\{ x^\alpha v \mid \alpha \in \mathbb{N}_0^d,\ |\alpha| \leq n \big\}$$

(it is not difficult to see that the maximal dimension of the vector space $\mathrm{span}\big\{x^\alpha v \,|\, \alpha \in \mathbb{N}_0^d,\ |\alpha| \le n\big\}$ *is* $\binom{d+n}{d}$*, see the proof of Theorem 1.1 below). On the other hand, the constant* $\mathrm{c}_{d,n}(v)$ *cannot be removed from the estimate of Theorem 1.1, in the sense that there are potentials* V *in* $\mathrm{C}_0^\infty(\mathbb{R}^d)$ *such that* H_s *has at least* $\mathrm{c}_{d,n}(v)$ *bound states. More precisely, we will prove that for all* $V \in \mathrm{C}_0^\infty(\mathbb{R}^d)$*,* $V \ge 0$*, the operator* H_s *has at least* $\mathrm{c}_{d,n}(v)$ *negative eigenvalues counting multiplicities. See Proposition 4.1 below.*

Remark 1.3. *In the endpoint case* $s = d/2$*, the bound stated in Theorem 1.1 does not hold. Indeed, if it were true, then it would imply that, for* $d = 2$*,*

$$N_{<0}(-\Delta - V(x)) - 1 \lesssim \|\sqrt{1+\ln\langle x\rangle}\, v\|_{L^2}^2,$$

which cannot hold, as discussed in the introduction and proven in [14].

Remark 1.4. *Since the operators* H_s *and* $(-\Delta)^s - V(x+x_0)$ *are unitarily equivalent, for any* $x_0 \in \mathbb{R}^d$*, the weights* $|x|^{s-\frac{d}{2}}$ *and* $\langle x\rangle^{s-\frac{d}{2}}\sqrt{1+\ln\langle x\rangle}$ *in the estimate of Theorem 1.1 can be replaced by* $|x-x_0|^{s-\frac{d}{2}}$ *and* $\langle x-x_0\rangle^{s-\frac{d}{2}}\sqrt{1+\ln\langle x-x_0\rangle}$*, respectively.*

We also note that, for $d = 1$ and $s = 1$, Theorem 1.1 gives, for the usual Schrödinger operator $H = -\Delta - V(x)$,

$$N_{<0}(H) - 1 \lesssim \int_{\mathbb{R}} |x| V(x)\mathrm{d}x, \quad d = 1.$$

The Bargmann estimate (1.2), which follows from the explicit expression of Green's operator in one-dimension, is of course stronger, since it gives the same estimate but with a constant equal to 1 in front of the integral in the right hand side (instead of the implicit constant we obtain). Likewise, for $d = 1$ and $s \in \mathbb{N}$, Theorem 1.1 gives

$$N_{<0}(H_s) - s \lesssim_s \int_{\mathbb{R}} |x|^{2s-1} V(x)\mathrm{d}x, \quad d = 1, \quad s \in \mathbb{N},$$

which is (1.7) in the particular case where $q = 1$. Our result therefore shows how (1.2), and (1.7) with $q = 1$, can be generalized to any dimension for the fractional Schrödinger operator H_s, with any real $s > d/2$.

For d odd and $\mathbb{N} \ni s \ge d/2$, our result also corresponds to the endpoint case $q = 1$ in the family of estimates (1.8) proven by Egorov and Kondratiev [10]. Note that the endpoint case $q = 1$ was left open in [10]. Note also that our proof is very different from that in [10], see the next subsection for a description of the strategy followed in this paper. In the case where d is even, and with $s > d/2$, our result corresponds again to $q = 1$ in (1.8), except for the local behavior of V, in that our bound requires that V is L^1 near the origin, while (1.8) with $q = 1$ would only require that $(\ln|x|)|x|^{2s-d}V(x)$ is L^1 near 0.

Our result in the critical case $s = d/2$ is stated in terms of the harmonic oscillator

$$\mathbf{h} := c_d(-\Delta + x^2),$$

where the constant c_d is chosen, for technical convenience, as $c_d := e^e/d$ (so that $\mathbf{h} \ge e^e$). We then have the following result.

Theorem 1.5 ("Critical case", $s = \frac{d}{2}$)**.** *Let* $d \ge 1$*,* $s = \frac{d}{2}$*,* $\varepsilon > 0$ *and set* $v := V^{\frac{1}{2}}$*. Then*

$$N_{<0}(H_s) - 1 \lesssim_{d,\varepsilon} \big\|(\ln \mathbf{h})^{\frac{1}{2}}(\ln\ln \mathbf{h})^{\frac{1}{2}+\varepsilon}\, v\big\|_{L^2}^2,$$

for all v such that the right hand side is finite.

Theorem 1.5 should be compared with the bounds (1.3)–(1.4) for $H = -\Delta - V(x)$ in dimension 2. In particular, similarly as in (1.3)–(1.4), our estimate requires both a logarithmic decay and a "logarithmic regularity" of v, encoded here in the condition that v belongs to the domain of $(\ln \mathbf{h})^{1/2}$. The slightly stronger requirement that v belongs to the (smaller) domain of $(\ln \mathbf{h})^{1/2}(\ln\ln \mathbf{h})^{1/2+\varepsilon}$ may be an artifact of our proof.

1.2. **Elements of the proof and auxiliary results.** Our proof of Theorems 1.1 and 1.5 starts with a usual application of the Birman-Schwinger principle [2, 30]. In our context, it states that, for all $E < 0$,

$$N_{\le E}(H_s) = N_{\ge 1}(K_E), \tag{1.10}$$

where $N_{\le E}(A)$ (respectively $N_{\ge E}(A)$) denote the number of eigenvalues less than or equal to E (respectively larger than or equal to E) of a self-adjoint operator A, and the Birman-Schwinger operator K_E is defined by

$$K_E := v(x)\big((-\Delta)^s - E\big)^{-1} v(x)\,, \quad E < 0\,.$$

Recall that we have set

$$v := V^{\frac12}.$$

For the convenience of the reader, a proof of the Birman-Schwinger principle (1.10) under our assumptions is recalled in Appendix B.

Next, recalling that $n = \lfloor s - \frac{d}{2} \rfloor$, we introduce the finite-dimensional vector space

$$\mathcal{F}_n := \operatorname{span}\big\{ x^\alpha v \,|\, \alpha \in \mathbb{N}_0^d,\ |\alpha| \le n \big\} \subset L^2\,. \tag{1.11}$$

The Birman-Schwinger operator is then split into its 'low- and high-frequencies' parts. More precisely, we set

$$K_{E,<1} := v(x)\,\big((-\Delta)^s - E\big)^{-1} \mathbb{1}_{|-i\nabla|<1}\, v(x)\,, \qquad K_{E,<1}^\perp := \Pi_{\mathcal{F}_n}^\perp K_{E,<1} \Pi_{\mathcal{F}_n}^\perp, \tag{1.12}$$

$$K_{E,>1} := v(x)\,\big((-\Delta)^s - E\big)^{-1} \mathbb{1}_{|-i\nabla|>1}\, v(x)\,, \qquad K_{E,>1}^\perp := \Pi_{\mathcal{F}_n}^\perp K_{E,>1} \Pi_{\mathcal{F}_n}^\perp, \tag{1.13}$$

where $\Pi_{\mathcal{F}_n}^\perp$ denotes the orthogonal projection onto $\mathcal{F}_n^\perp$.

The variational principle (which we recall in Appendix A) then yields

$$N_{\ge 1}(K_E) \le \dim(\mathcal{F}_n) + N_{\ge 1}(K_E^\perp), \tag{1.14}$$

where $K_E^\perp = K_{E,<1}^\perp + K_{E,>1}^\perp$. It is not difficult to verify that

$$\dim(\mathcal{F}_n) \le \binom{d+n}{d},$$

(see Eq. (4.3) in the proof of Theorem 1.1). Now the splitting into high- and low-frequencies comes into play, as we can write

$$N_{\ge 1}(K_E^\perp) \le 2\big\|K_{E,>1}^\perp\big\|_{\mathcal{L}^{1,\infty}}^* + 2\big\|K_{E,<1}^\perp\big\|_{\mathcal{L}^{1,\infty}}^* \le 2\big\|K_{E,>1}\big\|_{\mathcal{L}^{1,\infty}}^* + 2\big\|K_{E,<1}^\perp\big\|_{\mathcal{L}^{1}}. \tag{1.15}$$

Note that we have estimated $\|K_{E,>1}^\perp\|_{\mathcal{L}^{1,\infty}}^* \le \|K_{E,>1}\|_{\mathcal{L}^{1,\infty}}^*$, namely we do not use the orthogonal projection $\Pi_{\mathcal{F}_n}^\perp$ for the high-frequencies part. On the other hand, to estimate the low-frequencies part, the orthogonal projection $\Pi_{\mathcal{F}_n}^\perp$ plays a crucial role, but it suffices to estimate the trace norm of $K_{E,<1}^\perp$ instead of the more complicated quasi-norm in $\mathcal{L}^{1,\infty}$.

Theorems 1.1 and 1.5 are then consequences of the following two theorems.

Theorem 1.6 (Low-frequencies estimate)**.** *Let $d \geq 1$, $s \geq d/2$ and $E \leq 0$. Then*

$$\big\|K^{\perp}_{E,<1}\big\|_{\mathcal{L}^1} \lesssim_{d,s} \begin{cases} \big\|\langle x\rangle^{s-\frac{d}{2}}\sqrt{1+\ln\langle x\rangle}\, v\big\|^2_{L^2} & \text{if } s-\frac{d}{2}\in\mathbb{N}_0,\\ \big\|\langle x\rangle^{s-\frac{d}{2}}\, v\big\|^2_{L^2} & \text{if } s-\frac{d}{2}\notin\mathbb{N}_0,\end{cases} \tag{1.16}$$

for all v such that the right hand side is finite.

Theorem 1.7 (High-frequencies estimate)**.** *Let $d \geq 1$, $s \geq d/2$, $\varepsilon > 0$ and $E \leq 0$. Then*

$$\big\|K_{E,>1}\big\|^*_{\mathcal{L}^{1,\infty}} \begin{cases} \lesssim_{d,\varepsilon} \big\|(\ln\mathbf{h})^{\frac12}(\ln\ln\mathbf{h})^{\frac12+\varepsilon}\, v\big\|^2_{L^2} & \text{if } s=\frac{d}{2},\\ \lesssim_{d,s} \|v\|^2_{L^2} & \text{if } s>\frac{d}{2},\end{cases} \tag{1.17}$$

for all v such that the right hand side is finite.

The main ideas of the proof of Theorem 1.6 are as follows. We first use that

$$\big\|K^{\perp}_{E,<1}\big\|_{\mathcal{L}^1} = \int_{|\xi|<1}\big\|\Pi^{\perp}_{\mathcal{F}_n}e^{ix\cdot\xi}v(x)\big\|^2_{L^2_x}\frac{\mathrm{d}\xi}{|\xi|^{2s}-E} \leq \int_{|\xi|<1}\big\|\Pi^{\perp}_{\mathcal{F}_n}e^{ix\cdot\xi}v(x)\big\|^2_{L^2_x}\frac{\mathrm{d}\xi}{|\xi|^{2s}},$$

(see Lemma 2.1). For $s \geq \frac{d}{2}$, $\xi \mapsto |\xi|^{-2s}\mathbb{1}_{|\xi|<1}$ is not integrable. We decompose the region $|\xi| < 1$ into annuli $e^{-k-1} \leq |\xi| < e^{-k}$ for $k \in \mathbb{N}_0$, which we combine with a splitting of v in each annuli, of the form $v = v_k^< + v_k^>$, with $v_k^<(x) = \mathbb{1}_{|x|\leq e^k}v(x)$, $v_k^>(x) = \mathbb{1}_{|x|\geq e^k}v(x)$. For the terms with $v_k^>$, we can use the decay of v at infinity to 'gain' powers of ξ since

$$\big\||\xi|^{-2s}\mathbb{1}_{e^{-k-1}\leq|\xi|<e^{-k}}\mathbb{1}_{|x|\geq e^k}v\big\|_{L^2} \lesssim e^{2ks}\big\|\mathbb{1}_{|x|\geq e^k}v\big\|_{L^2} \leq \big\||x|^{2s}v\big\|_{L^2}.$$

A refined estimate shows that the decay conditions imposed in the right-hand side of (1.16) are enough to have summability with respect to k. To estimate the terms with $v_k^<$, we use that $\Pi^{\perp}_{\mathcal{F}_n}x^\alpha v = 0$ for all $|\alpha| \leq n$. Expanding the exponential $e^{ix\cdot\xi}$ into a series then allows us again to gain powers of ξ and reach integrability.

In the case where $s > \frac{d}{2}$, the proof of Theorem 1.7 is straightforward (using that $\xi \mapsto |\xi|^{-2s}\mathbb{1}_{|\xi|>1}$ is integrable). In the critical case where $s = \frac{d}{2}$, Theorem 1.7 is a corollary of the following Cwikel-type estimate (Theorem 1.8). Before stating it we recall a few notations.

For $1 \leq p < \infty$, the weak spaces $L^{p,\infty}$ are defined as the sets of all measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ such that the quasinorm

$$\|f\|^*_{L^{p,\infty}} := \sup_{t>0}\lambda(\{|f|>t\})^{1/p}\,t$$

is finite (here λ stands for the Lebesgue measure). For $1 \leq p, q < \infty$, the spaces $\ell^q(L^p)$ are defined as follows. For any $\mathbf{m} \in \mathbb{Z}^d$, let $\chi_{\mathbf{m}}$ be the characteristic function of the unit hypercube of $\mathbb{R}^d$ with center $\mathbf{m}$ and, for all function $f : \mathbb{R}^d \to \mathbb{C}$, let $f_{\mathbf{m}} := \chi_{\mathbf{m}}f$. The space $\ell^q(L^p)$ is the set of measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ such that $(\|f_{\mathbf{m}}\|_{L^p})_{\mathbf{m}} \in \ell^q$, equipped with the norm

$$\|f\|_{\ell^q(L^p)} := \Bigg(\sum_{\mathbf{m}\in\mathbb{Z}^d}\|f_{\mathbf{m}}\|^q_{L^p}\Bigg)^{1/q}. \tag{1.18}$$

Likewise, $\ell^{p,\infty}(\mathbb{Z}^d)$ are the spaces of families of complex numbers $u = (u_{\mathbf{m}})_{\mathbb{Z}^d}$ such that the quasinorm

$$\|u\|^*_{\ell^{p,\infty}} := \sup_{j\geq 0}(j+1)^{1/p} u^*_j$$

is finite, where $(u^*_j)_{j\in\mathbb{N}_0}$ is the sequence of the $|u_{\mathbf{m}}|$ sorted in decreasing order. The space $\ell^{q,\infty}(L^p)$ is defined analogously to the space $\ell^q(L^p)$ in (1.18). The Fourier transform on $\mathbb{R}^d$ is denoted by

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\mathrm{d}x.$$

For $f : \mathbb{R}^d \to \mathbb{R}$ a measurable function, $f(-i\nabla)$ denotes the operator defined by $f(-i\nabla)\varphi = \mathcal{F}^{-1}(f\hat{\varphi})$.

Theorem 1.8 (Cwikel-type estimate in $\mathcal{L}^{2,\infty}$)**.** *Let $d \geq 1$, $\delta > 0$ and $\varepsilon > 0$. Then*

$$\|f(x)g(-i\nabla)\|^*_{\mathcal{L}^{2,\infty}} \lesssim_{d,\delta,\varepsilon} \left\|(\ln \mathbf{h})^{\frac{1}{2}}(\ln\ln \mathbf{h})^{\frac{1}{2}+\varepsilon} f\right\|_{L^2} \sup_{\substack{2<p\leq 2+\delta \\ \frac{1}{p}+\frac{1}{p'}=1}} \inf_{\substack{g_p, g_{p'} \\ g^2 = g_p g_{p'}}} \sqrt{\|g_p\|_{L^{p,\infty}} \|g_{p'}\|_{\ell^{p',\infty}(L^2)}}$$

for any f and g such that the right hand side is finite.

Remark 1.9. *One can state a slightly stronger estimate, involving the norm of g in a suitably defined vector space, as follows. Given $\mathcal{E}_1$ and $\mathcal{E}_2$ two quasi-normed subspaces of the measurable functions from $\mathbb{R}^d$ to $\mathbb{R}$, endowed with quasi-norms $\|\cdot\|_{\mathcal{E}_1}$ and $\|\cdot\|_{\mathcal{E}_2}$, consider the vector space*

$$\sqrt{\mathcal{E}_1\cdot\mathcal{E}_2} := \left\{\varphi \mid \exists J \in \mathbb{N}, \exists (a,b) \in \mathcal{E}_1^J \times \mathcal{E}_2^J, \varphi^2 \leq \sum_{j=1}^J a_j b_j\right\}$$

endowed with the quasi-norm

$$\|\varphi\|^*_{\sqrt{\mathcal{E}_1\cdot\mathcal{E}_2}} := \inf\left\{\sqrt{\sum_{j=1}^J \|a_j\|_{\mathcal{E}_1}\|b_j\|_{\mathcal{E}_2}} \;\Big|\; J\in\mathbb{N}, (a,b)\in\mathcal{E}_1^J\times\mathcal{E}_2^J, \varphi^2 \leq \sum_{j=1}^J a_j b_j\right\}.$$

Then the following holds: for all $d \geq 1$, $\delta > 0$ and $\varepsilon > 0$,

$$\|f(x)g(-i\nabla)\|^*_{\mathcal{L}^{2,\infty}} \lesssim_{d,\delta,\varepsilon} \left\|(\ln \mathbf{h})^{\frac{1}{2}}(\ln\ln \mathbf{h})^{\frac{1}{2}+\varepsilon} f\right\|_{L^2} \sup_{\substack{2<p\leq 2+\delta \\ \frac{1}{p}+\frac{1}{p'}=1}} \|g\|^*_{\sqrt{L^{p,\infty}\cdot \ell^{p',\infty}(L^2)}}$$

for any f and g such that the right hand side is finite.

Theorem 1.8 is obtained by first decomposing f as

$$f = \sum_{k\in\mathbb{N}} \pi_k f, \quad \pi_k := \mathbb{1}_{\Lambda_k \leq \mathbf{h} < \Lambda_{k+1}}, \quad \Lambda_k := e^{e^k},$$

and then using Hölder's inequality in weak trace ideals in each spectral region:

$$\left(\|(\pi_k f)(x) g(-i\nabla)\|^*_{\mathcal{L}^{2,\infty}}\right)^2 \leq \|(\pi_k f)(x) g_p(-i\nabla)\|^*_{\mathcal{L}^{p,\infty}} \|(\pi_k f)(x) g_{p'}(-i\nabla)\|^*_{\mathcal{L}^{p',\infty}}.$$

Applying the usual Cwikel estimate [7, Theorem 4.2] and an estimate due to Simon [32, Theorem 4.6], we are then able to obtain Theorem 1.8 by suitably choosing p (depending on k).

1.3. **Organization of the paper.** Apart from Cwikel's estimate just mentioned, our paper is self-contained. It is organized as follows. Sections 2 and 3 are devoted to the proofs of Theorems 1.6 and 1.7 respectively. In Section 4, we combine Theorems 1.6 and 1.7 to deduce our main results, Theorem 1.1 and Theorem 1.5. Appendices B, A and C recall proofs of the Birman-Schwinger principle, the variational principle and Simon's result [32, Theorem 4.6], respectively.

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2. Low-frequencies estimate

In this section we prove Theorem 1.6. We will use the following notations. For $t \geq 0$, let $\sigma_t := e^{-t}$. We decompose v into

$$v_t^<(x) := \mathbb{1}_{|x|\leq e^t} v(x) \quad \text{and} \quad v_t^>(x) := \mathbb{1}_{|x|\geq e^t} v(x). \tag{2.1}$$

Before we turn to the proof of Theorem 1.6, we prove the following easy lemma which gives a convenient formula for the trace of $\Pi_{\mathcal{F}_n}^\perp K_{E,<1} \Pi_{\mathcal{F}_n}^\perp$ (recall that $K_{E,<1}$ has been defined in (1.12) and $\mathcal{F}_n$ has been defined in (1.11)). Note that taking $B = \mathrm{Id}$ in the next lemma, we obtain the well-known formula for the Hilbert-Schmidt norm of an operator of the form $g(-i\nabla)f(x)$.

Lemma 2.1. *Let f, g be two functions in L^2 and B be a bounded operator on L^2. Then*

$$\left\|\bar{g}(-i\nabla)\bar{f}(x)B^*\right\|_{\mathcal{L}^2}^2 = (2\pi)^{-d}\int_{\mathbb{R}^d} |g(\xi)|^2 \|Be^{ix\cdot\xi}f(x)\|_{L_x^2}^2 \mathrm{d}\xi\,. \tag{2.2}$$

Proof. Let $(\varphi_j)_{j\in\mathbb{N}_0}$ be an orthonormal basis of L^2. For all $j \in \mathbb{N}_0$, we have

$$\left\|\bar{g}(-i\nabla)\bar{f}(x)B^*\varphi_j\right\|_{L^2}^2 = \int_{\mathbb{R}^d} |g(\xi)|^2 \left|\mathcal{F}(\bar{f}(x)B^*\varphi_j)(\xi)\right|^2 \mathrm{d}\xi. \tag{2.3}$$

Now, for all $\xi \in \mathbb{R}^d$, we can rewrite

$$\mathcal{F}(\bar{f}(x)B^*\varphi_j)(\xi) = (2\pi)^{-\frac{d}{2}}\left\langle e^{ix\cdot\xi}f(x), B^*\varphi_j\right\rangle_{L_x^2} = (2\pi)^{-\frac{d}{2}}\left\langle Be^{ix\cdot\xi}f(x), \varphi_j\right\rangle_{L_x^2}.$$

Summing (2.3) over j, we obtain

$$\left\|\bar{g}(-i\nabla)\bar{f}(x)B^*\right\|_{\mathcal{L}^2}^2 = (2\pi)^{-d}\sum_{j\in\mathbb{N}_0}\int_{\mathbb{R}^d} |g(\xi)|^2 \left|\left\langle Be^{ix\cdot\xi}f(x), \varphi_j\right\rangle_{L_x^2}\right|^2 \mathrm{d}\xi,$$

which implies the statement of the lemma by Parseval's equality. □

Now we are ready to prove Theorem 1.6.

Proof of Theorem 1.6. Applying Lemma 2.1 we can express the trace of $K_{E,<1}^\perp$ as

$$\left\|\Pi_{\mathcal{F}_n}^\perp v(x)\left((-\Delta)^s - E\right)^{-1}\mathbb{1}_{|-i\nabla|<1} v(x)\Pi_{\mathcal{F}_n}^\perp\right\|_{\mathcal{L}^1} = \int_{|\xi|<1} \left\|\Pi_{\mathcal{F}_n}^\perp e^{ix\cdot\xi}v(x)\right\|_{L_x^2}^2 \frac{\mathrm{d}\xi}{|\xi|^{2s} - E}\,.$$

Using the decompositions $e^{i\theta} = \sum_{j=0}^{n} \frac{(i\theta)^j}{j!} + \sum_{j\geq n+1} \frac{(i\theta)^j}{j!}$ and $v = v_k^> + v_k^<$, we obtain, since $E \leq 0$,

$$\int_{|\xi|<1} \left\| \Pi_{\mathcal{F}_n}^{\perp} e^{ix\cdot\xi} v(x) \right\|_{L_x^2}^2 \frac{\mathrm{d}\xi}{|\xi|^{2s}} \lesssim \sum_{k\in\mathbb{N}_0} \int_{\sigma_{k+1}\leq|\xi|<\sigma_k} (A_1(k,\xi) + A_2(k,\xi) + B(k,\xi)) \frac{\mathrm{d}\xi}{|\xi|^{2s}},$$

where we have set

$$A_1(k,\xi) := \left\| \Pi_{\mathcal{F}_n}^{\perp} e^{ix\cdot\xi} v_k^>(x) \right\|_{L_x^2}^2, \quad A_2(k,\xi) := \sum_{j=0}^{n} \left\| \Pi_{\mathcal{F}_n}^{\perp} \frac{(ix\cdot\xi)^j}{j!} v_k^<(x) \right\|_{L_x^2}^2,$$

and

$$B(k,\xi) := \left\| \Pi_{\mathcal{F}_n}^{\perp} \sum_{j\geq n+1} \frac{(ix\cdot\xi)^j}{j!} v_k^<(x) \right\|_{L_x^2}^2.$$

The estimate of $A_1(k,\xi)$ is straightforward:

$$A_1(k,\xi) \leq \| e^{ix\cdot\xi} v_k^>(x) \|_{L_x^2}^2 \leq \| v_k^> \|_{L_x^2}^2 \leq \sum_{|\alpha|\leq n} |\xi|^{2|\alpha|} \| x^\alpha v_k^> \|_{L^2}^2.$$

The purpose of the last inequality is only to bound $A_1(k,\xi)$ and $A_2(k,\xi)$ by the same term. To estimate $A_2(k,\xi)$, using $|\xi| \leq 1$ and $\Pi_{\mathcal{F}_n}^{\perp} x^\alpha v = 0$ for $|\alpha| \leq n$, we write

$$A_2(k,\xi) \leq \sum_{|\alpha|\leq n} \xi^{2\alpha} \left\| \Pi_{\mathcal{F}_n}^{\perp} x^\alpha (v - v_k^>) \right\|_{L^2}^2 = \sum_{|\alpha|\leq n} \xi^{2\alpha} \left\| \Pi_{\mathcal{F}_n}^{\perp} x^\alpha v_k^> \right\|_{L^2}^2 \leq \sum_{|\alpha|\leq n} |\xi|^{2|\alpha|} \| x^\alpha v_k^> \|_{L^2}^2.$$

Integrating over ξ and summing over k yields

$$\begin{aligned} \sum_{k\in\mathbb{N}_0} \int_{\sigma_{k+1}\leq|\xi|<\sigma_k} (A_1(k,\xi) + A_2(k,\xi)) \frac{\mathrm{d}\xi}{|\xi|^{2s}} &\lesssim \sum_{\substack{k\in\mathbb{N}_0 \\ |\alpha|\leq n}} \int_{\sigma_{k+1}\leq|\xi|<\sigma_k} \frac{\mathrm{d}\xi}{|\xi|^{2s-2|\alpha|}} \| x^\alpha v_k^> \|_{L^2}^2 \\ &\lesssim_{d,s} \sum_{\substack{k\in\mathbb{N}_0 \\ |\alpha|\leq n}} \sigma_k^{d-2s+2|\alpha|} \| x^\alpha v_k^> \|_{L^2}^2. \end{aligned}$$

To bound this sum by an integral we isolate the term for $k = 0$, shift the indexes for $k \geq 1$ and use that $\sigma_{k+1} = e^{-1}\sigma_k$ to obtain

$$\sum_{k\in\mathbb{N}_0} \int_{\sigma_{k+1}\leq|\xi|<\sigma_k} (A_1(k,\xi) + A_2(k,\xi)) \frac{\mathrm{d}\xi}{|\xi|^{2s}} \lesssim_{d,s} \| \langle x\rangle^n v \|_{L^2}^2 + \sum_{\substack{k\in\mathbb{N}_0 \\ |\alpha|\leq n}} \sigma_k^{d-2s+2|\alpha|} \| x^\alpha v_{k+1}^> \|_{L^2}^2.$$

Now, since $k \mapsto \sigma_k^{d-2s+2|\alpha|}$ is non-decreasing (given that $s - \frac{d}{2} \geq n \geq |\alpha|$) and $k \mapsto \| x^\alpha v_k^> \|_{L^2}^2$ is decreasing, we can estimate

$$\begin{aligned} \sum_{k\in\mathbb{N}_0} \sigma_k^{d-2s+2|\alpha|} \| x^\alpha v_{k+1}^> \|_{L^2}^2 &= \sum_{k\in\mathbb{N}_0} \int_k^{k+1} \sigma_k^{d-2s+2|\alpha|} \| x^\alpha v_{k+1}^> \|_{L^2}^2 \mathrm{d}t \\ &\leq \sum_{k\in\mathbb{N}_0} \int_k^{k+1} \sigma_t^{d-2s+2|\alpha|} \| x^\alpha v_t^> \|_{L^2}^2 \mathrm{d}t = \int_0^\infty \sigma_t^{d-2s+2|\alpha|} \int_{|x|\geq e^t} |x^\alpha v(x)|^2 \mathrm{d}x\mathrm{d}t. \end{aligned}$$

By Fubini's Theorem, this gives

$$\begin{aligned}\sum_{k\in\mathbb{N}_0}\int_{\sigma_{k+1}\le|\xi|<\sigma_k}&(A_1(k,\xi)+A_2(k,\xi))\frac{\mathrm{d}\xi}{|\xi|^{2s}}\\ &\lesssim_{d,s}\|\langle x\rangle^n v\|_{L^2}^2+\sum_{|\alpha|\le n}\int_{|x|\ge1}|x^\alpha v(x)|^2\int_0^{\ln|x|}\sigma_t^{d-2s+2|\alpha|}\,\mathrm{d}t\,\mathrm{d}x\\ &\lesssim_{d,s}\begin{cases}\|\langle x\rangle^{s-\frac{d}{2}}\sqrt{1+\ln\langle x\rangle}v\|_{L^2}^2 & \text{if } s-\frac{d}{2}\in\mathbb{N}_0,\\ \|\langle x\rangle^{s-\frac{d}{2}}v\|_{L^2}^2 & \text{if } s-\frac{d}{2}\notin\mathbb{N}_0,\end{cases}\end{aligned}\tag{2.4}$$

where we used that $\sigma_t=e^{-t}$ in the last inequality.

It remains to estimate $B(k,\xi)$. We write

$$B(k,\xi)\le\Big\|\sum_{j\ge n+1}\frac{(ix\cdot\xi)^j}{j!}v_k^<(x)\Big\|_{L_x^2}^2\lesssim_d\sum_{|\alpha|=n+1}\xi^{2\alpha}\|x^\alpha v_k^<\|_{L^2}^2\lesssim_d|\xi|^{2n+2}\big\||x|^{n+1}v_k^<\big\|_{L^2}^2.$$

Integrating over ξ and summing over k yields

$$\begin{aligned}\sum_{k\in\mathbb{N}_0}\int_{\sigma_{k+1}\le|\xi|<\sigma_k}B(k,\xi)\frac{\mathrm{d}\xi}{|\xi|^{2s}}&\lesssim_d\sum_{k\in\mathbb{N}_0}\int_{\sigma_{k+1}\le|\xi|<\sigma_k}|\xi|^{2n+2-2s}\mathrm{d}\xi\,\big\||x|^{n+1}v_k^<\big\|_{L^2}^2\\ &\lesssim_d\sum_{k\in\mathbb{N}_0}\sigma_k^{d-2s+2n+2}\,\big\||x|^{n+1}v_k^<\big\|_{L^2}^2.\end{aligned}$$

Since $k\mapsto\sigma_k^{d-2s+2n+2}$ is decreasing (as $d-2s+2n+2>0$) and $t\mapsto\||x|^{n+1}v_t^<\|_{L^2}^2$ is increasing, we can estimate

$$\begin{aligned}\sum_{k\in\mathbb{N}_0}\int_{\sigma_{k+1}\le|\xi|<\sigma_k}B(k,\xi)\frac{\mathrm{d}\xi}{|\xi|^{2s}}&\lesssim_d\sum_{k\in\mathbb{N}_0}\int_k^{k+1}\sigma_{k+1}^{d-2s+2n+2}\,\big\||x|^{n+1}v_k^<\big\|_{L^2}^2\mathrm{d}t\\ &\lesssim_d\sum_{k\in\mathbb{N}_0}\int_k^{k+1}\sigma_t^{d-2s+2n+2}\,\big\||x|^{n+1}v_t^<\big\|_{L^2}^2\mathrm{d}t\\ &\lesssim_d\int_0^\infty\sigma_t^{d-2s+2n+2}\int_{|x|<e^t}|x|^{2n+2}|v(x)|^2\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

An application of Fubini's theorem yields, as $\sigma_t=e^{-t}$,

$$\begin{aligned}\sum_{k\in\mathbb{N}_0}\int_{\sigma_{k+1}\le|\xi|<\sigma_k}B(k,\xi)\frac{\mathrm{d}\xi}{|\xi|^{2s}}&\lesssim_d\int_{\mathbb{R}^d}\int_{\ln|x|}^\infty e^{-t(d-2s+2n+2)}\mathrm{d}t\,|x|^{2n+2}|v(x)|^2\,\mathrm{d}x\\ &\lesssim_{d,s}\int_{\mathbb{R}^d}|x|^{-(d-2s+2n+2)}\,|x|^{2n+2}|v(x)|^2\,\mathrm{d}x\\ &\lesssim_{d,s}\int_{\mathbb{R}^d}|x|^{2s-d}|v(x)|^2\,\mathrm{d}x\,,\end{aligned}$$

and therefore

$$\sum_{k\in\mathbb{N}_0}\int_{\sigma_{k+1}\le|\xi|<\sigma_k}B(k,\xi)\frac{\mathrm{d}\xi}{|\xi|^{2s}}\lesssim_{d,s}\big\|\langle x\rangle^{s-\frac{d}{2}}v\big\|_{L^2}^2.\tag{2.5}$$

Putting together (2.4) and (2.5) we obtain the statement of Theorem 1.6. □

3. High-frequencies estimate

This section is devoted to the proof of the Cwikel-type estimate in $\mathcal{L}^{2,\infty}$ given in Theorem 1.8, as well as its consequence stated in Theorem 1.7. Before giving the proof of Theorem 1.8, we show that it indeed implies Theorem 1.7.

Proof of Theorem 1.7 using Theorem 1.8. Recall that $K_{E,>1}$ has been defined in (1.13). As $x \mapsto 1/x$ is operator monotone, we have

$$\|K_{E,>1}\|^*_{\mathcal{L}^{1,\infty}} \le \|v(x)\,(-\Delta)^{-s}\mathbb{1}_{|-i\nabla|>1}\,v(x)\|^*_{\mathcal{L}^{1,\infty}}, \tag{3.1}$$

for all $E \le 0$, the operator $(-\Delta)^{-s}\mathbb{1}_{|-i\nabla|>1}$ being bounded.

For $s > \frac{d}{2}$, the map $\xi \mapsto |\xi|^{-s}\mathbb{1}_{|\xi|\ge 1}$ belongs to L^2. Hence the statement of Theorem 1.7 is straightforward since the trace norm dominates the $\|\cdot\|^*_{\mathcal{L}^{1,\infty}}$-norm and

$$\begin{aligned}\big\|v(x)(-\Delta)^{-s}\mathbb{1}_{|-i\nabla|>1}v(x)\big\|_{\mathcal{L}^1} &= \big\|(-\Delta)^{-\frac{s}{2}}\mathbb{1}_{|-i\nabla|>1}v(x)\big\|^2_{\mathcal{L}^2}\\ &= \big\||\xi|^{-s}\mathbb{1}_{|\xi|\ge 1}\big\|^2_{L^2}\|v\|^2_{L^2} = C_s\|v\|^2_{L^2}.\end{aligned}$$

For $s = \frac{d}{2}$, writing $v(x)(-\Delta)^{-d/2}\mathbb{1}_{|-i\nabla|>1}v(x) = AA^*$ with $A = v(x)(-\Delta)^{-d/4}\mathbb{1}_{|-i\nabla|>1}$, together with the relation $\|A^*A\|^*_{\mathcal{L}^{1,\infty}} = (\|A\|^*_{\mathcal{L}^{2,\infty}})^2$, yields

$$\big\|v(x)(-\Delta)^{-d/2}\mathbb{1}_{|-i\nabla|>1}v(x)\big\|^*_{\mathcal{L}^{1,\infty}} = \Big(\big\|v(x)(-\Delta)^{-d/4}\mathbb{1}_{|-i\nabla|>1}\big\|^*_{\mathcal{L}^{2,\infty}}\Big)^2. \tag{3.2}$$

Now we apply Theorem 1.8 with $f(x) = v(x)$, $g(\xi) = \frac{1}{|\xi|^{d/2}}\mathbb{1}_{|\xi|\ge 1}$. Setting

$$g_p(\xi) := |\xi|^{-d/p}\mathbb{1}_{|\xi|\ge 1},$$

we have $g^2 = g_p g_{p'}$ for any $p \ge 2$ and $\frac{1}{p'} = 1 - \frac{1}{p}$. We claim that the quasinorms

$$\|g_p\|^*_{L^{p,\infty}}, \quad \|g_{p'}\|^*_{\ell^{p',\infty}(L^2)}$$

are uniformly bounded with respect to $p \ge 2$. Indeed, an easy computation shows that

$$\|g_p\|^*_{L^{p,\infty}} = \sup_{t>0} t\lambda(\{1 \le |\xi| \le t^{-p/d}\})^{1/p} \lesssim_d 1. \tag{3.3}$$

Similarly, for $g_{p'}$,

$$\begin{aligned}\|g_{p'}\|^*_{\ell^{p',\infty}(L^2)} =&\|\,\|\chi_{\mathbf{m}}g_{p'}\|_{L^2}\|^*_{\ell^{p',\infty}} \lesssim \|\langle\mathbf{m}\rangle^{-d/p'}\|^*_{\ell^{p',\infty}} = \sup_{j\ge 0}(j+1)^{1/p'}(\langle\mathbf{m}\rangle^{-d/p'})^*_j\\ &\lesssim_d \sup_{j\ge 1}(j+1)^{1/p'}j^{-1/p'} \lesssim_d 2.\end{aligned} \tag{3.4}$$

Hence we can apply Theorem 1.8 with

$$\sup_{\substack{2<p\le 2+\delta\\ \frac{1}{p}+\frac{1}{p'}=1}} \inf_{\substack{g_p,g_{p'}\\ g^2=g_pg_{p'}}} \sqrt{\|g_p\|_{L^{p,\infty}}\|g_{p'}\|_{\ell^{p',\infty}(L^2)}} \lesssim_d 1,$$

for any $\delta > 0$. This concludes the proof of Theorem 1.7. □

Now we turn to the proof of Theorem 1.8. It is based on the following results.

Theorem 3.1 (Cwikel [7])**.** *Let $d \geq 1$. Then*

$$\|f(x)g(-i\nabla)\|^*_{\mathcal{L}^{p,\infty}} \lesssim_d (p-2)^{-\frac{1}{p}} \|f\|_{L^p} \|g\|^*_{L^{p,\infty}}$$

for all $p \in (2,\infty)$, $f \in L^p(\mathbb{R}^d)$ and $g \in L^{p,\infty}(\mathbb{R}^d)$.

Lemma 3.2 (Sobolev embedding)**.** *Let $d \geq 1$, $\delta > 0$. Then*

$$\|f\|_{L^p} \lesssim_{d,\delta} \|f\|_{H^t}$$

for all $p \in [2, 2+\delta]$, $t \geq d(\frac{1}{2} - \frac{1}{p})$ and $f \in H^t$.

Theorem 3.3 (Simon [34])**.** *Let $d \geq 1$, $0 < \delta' < 1$. Then*

$$\|f(x)g(-i\nabla)\|^*_{\mathcal{L}^{p',\infty}} \lesssim_{d,\delta'} (2-p')^{\frac{1}{p'}-1} \|f\|_{L^{p'}} \|g\|^*_{\ell^{p',\infty}(L^2)}$$

for all $p' \in [2-\delta', 2)$, $f \in L^{p'}(\mathbb{R}^d)$ and $g \in \ell^{p',\infty}(L^2(\mathbb{R}^d))$.

Lemma 3.4 (Embedding of $L^2(\langle x\rangle^{2r}\mathrm{d}x)$ into $\ell^{p'}(L^2)$)**.** *Let $d \geq 1$. Then*

$$\|f\|_{\ell^{p'}(L^2)} \lesssim_d \|\langle x\rangle^r f\|_{L^2}.$$

for all $1 \leq p' < 2$, $r > d(\frac{1}{p'} - \frac{1}{2})$ and $f \in L^2(\langle x\rangle^{2r}\mathrm{d}x)$.

Theorem 3.1 is a direct consequence of [7, Theorem 4.2], Lemma 3.2 is the usual Sobolev embedding, Theorem 3.3 is [34, Theorem 4.6] with an explicit dependence on the parameter p', and Lemma 3.4 follows from a direct computation. In Appendix C, for the convenience of the reader, we prove Theorem 3.3, reproducing the proof of [34, Theorem 4.6] and following the dependence on p' in each estimate, and we prove Lemma 3.4.

We recall from the introduction the definition of the harmonic oscillator

$$\mathbf{h} := c_d(-\Delta + x^2),$$

where the constant c_d is chosen such that $\mathbf{h} \geq e^e$.

Proof of Theorem 1.8. Without loss of generality, we assume that $0 < \delta < 1$. Let $\Lambda_k := e^{e^k}$. We will use the following decomposition:

$$f = \sum_{k\in\mathbb{N}} \pi_k f,$$

where π_k stands for the spectral projection

$$\pi_k := \mathbb{1}_{\Lambda_k \leq \mathbf{h} < \Lambda_{k+1}}.$$

Using that $\|\cdot\|^*_{\mathcal{L}^{2,\infty}}$ is equivalent to a certain norm $\|\cdot\|_{\mathcal{L}^{2,\infty}}$, we can write

$$\begin{aligned}\big\|f(x)g(-i\nabla)\big\|^*_{\mathcal{L}^{2,\infty}} &\lesssim \big\|f(x)g(-i\nabla)\big\|_{\mathcal{L}^{2,\infty}} \\ &\lesssim \sum_{k\in\mathbb{N}} \big\|(\pi_k f)(x)g(-i\nabla)\big\|_{\mathcal{L}^{2,\infty}} \lesssim \sum_{k\in\mathbb{N}} \big\|(\pi_k f)(x)g(-i\nabla)\big\|^*_{\mathcal{L}^{2,\infty}}. \end{aligned} \tag{3.5}$$

Let $p \in (2, 2+\delta]$, $\frac{1}{p} + \frac{1}{p'} = 1$, and let $g_p \in L^{p,\infty}$, $g_{p'} \in \ell^{p',\infty}(L^2)$ be such that $g^2 = g_p g_{p'}$. Thanks to the relation $(\|A\|^*_{\mathcal{L}^{2,\infty}})^2 = \|A^*A\|^*_{\mathcal{L}^{1,\infty}}$ for any operator A in $\mathcal{L}^{2,\infty}$, we have, for all $k \in \mathbb{N}$,

$$\begin{aligned}\left(\left\|(\pi_k f)(x)\, g(-i\nabla)\right\|^*_{\mathcal{L}^{2,\infty}}\right)^2 &= \left\|g(-i\nabla)(\pi_k f(x))^2 g(-i\nabla)\right\|^*_{\mathcal{L}^{1,\infty}} \\ &= \|\pi_k f(x)\, g^2(-i\nabla)\, \pi_k f(x)\|^*_{\mathcal{L}^{1,\infty}} \\ &= \|\pi_k f(x)\, g_p(-i\nabla)\left(\pi_k f(x) g_{p'}(-i\nabla)\right)^*\|^*_{\mathcal{L}^{1,\infty}} \\ &\lesssim \|\pi_k f(x)\, g_p(-i\nabla)\|^*_{\mathcal{L}^{p,\infty}}\, \|\pi_k f(x) g_{p'}(-i\nabla)\|^*_{\mathcal{L}^{p',\infty}},\end{aligned} \tag{3.6}$$

for any $p \in (2, 2+\delta]$, thanks to Hölder's inequality in weak trace ideals ([32, Theorem 2.1]).

Since $p > 2$, the usual Cwikel estimate, Theorem 3.1, yields

$$\|\pi_k f(x)\, g_p(-i\nabla)\|^*_{\mathcal{L}^{p,\infty}} \lesssim_d (p-2)^{-\frac{1}{p}} \|\pi_k f\|_{L^p} \|g_p\|^*_{L^{p,\infty}}. \tag{3.7}$$

On the other hand, since $p' < 2$, Simon's result, Theorem 3.3, implies

$$\|\pi_k f(x)\, g_{p'}(-i\nabla)\|^*_{\mathcal{L}^{p',\infty}} \lesssim_{d,\delta} (2-p')^{\frac{1}{p'}-1} \|\pi_k f\|_{\ell^{p'}(L^2)} \|g_{p'}\|^*_{\ell^{p',\infty}(L^2)}. \tag{3.8}$$

It follows from (3.6)–(3.8) that, for all $p > 2$,

$$\begin{aligned}\left(\left\|(\pi_k f)(x)\, g(-i\nabla)\right\|^*_{\mathcal{L}^{2,\infty}}\right)^2 \lesssim_{d,\delta} &\ (p-2)^{-\frac{1}{p}} (2-p')^{\frac{1}{p'}-1} \|\pi_k f\|_{L^p} \|\pi_k f\|_{\ell^{p'}(L^2)} \\ &\times \inf_{\substack{g_p, g_{p'} \\ g^2 = g_p g_{p'}}} \left(\|g_p\|^*_{L^{p,\infty}} \|g_{p'}\|^*_{\ell^{p',\infty}(L^2)}\right).\end{aligned} \tag{3.9}$$

We now give bounds on the $\pi_k f$ terms. Choosing $p \in (2, 2+\delta]$ in such a way that $t_p := d(\frac{1}{2} - \frac{1}{p}) \le 2$, the usual Sobolev embedding, Lemma 3.2, together with the quadratic form inequality $\langle -i\nabla\rangle^{t_p} \le \langle \mathbf{h}\rangle^{t_p/2}$ give

$$\|\pi_k f\|_{L^p} \lesssim_d \|\langle -i\nabla\rangle^{t_p} \pi_k f\|_{L^2} \lesssim_d \Lambda_{k+1}^{t_p/2} \|\pi_k f\|_{L^2}.$$

At this point we take

$$\frac{1}{p} = \frac{1}{2} - \frac{\delta}{d \ln \Lambda_{k+1}}$$

so that $p \in (2, 2+\delta]$ and $t_p = \frac{\delta}{\ln \Lambda_{k+1}}$, which in turn gives $\Lambda_{k+1}^{t_p/2} = e^{\delta/2}$.

To treat the contribution of $\|\pi_k f\|_{\ell^{p'}(L^2)}$, we use the embedding $L^2(\langle x\rangle^{2r} \mathrm{d}x) \hookrightarrow \ell^{p'}(L^2)$ for any $r > d(\frac{1}{p'} - \frac{1}{2})$, see Lemma 3.4. With our choice of p, we have

$$\frac{1}{p'} = \frac{1}{2} + \frac{\delta}{d \ln \Lambda_{k+1}}.$$

Hence we can choose $r = \frac{2\delta}{\ln \Lambda_{k+1}} \le 2$ yielding $\langle x\rangle^r \le \langle \mathbf{h}\rangle^{r/2}$ and

$$\|\pi_k f\|_{\ell^{p'}(L^2)} \lesssim_{d,\delta} \|\langle x\rangle^r \pi_k f\|_{L^2} \lesssim_{d,\delta} \Lambda_{k+1}^{r/2} \|\pi_k f\|_{L^2}.$$

Similarly as before, we observe that $\Lambda_{k+1}^{r/2} = e^{\delta}$. Hence our previous estimates imply

$$\|\pi_k f\|_{L^p} \|\pi_k f\|_{\ell^{p'}(L^2)} \lesssim_{d,\delta} \|\pi_k f\|^2_{L^2}. \tag{3.10}$$

Next, using the relations

$$p-2=\frac{4\delta}{d\ln\Lambda_{k+1}-2\delta},\quad 2-p'=\frac{4\delta}{d\ln\Lambda_{k+1}+2\delta},$$

yields the bound

$$\big[(p-2)(2-p')\big]^{-\frac{1}{p}}\lesssim_{d,\delta}\ln\Lambda_{k+1}. \tag{3.11}$$

Putting together (3.9), (3.10) and (3.11) gives

$$\big\|(\pi_k f)(x)\,g(-i\nabla)\big\|^*_{\mathcal{L}^{2,\infty}}\lesssim_{d,\delta}(\ln\Lambda_{k+1})^{\frac12}\|\pi_k f\|_{L^2}\Bigg(\sup_{\substack{2<p\le 2+\delta\\ \frac1p+\frac1{p'}=1}}\ \inf_{\substack{g_p,g_{p'}\\ g^2=g_pg_{p'}}}\ \|g_p\|^*_{L^{p,\infty}}\|g_{p'}\|^*_{\ell^{p',\infty}(L^2)}\Bigg)^{\frac12}.$$

Since $\ln\Lambda_{k+1}=e\ln\Lambda_k$, the k-dependent part of the right hand side can then be summed over k as follows:

$$\begin{aligned}\sum_{k\in\mathbb{N}}(\ln\Lambda_{k+1})^{\frac12}\|\pi_k f\|_{L^2}&\lesssim\sum_{k\in\mathbb{N}}\big\|\pi_k((\ln\mathbf{h})^{\frac12}f)\big\|_{L^2}\\&\lesssim_{d,\varepsilon}\sum_{k\in\mathbb{N}}k^{-\frac12-\varepsilon}\big\|\pi_k((\ln\mathbf{h})^{\frac12}(\ln\ln\mathbf{h})^{\frac12+\varepsilon}f)\big\|_{L^2}\\&\lesssim_{d,\varepsilon}\big\|(\ln\mathbf{h})^{\frac12}(\ln\ln\mathbf{h})^{\frac12+\varepsilon}f\big\|_{L^2},\end{aligned}$$

where we used that $0<\varepsilon$ and the Cauchy-Schwarz inequality in the last inequality. This along with (3.5) implies the statement of Theorem 1.8. ☐

4. Proof of Theorems 1.1 and 1.5

In this section we prove Theorem 1.1 using the Birman-Schwinger principle, the variational principle, Theorem 1.6 and Theorem 1.7.

Proof of Theorem 1.1 and 1.5. Let $E<0$. To estimate $N_{\le E}(H_s)$ we use the Birman-Schwinger principle (see Proposition B.3) which shows that

$$N_{\le E}(H_s)=N_{\ge 1}(K_E), \tag{4.1}$$

where we recall that the Birman-Schwinger operator K_E is given by

$$K_E=v(x)\big((-\Delta)^s-E\big)^{-1}v(x),$$

with $v(x)=\sqrt{V(x)}$. We recall also that $n=\lfloor s-\frac d2\rfloor$ and

$$\mathcal{F}_n:=\operatorname{span}\big\{x^\alpha v\,|\,\alpha\in\mathbb{N}_0^d,\ |\alpha|\le n\big\}, \tag{4.2}$$

where $|\alpha|=\sum_{j=1}^d\alpha_j$ and $x^\alpha=\prod_{j=1}^d x_j^{\alpha_j}$. Note that, with $S_1:=\{\alpha\in\mathbb{N}_0^d\mid|\alpha|\le n\}$,

$$\dim(\mathcal{F}_n)\le|S_1|=\binom{d+n}{d}. \tag{4.3}$$

Indeed, set $S_2:=\{X\subseteq\{0,1,2,\dots,1,d+n-1\}\mid|X|=d\}$. Then

$$S_1\ni\alpha\mapsto\{k-1+\alpha_1+\cdots+\alpha_k\mid 1\le k\le d\}\in S_2$$

and

$$S_2\ni\{\beta_1<\cdots<\beta_d\}\mapsto(\beta_1,\beta_2-\beta_1-1,\dots,\beta_d-\beta_{d-1}-1)\in S_1$$

are inverse functions of each other and hence bijections. It follows that $|S_1| = |S_2| = \binom{d+n}{d}$.

By the variational principle recalled in Proposition A.1, if $\Pi^{\perp}_{\mathcal{F}_n}$ denotes the orthogonal projection onto $\mathcal{F}_n^{\perp}$, we have

$$N_{\geq 1}(K_E) \leq \binom{d+n}{d} + N_{\geq 1}(K_E^{\perp}), \tag{4.4}$$

with $K_E^{\perp} = \Pi^{\perp}_{\mathcal{F}_n} K_E \Pi^{\perp}_{\mathcal{F}_n}$.

Let $j_{\max} := \max\{j \geq 0 \mid \lambda_j(K_E^{\perp}) \geq 1\}$. Using that $\lambda_j(K_E^{\perp})$ is a decreasing sequence and actually coincides with the singular values of $K_E^{\perp}$ (as $K_E^{\perp} \geq 0$), we have

$$N_{\geq 1}(K_E^{\perp}) = (j_{\max} + 1) \leq (j_{\max} + 1)\lambda_{j_{\max}}(K_E^{\perp}) \leq \|K_E^{\perp}\|^*_{\mathcal{L}^{1,\infty}} . \tag{4.5}$$

We now use the decomposition in low- and high-frequencies parts of $K_E^{\perp}$ as defined in (1.12)–(1.13), obtaining

$$\|K_E^{\perp}\|^*_{\mathcal{L}^{1,\infty}} \leq 2\|K^{\perp}_{E,<}\|^*_{\mathcal{L}^{1,\infty}} + 2\|K^{\perp}_{E,>}\|^*_{\mathcal{L}^{1,\infty}} \leq 2\|K^{\perp}_{E,<}\|_{\mathcal{L}^1} + 2\|K_{E,>}\|^*_{\mathcal{L}^{1,\infty}}. \tag{4.6}$$

It follows from (4.1)-(4.6), Theorem 1.6 and Theorem 1.7 that

$$N_{<0}(H_s) - \binom{d+n}{d} \lesssim_{d,s} \begin{cases} \left\| \langle x \rangle^{s-\frac{d}{2}} v \right\|^2_{L^2} & \text{if } s - \frac{d}{2} \notin \mathbb{N}_0, \\ \left\| \langle x \rangle^{s-\frac{d}{2}} \sqrt{1 + \ln\langle x \rangle}\, v \right\|^2_{L^2} & \text{if } s - \frac{d}{2} \in \mathbb{N}, \\ \left\| (\ln \mathbf{h})^{\frac{1}{2}} (\ln\ln \mathbf{h})^{\frac{1}{2}+\varepsilon} v \right\|^2_{L^2} & \text{if } s = \frac{d}{2}. \end{cases}$$

This proves Theorem 1.1 in the case where $s - d/2 \in \mathbb{N}$ and Theorem 1.5. In the case where $s - d/2 \notin \mathbb{N}_0$, it remains to show that we can replace $\langle x \rangle$ by $|x|$. To this end, we argue as follows.[1] By scaling, the operators H_s and $R^{2s}(-\Delta)^s - V(R^{-1}x)$ are unitarily equivalent, for any $R > 0$. Hence

$$N_{<0}(H_s) = N_{<0}\big((-\Delta)^s - R^{-2s} V(R^{-1}x)\big).$$

Applying the previous estimate (for $s - d/2 \notin \mathbb{N}_0$), we obtain that

$$N_{<0}(H_s) - \binom{d+n}{d} \lesssim_{d,s} R^{-2s} \int_{\mathbb{R}^d} \langle x \rangle^{2s-d} V(R^{-1}x) \mathrm{d}x = \int_{\mathbb{R}^d} (R^{-2} + x^2)^{s-\frac{d}{2}} V(x) \mathrm{d}x.$$

Letting $R \to \infty$, using the monotone convergence theorem, we deduce that

$$N_{<0}(H_s) - \binom{d+n}{d} \lesssim_{d,s} \int_{\mathbb{R}^d} |x|^{2s-d} V(x) \mathrm{d}x,$$

which proves Theorem 1.1 in the case where $s - d/2 \notin \mathbb{N}_0$. □

We conclude this section with a proposition showing that $H_s = (-\Delta)^s - V$ has at least $\dim \mathcal{F}_n$ negative eigenvalues for smooth compactly supported V (with $\mathcal{F}_n$ defined in (4.2)). Taking V such that $\dim \mathcal{F}_n$ is maximal, i.e. $\dim(\mathcal{F}_n) = |\{\alpha \in \mathbb{N}_0^d \mid |\alpha| \leq n\}|$, shows that the constant $\binom{d+n}{d}$ cannot be removed from the statement of Theorem 1.1. The proof of Proposition 4.1 is a fairly direct generalization of that given in [27, Theorem XIII.11].

Proposition 4.1. *Let $d \geq 1$, $s \geq \frac{d}{2}$ and $n = \lfloor s - \frac{d}{2} \rfloor$. Let $V \in \mathrm{C}_0^\infty(\mathbb{R}^d)$ be such that $V \geq 0$. Then the operator $H_s = (-\Delta)^s - V$ has at least $\dim \mathcal{F}_n$ negative eigenvalues.*

[1] We are grateful to R. Frank for pointing out this argument to us.

Proof. By the Birman-Schwinger principle (see Proposition B.3), it suffices to show that $N_{\geq 1}(K_E) \geq \dim \mathcal{F}_n$ for $E < 0$, $|E|$ small enough, where K_E is the Birman-Schwinger operator defined as above, namely $K_E = v(x)((-\Delta)^s - E)^{-1} v(x)$.

Let $\varphi \in \mathcal{F}_n$, $\varphi \neq 0$. Then $\varphi \in \mathrm{C}_0^\infty(\mathbb{R}^d)$ and we claim that there exist $\varepsilon > 0$ and $c > 0$ (which depends on V, φ, n) such that, for all $\xi \in \mathbb{R}^d$ with $|\xi| \leq \varepsilon$,

$$|\widehat{v\varphi}(\xi)| \geq c|\xi|^n. \tag{4.7}$$

Indeed, if this property did not hold, then we would have that for all $\alpha \in \mathbb{N}_0^d$ such that $|\alpha| \leq n$,

$$0 = \partial_\xi^\alpha \widehat{v\varphi}(0) = (-i)^\alpha \widehat{x^\alpha v\varphi}(0) = (-i)^\alpha \int_{\mathbb{R}^d} x^\alpha v(x)\varphi(x)\mathrm{d}x,$$

which contradicts the facts that $\varphi \in \mathcal{F}_n$ and $\varphi \neq 0$.

Now using (4.7), we write, for all $\varphi \in \mathcal{F}_n$, $\varphi \neq 0$,

$$\langle \varphi, K_E\, \varphi\rangle = \int_{\mathbb{R}^d} \big(|\xi|^{2s} - E\big)^{-1} \big|\widehat{v\varphi}(\xi)\big|^2 \mathrm{d}\xi \geq c \int_{|\xi|\leq\varepsilon} |\xi|^{2n} \big(|\xi|^{2s} - E\big)^{-1} \mathrm{d}\xi.$$

Since $2s - 2n \geq d$, the previous integral tends to infinity as $E \to 0$. Hence it follows from the min-max principle (see Theorem A.2) that, for $|E|$ small enough, K_E has at least $\dim \mathcal{F}_n$ eigenvalues larger than 1. This concludes the proof. ☐

Appendix A. Variational principle

In this appendix, we recall how to estimate the number of eigenvalues larger than 1 of an operator, by the number of eigenvalues larger than 1 of the restriction of this operator to a linear subspace, up to the dimension of the subspace itself. We refer to e.g. [13, Section 1.2.3] for general versions of the variational principle.

If $\mathcal{F}$ is a closed linear subspace of a Hilbert space $\mathcal{H}$, $\Pi_{\mathcal{F}}$ denotes the orthogonal projection onto $\mathcal{F}$.

Proposition A.1. *Let K a compact self-adjoint non-negative operator on a Hilbert space $\mathcal{H}$. Then, for any linear subspace $\mathcal{F}$ of $\mathcal{H}$ of finite dimension,*

$$N_{\geq 1}(K) \leq \dim \mathcal{F} + N_{\geq 1}(\Pi_{\mathcal{F}^\perp} K \Pi_{\mathcal{F}^\perp})\,. \tag{A.1}$$

To prove this result we use the following simple version of the min-max principle. (See e.g. [27] for a more general version.)

Theorem A.2. *Let K a compact selfadjoint non-negative operator on a Hilbert space $\mathcal{H}$. Then the sequence defined for $j \geq 0$ by*

$$\lambda_j(K) = \min_{\dim \mathcal{S} = j} \max_{\substack{u \in \mathcal{S}^\perp \\ \|u\|=1}} \langle u, Ku\rangle$$

coincides with the non-increasing sequence either of the positive eigenvalues of K if K is of infinite rank, or, otherwise, of all its eigenvalues. Here the minimum is taken over all linear subspaces $\mathcal{S}$ of $\mathcal{H}$ of dimension $\dim \mathcal{S} = j$.

Proof of Proposition A.1. Let $D = \dim \mathcal{F}$. By the min-max principle in Theorem A.2,

$$\lambda_{D+k}(K) = \min_{\dim \mathcal{S}=D+k} \max_{\substack{u\in\mathcal{S}^\perp \\ \|u\|=1}} \langle u, Ku\rangle \,.$$

For any subspace $\mathcal{V}$ of $\mathcal{F}^\perp$ of dimension k, we have $\dim(\mathcal{F}\oplus\mathcal{V}) = D+k$. As $u = \Pi_{\mathcal{F}^\perp} u$ for $u \in \mathcal{F}^\perp$,

$$\lambda_{D+k}(K) \leq \max_{\substack{u\in(\mathcal{F}\oplus\mathcal{V})^\perp \\ \|u\|=1}} \langle u, Ku\rangle = \max_{\substack{u\in\mathcal{F}^\perp\cap\mathcal{V}^\perp \\ \|u\|=1}} \langle \Pi_{\mathcal{F}^\perp} u, K\Pi_{\mathcal{F}^\perp} u\rangle = \max_{\substack{u\in\mathcal{V}^\perp \\ \|u\|=1}} \langle u, \Pi_{\mathcal{F}^\perp} K \Pi_{\mathcal{F}^\perp} u\rangle,$$

for any subspace $\mathcal{V}$ of $\mathcal{F}^\perp$ of dimension k. This implies

$$\lambda_{D+k}(K) \leq \min_{\substack{\dim\mathcal{V}=k \\ \mathcal{V}\subseteq\mathcal{F}^\perp}} \max_{\substack{u\in\mathcal{V}^\perp \\ \|u\|=1}} \langle u, \Pi_{\mathcal{F}^\perp} K \Pi_{\mathcal{F}^\perp} u\rangle = \lambda_k(\Pi_{\mathcal{F}^\perp} K \Pi_{\mathcal{F}^\perp}) \,. \tag{A.2}$$

As eigenvalues given by the min-max principle are sorted in non-increasing order, we deduce that

$$N_{\geq 1}(K) - D \leq |\{k \geq 0 \mid \lambda_{D+k}(K) \geq 1\}| \leq |\{k\geq 0 \mid \lambda_k(\Pi_{\mathcal{F}^\perp} K \Pi_{\mathcal{F}^\perp}) \geq 1\}| = N_{\geq 1}(\Pi_{\mathcal{F}^\perp} K \Pi_{\mathcal{F}^\perp}),$$

which yields (A.1). ☐

Appendix B. Birman-Schwinger principle

In this section, for the convenience of the reader, we recall a proof of the Birman-Schwinger principle for $H_s = (-\Delta)^s - v^2$ and $K_E := v\big((-\Delta)^s - E\big)^{-1} v$, under the following assumptions:

Hypothesis B.1. *Let $d \geq 1$, $s \geq d/2$, and v measurable and real-valued, such that*

- *either $v \in L^2$, if $s > d/2$,*
- *or $v \in \mathcal{D}((\ln \mathbf{h})^{\frac{1}{2}} (\ln\ln \mathbf{h})^{\frac{1}{2}+\varepsilon}) \subset L^2$ for some $\varepsilon > 0$ if $s = d/2$.*

Here we denote by $\mathcal{D}(A)$ the domain of an operator A. We refer to e.g. [13, Section 1.2.8] for a proof of the Birman-Schwinger principle in a general abstract setting.

Remark B.2. *Hypothesis B.1 and $E < 0$ ensure that the chain*

$$L^2 \xrightarrow{v\times} L^1 \hookrightarrow (L^\infty)^* \hookrightarrow H^{-s} \xrightarrow{((-\Delta)^s - E)^{-1}} H^s \hookrightarrow L^\infty \xrightarrow{v\times} L^2, \tag{B.1}$$

holds for $s > d/2$. For $s = d/2$, we observe that, for all $p > 2$, $\frac{1}{p} + \frac{1}{p'} = 1$,

$$(|\xi|^d - E)^{-1} = (|\xi|^d - E)^{-\frac{1}{p}} (|\xi|^d - E)^{-\frac{1}{p'}},$$

with

$$\|(|\xi|^d - E)^{-\frac{1}{p}}\|^*_{L^{p,\infty}} \lesssim_d 1 \quad \text{ and } \quad \|(|\xi|^d - E)^{-\frac{1}{p'}}\|^*_{\ell^{p',\infty}(L^2)} \lesssim_d 1,$$

uniformly in $p \in (2, 2+\delta]$ for any $\delta > 0$ (this follows from a similar calculation as in (3.3)–(3.4)). Theorem 1.8 then shows that $((-\Delta)^{d/2} - E)^{-\frac{1}{2}} v(x)$ belongs to $\mathcal{L}^{2,\infty}$ and hence is bounded. Its adjoint is then also bounded. This shows that the operator of multiplication by v is bounded from $H^{d/2}$ to L^2 and from L^2 to $H^{-d/2}$. Therefore the chain

$$L^2 \xrightarrow{v\times} H^{-d/2} \xrightarrow{((-\Delta)^{d/2} - E)^{-1}} H^{d/2} \xrightarrow{v\times} L^2, \tag{B.2}$$

holds. In particular K_E is a bounded operator on L^2.

Moreover K_E is also compact. For $s > d/2$ it is Hilbert-Schmidt, since its integral kernel is given by the L^2 function

$$-v(x)v(y)\int e^{-i(x-y)\xi}\frac{1}{|\xi|^{2s}-E}\mathrm{d}\xi\,.$$

For $s = d/2$, this follows again from Theorem 1.8.

Note that this also implies that H_s is self-adjoint by the KLMN theorem [26, Theorem X.17] and that the essential spectrum of H_s is equal to $[0,\infty)$ thanks to Weyl's essential spectrum theorem [27, Theorem XIII.14] (see also [27, Section XIII.4, Example 7]).

Recall that $N_{\le r}(A)$ (respectively $N_{\ge r}(A)$) denotes the number of eigenvalues less or equal (respectively larger or equal) than r of a self-adjoint operator A, counted with multiplicity.

Proposition B.3 (Birman-Schwinger principle)**.** *Assume $E < 0$ and Hypothesis B.1 holds. Then*

$$N_{\le E}(H_s) = N_{\ge 1}(K_E)\,. \tag{B.3}$$

The non-increasing sequence of eigenvalues $(\lambda_j(K))_{j\ge 0}$ is rigorously defined in the statement of Theorem A.2. We prove Proposition B.3 following the arguments of [22], using properties of the maps $E \mapsto \lambda_j(K_E)$ that we collect in the following lemma.

Lemma B.4. *Assume Hypothesis B.1 holds. For any $j \ge 0$, the map $E \mapsto \lambda_j(K_E)$ is non-decreasing, continuous on $(-\infty, 0)$ and goes to 0 as $E \to -\infty$.*

Proof. As $x \mapsto 1/x$ is operator monotone, the expression of $\lambda_j(K_E)$ given in the min-max principle (Theorem A.2) yields that $E \mapsto \lambda_j(K_E)$ is non-decreasing.

To prove continuity we use first the resolvent identity: let $E' < E < 0$. Then

$$K_E - K_{E'} = (E-E')\,v\left((-\Delta)^s - E\right)^{-1}\left((-\Delta)^s - E'\right)^{-1}v \le \frac{E-E'}{-E'}K_E,$$

and hence, for all $u \in L^2$,

$$\langle K_E\,u,u\rangle \le \langle K_{E'}\,u,u\rangle + \frac{E-E'}{-E'}\|K_E\|_{\mathcal{L}^\infty}\|u\|_{L^2}^2.$$

The min-max principle (Theorem A.2) and the previous inequality then yield

$$\lambda_j(K_E) \le \max_{\substack{u\in\mathcal{S}^\perp\\ \|u\|_{L^2}=1}}\langle K_E\,u,u\rangle \le \max_{\substack{u\in\mathcal{S}^\perp\\ \|u\|_{L^2}=1}}\langle K_{E'}\,u,u\rangle + \frac{E-E'}{-E'}\|K_E\|_{\mathcal{L}^\infty},$$

for all subspace $\mathcal{S}$ of L^2 of dimension j. Hence, taking the minimum over all such spaces and using again the min-max principle, we obtain

$$\lambda_j(K_E) \le \lambda_j(K_{E'}) + \frac{E-E'}{-E'}\|K_E\|_{\mathcal{L}^\infty}.$$

Together with $\lambda_j(K_{E'}) \le \lambda_j(K_E)$, this gives the continuity with respect to E of $\lambda_j(K_E)$.

To prove that $\lambda_j(K_E) \to 0$ as $E \to -\infty$, since $\lambda_j(K_E) \le \|K_E\|_{\mathcal{L}^\infty}$, it suffices to show that $\|K_E\|_{\mathcal{L}^\infty} = \|[(-\Delta)^s - E]^{-1/2}v(x)\|_{\mathcal{L}^\infty}^2 \to 0$.

Suppose that $s = d/2$. Recall that $v \in \mathcal{D}((\ln \mathbf{h})^{\frac12}(\ln\ln \mathbf{h})^{\frac12+\varepsilon})$ by assumption. Let $\varepsilon' > 0$ and let $R_{\varepsilon'} > 0$ be such that

$$\big\| \mathbb{1}_{\mathbf{h}\geq R_{\varepsilon'}}(\ln \mathbf{h})^{\frac12}(\ln\ln \mathbf{h})^{\frac12+\varepsilon} v\big\|_{L^2} \leq \varepsilon'. \tag{B.4}$$

Setting $v_{\varepsilon'} := \mathbb{1}_{\mathbf{h}<R_{\varepsilon'}} v$, we have $v_{\varepsilon'} \in L^\infty$ (as $v_{\varepsilon'}$ is a finite linear combination of bound states of $\mathbf{h}$) and therefore we can write

$$\begin{aligned}\big\|[(-\Delta)^s - E]^{-\frac12} v(x)\big\|_{\mathcal{L}^\infty} &\leq \big\|[(-\Delta)^s - E]^{-\frac12} v_{\varepsilon'}(x)\big\|_{\mathcal{L}^\infty} + \big\|[(-\Delta)^s - E]^{-\frac12}(v(x) - v_{\varepsilon'}(x))\big\|_{\mathcal{L}^\infty}\\ &\leq (-E)^{-\frac12}\|v_{\varepsilon'}\|_{L^\infty} + C\big\|(\ln \mathbf{h})^{\frac12}(\ln\ln \mathbf{h})^{\frac12+\varepsilon}(v - v_{\varepsilon'})\big\|_{L^2},\end{aligned} \tag{B.5}$$

for some $C > 0$, uniformly in $E \leq -1$. In the second inequality, we used that $\|A\|_{\mathcal{L}^\infty} \lesssim \|A\|_{\mathcal{L}^{2,\infty}}$, for any operator $A \in \mathcal{L}^{2,\infty}$, together with Theorem 1.8 (applied with $g(\xi) = [|\xi|^{2s} - E]^{-1/2}$, so that g^2 can be decomposed as $g^2(\xi) = g_p(\xi)g_{p'}(\xi)$ with $p > 2$, $\frac1p + \frac1{p'} = 1$ and $g_p(\xi) = [|\xi|^{2s} - E]^{-1/p}$; a similar calculation as in (3.3)–(3.4) then shows that $\|g_p\|_{L^{p,\infty}}$, $\|g_{p'}\|_{\ell^{p',\infty}(L^2)}$ are uniformly bounded in $p \in (2, 2+\delta]$ and $E \leq -1$ for any $\delta > 0$). Combining (B.4) and (B.5) shows that $\|[(-\Delta)^s - E]^{-1/2}v(x)\|_{\mathcal{L}^\infty} \to 0$ as $E \to -\infty$.

In the case where $s > d/2$, it suffices to write

$$\begin{aligned}\|K_E\|_{\mathcal{L}^\infty} = \sup_{\|u\|_{L^2}=1}\langle u, K_E u\rangle &= \sup_{\|u\|_{L^2}=1}\int_{\mathbb{R}^d}\big(|\xi|^{2s} - E\big)^{-1}|\widehat{vu}(\xi)|^2\mathrm{d}\xi\\ &\leq \big\|\big(|\xi|^{2s} - E\big)^{-1}\big\|_{L^1}\sup_{\|u\|_{L^2}=1}\big\||\widehat{vu}|^2\big\|_{L^\infty}.\end{aligned}$$

Now we have

$$\big\||\widehat{vu}|^2\big\|_{L^\infty} = \big\|\widehat{vu}\big\|_{L^\infty}^2 \lesssim_d \|u\|_{L^2}^2\|v\|_{L^2}^2,$$

and the dominated convergence theorem shows that $\|(|\xi|^{2s} - E)^{-1}\|_{L^1} \to 0$ as $E \to -\infty$. This concludes the proof. ☐

Now we are ready to prove Proposition B.3.

Proof of Proposition B.3. Any eigenfunction ψ of H_s associated to an eigenvalue $E' < 0$ is in particular in the domain of H_s (hence in H^s, the form domain of H_s), and satisfies

$$((-\Delta)^s - E')\psi = v^2\,\psi.$$

We set $\phi = v\psi \in H^{-s}$ (see Remark B.2). The resolvent $((-\Delta)^s - E')^{-1}$ applied to the equality above yields

$$\psi = \big((-\Delta)^s - E'\big)^{-1} v\phi \in H^s\,,$$

which in turn implies that $\phi \neq 0$. Multiplying by v then gives

$$\phi = v\big((-\Delta)^s - E'\big)^{-1} v\phi \in L^2,$$

so that ϕ is an eigenvector of $K_{E'}$ corresponding to the eigenvalue 1.

Viceversa, for any eigenfunction $\phi \in L^2$ of $K_{E'}$ associated to the eigenvalue 1, we set $\psi = ((-\Delta)^s - E')^{-1}v\phi \in H^s \subset L^2$. Multiplying by v yields $v\psi = \phi \neq 0$, so that $\psi \neq 0$ and

$$((-\Delta)^s - E')\psi = v\phi = v^2((-\Delta)^s - E')^{-1}v\phi = v^2\psi\,.$$

It follows that ψ is an eigenvector of H_s associated to the eigenvalue E'.

We have thus, for any $E' < 0$, a bijection between the eigenfunctions ϕ of $K_{E'}$ corresponding to the eigenvalue 1, and the eigenfunctions ψ of H_s corresponding to E'. Hence

$$N_{\leq E}(H_s) = \sum_{E' \leq E} \dim \ker(H_s - E') = \sum_{E' \leq E} |\{j \mid \lambda_j(K_{E'}) = 1\}| . \tag{B.6}$$

Now, for every j, the map $E \mapsto \lambda_j(K_E)$ takes at most once the value 1, because otherwise the set of eigenvalues of H_s would contain an interval $[E_1, E_2] \subset (-\infty, 0)$, which is impossible. It follows that

$$\sum_{E' \leq E} |\{j \mid \lambda_j(K_{E'}) = 1\}| = |\{j \mid \exists E' \leq E, \lambda_j(K_{E'}) = 1\}| . \tag{B.7}$$

As, for any j, $E' \mapsto \lambda_j(K_{E'})$ is continuous and $\lambda_j(K_{E'}) \to 0$ as $E' \to -\infty$, we deduce that

$$|\{j \mid \exists E' \leq E, \lambda_j(K_{E'}) = 1\}| = |\{j \mid \lambda_j(K_E) \geq 1\}| = N_{\geq 1}(K_E) . \tag{B.8}$$

The bound (B.3) then follows from (B.6), (B.7) and (B.8). □

Appendix C. Proofs of Theorem 3.3 and Lemma 3.4

In this section we prove Theorem 3.3 and Lemma 3.4 which were used in the proof of Theorem 1.8. To obtain Theorem 3.3, we reproduce the proof of [34, Theorem 4.6], carefully following the dependence on the parameter p' in all the estimates.

Proof of Theorem 3.3. Assume that $\|f\|_{\ell^{p'}(L^2)} = \|g\|^*_{\ell^{p'},\infty(L^2)} = 1$. Recall that $\chi_{\mathbf{m}}$ stands for the characteristic function of the unit hypercube of $\mathbb{R}^d$ with center $\mathbf{m} \in \mathbb{Z}^d$ and, for all function $f : \mathbb{R}^d \to \mathbb{C}$, $f_{\mathbf{m}} := \chi_{\mathbf{m}} f$. We set $\tilde{f}_{\mathbf{m}} := \frac{f_{\mathbf{m}}}{\|f_{\mathbf{m}}\|_{L^2}}$, $\tilde{g}_{\mathbf{m}} := \frac{g_{\mathbf{m}}}{\|g_{\mathbf{m}}\|_{L^2}}$ and write

$$f = \sum_{\mathbf{m} \in \mathbb{Z}^d} a_{\mathbf{m}} \tilde{f}_{\mathbf{m}}, \quad a_{\mathbf{m}} := \|f_{\mathbf{m}}\|_{L^2}, \quad g = \sum_{\mathbf{m} \in \mathbb{Z}^d} b_{\mathbf{m}} \tilde{g}_{\mathbf{m}}, \quad b_{\mathbf{m}} := \|g_{\mathbf{m}}\|_{L^2},$$

so that $\|a_{\mathbf{m}}\|_{\ell^{p'}} = \|b_{\mathbf{m}}\|^*_{\ell^{p'},\infty} = 1$. As in [34, Theorem 4.6], for any $n \in \mathbb{Z}$, we define

$$\begin{aligned} f_n &:= \sum_{2^{n-1} < a_{\mathbf{m}} \leq 2^n} a_{\mathbf{m}} \tilde{f}_{\mathbf{m}}, \qquad g_n := \sum_{2^{n-1} < b_{\mathbf{m}} \leq 2^n} b_{\mathbf{m}} \tilde{g}_{\mathbf{m}} \\ A_n &:= \sum_{l+k \leq n} f_l(x) g_k(-i\nabla), \qquad B_n := \sum_{l+k > n} f_l(x) g_k(-i\nabla), \end{aligned}$$

so that $f(x)g(-i\nabla) = A_n + B_n$. Then using Fan's inequality [34, Theorem 1.7]:

$$\mu_m(f(x)g(-i\nabla)) \leq \mu_{m/2+1/2}(A_n) + \mu_{m/2+1/2}(B_n), \quad m \text{ odd}, \tag{C.1}$$

and

$$\mu_m(f(x)g(-i\nabla)) \leq \mu_{m/2+1}(A_n) + \mu_{m/2}(B_n) \leq \mu_{m/2}(A_n) + \mu_{m/2}(B_n), \quad m \text{ even}. \tag{C.2}$$

By estimating the norms $\|A_n\|_{\mathcal{L}^2}$ and $\|B_n\|_{\mathcal{L}^1}$ we obtain bounds on the singular values of A_n and B_n which will allow us to conclude. Since, f_l and g_k have disjoint supports, we first obtain that

$$\|A_n\|_{\mathcal{L}^2}^2 = \mathrm{Tr}(A_n^* A_n) = \mathrm{Tr}\Big(\sum_{\substack{l+k\leq n \\ l'+k'\leq n}} f_l(x)\overline{f_{l'}(x)} g_k(-i\nabla)\overline{g_{k'}(-i\nabla)} \Big)$$

$$= \mathrm{Tr}\Big(\sum_{l+k\leq n} \overline{g_k(-i\nabla)}|f_l(x)|^2 g_k(-i\nabla) \Big).$$

This expression can be computed thanks to the formula $\|f(x)g(-i\nabla)\|_{\mathcal{L}^2} = (2\pi)^{-d/2}\|f\|_{L^2}\|g\|_{L^2}$:

$$\|A_n\|_{\mathcal{L}^2}^2 = \sum_{l+k\leq n} \|f_l(x)g_k(-i\nabla)\|_{\mathcal{L}^2}^2 = c_d \sum_{l+k\leq n} \|f_l\|_{L^2}^2\|g_k\|_{L^2}^2 = c_d \sum_{\substack{l+k\leq n \\ 2^{l-1}<a_{\mathbf{m}}\leq 2^l \\ 2^{k-1}<b_{\mathbf{p}}\leq 2^k}} a_{\mathbf{m}}^2 b_{\mathbf{p}}^2 .$$

The number of $b_{\mathbf{p}}$ in the interval $(2^{k-1}, 2^k]$ is bounded by

$$|\{\mathbf{p} : b_{\mathbf{p}} \geq 2^{k-1}\}| \leq 2^{-p'(k-1)}\|b_{\mathbf{p}}\|_{\ell^{p'},\infty}^* \leq 2^2 2^{-kp'}. \tag{C.3}$$

Using this in the norm of A_n gives

$$\|A_n\|_{\mathcal{L}^2}^2 \lesssim_d \sum_{\substack{l+k\leq n \\ 2^{l-1}<a_{\mathbf{m}}\leq 2^l}} a_{\mathbf{m}}^2 2^{2k} \sum_{2^{k-1}<b_{\mathbf{p}}\leq 2^k} 1$$

$$\lesssim_d \sum_{\substack{l\in\mathbb{Z} \\ 2^{l-1}<a_{\mathbf{m}}\leq 2^l}} a_{\mathbf{m}}^2 \sum_{k\leq n-l} 2^{2k-kp'} = \frac{2^{(2-p')n}}{1-2^{p'-2}} \sum_{\substack{l\in\mathbb{Z} \\ 2^{l-1}<a_{\mathbf{m}}\leq 2^l}} 2^{-(2-p')l} a_{\mathbf{m}}^2 \leq \frac{2^{(2-p')n}}{1-2^{p'-2}} ,$$

where in the last inequality we have used the bound

$$\sum_{\substack{l\in\mathbb{Z} \\ 2^{l-1}<a_{\mathbf{m}}\leq 2^l}} 2^{-(2-p')l} a_{\mathbf{m}}^2 = \sum_{\substack{l\in\mathbb{Z} \\ 2^{l-1}<a_{\mathbf{m}}\leq 2^l}} 2^{-(2-p')l} a_{\mathbf{m}}^{2-p'} a_{\mathbf{m}}^{p'} \leq \sum_{\mathbf{m}} a_{\mathbf{m}}^{p'} = 1.$$

By [34, Theorem 4.5] we also have

$$\|B_n\|_{\mathcal{L}^1} \lesssim \sum_{\substack{l+k>n \\ 2^{l-1}<a_{\mathbf{m}}\leq 2^l \\ 2^{k-1}<b_{\mathbf{p}}\leq 2^k}} a_{\mathbf{m}} b_{\mathbf{p}}.$$

Using again (C.3) we have

$$\|B_n\|_{\mathcal{L}^1} \lesssim \sum_{\substack{l+k>n \\ 2^{l-1}<a_{\mathbf{m}}\leq 2^l}} a_{\mathbf{m}} 2^{k-kp'} = \sum_{\substack{l\in\mathbb{Z} \\ 2^{l-1}<a_{\mathbf{m}}\leq 2^l}} a_{\mathbf{m}} \sum_{k\geq n-l+1} 2^{(1-p')k}$$

$$= \sum_{\substack{l\in\mathbb{Z} \\ 2^{l-1}<a_{\mathbf{m}}\leq 2^l}} a_{\mathbf{m}} \frac{2^{(1-p')(n-l+1)}}{1-2^{1-p'}} \leq \frac{2^{(1-p')n}}{1-2^{1-p'}},$$

where we have used the following inequality

$$\sum_{\substack{l\in\mathbb{Z} \\ 2^{l-1}<a_{\mathbf{m}}\leq 2^l}} a_{\mathbf{m}} 2^{(1-p')(1-l)} = \sum_{\substack{l\in\mathbb{Z} \\ 2^{l-1}<a_{\mathbf{m}}\leq 2^l}} a_{\mathbf{m}}^{p'} a_{\mathbf{m}}^{1-p'} 2^{-(1-p')(l-1)} \leq \sum_{\mathbf{m}} a_{\mathbf{m}}^{p'} = 1.$$

Going back to (C.1) and (C.2), it suffices to consider m even. By the definition of the norms on the trace ideals $\mathcal{L}^1, \mathcal{L}^2$ and since the singular values are arranged in decreasing order, we have

$$\|B_n\|_{\mathcal{L}^1} \geq \sum_{k=1}^{m/2} \mu_k(B_n) \geq \frac{m}{2}\mu_{m/2}(B_n),$$

which implies

$$\mu_{m/2}(B_n) \lesssim \frac{2}{m}\ \frac{2^{(1-p')n}}{1-2^{1-p'}} \lesssim \frac{2^{(1-p')n}}{m}.$$

Analogously

$$\mu_{m/2}(A_n) \lesssim \sqrt{\frac{1}{m}}\ \frac{2^{(1-p'/2)n}}{\sqrt{1-2^{p'-2}}},$$

and hence

$$\mu_m(f(x)g(-i\nabla)) \lesssim m^{-1}2^{(1-p')n} + m^{-\frac{1}{2}}\frac{2^{(1-p'/2)n}}{\sqrt{1-2^{p'-2}}}. \tag{C.4}$$

Optimizing with respect to n yields

$$\mu_m(f(x)g(-i\nabla)) \lesssim m^{-\frac{1}{p'}}(1-2^{p'-2})^{\frac{1}{p'}-1} \lesssim m^{-\frac{1}{p'}}(2-p')^{\frac{1}{p'}-1}, \tag{C.5}$$

which proves the statement of the theorem. □

We conclude with the proof of Lemma 3.4.

Proof of Lemma 3.4. Let q be defined by $\frac{1}{q}+\frac{1}{2}=\frac{1}{p'}$. We have

$$\|f\|_{\ell^{p'}(L^2)} = \| \, \|\chi_{\mathbf{m}} f\|_{L^2}\|_{\ell^{p'}} \lesssim \|\langle \mathbf{m}\rangle^{-r}\|\chi_{\mathbf{m}} f\|_{L^2(\langle x\rangle^{2r}\mathrm{d}x)}\|_{\ell^{p'}} \lesssim \|\langle \mathbf{m}\rangle^{-r}\|_{\ell^q}\|f\|_{L^2(\langle x\rangle^{2r}\mathrm{d}x)},$$

where we choose r such that $rq > d$, so that $\langle \mathbf{m}\rangle^{-r}$ indeed belongs to $\ell^q(\mathbb{Z}^d)$. By straightforward computations one obtains

$$\|f\|_{\ell^{p'}(L^2)} \lesssim_d \frac{(rq-d+1)^{1/q}}{(rq-d)^{1/q}}\|f\|_{L^2(\langle x\rangle^{2r}\mathrm{d}x)}.$$

It then suffices to observe that, given $0<\delta<1$, the constant appearing in the right hand side of the previous inequality is uniformly bounded in $p' \in [2-\delta, 2)$. □

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