LIGHT CONES FOR OPEN QUANTUM SYSTEMS IN THE CONTINUUM

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ABSTRACT. We consider Markovian open quantum dynamics (MOQD) in the continuum. We show that, up to small-probability tails, the supports of quantum states evolving under such dynamics propagate with finite speed in any finite-energy subspace.

More precisely, we prove that if the initial quantum state is localized in space, then any finite-energy part of the solution of the von Neumann-Lindblad equation is approximately localized inside an energy-dependent light cone. We also obtain an explicit upper bound for the slope of this light cone.

1. INTRODUCTION

While non-relativistic quantum theory does not possess the strict light cone of relativistic theories, it has been shown in many contexts that its dynamics nonetheless exhibits a maximal speed bound up to small-probability leakage. By analogy, one speaks of a (system-dependent) *light cone* also in these cases. Existence of such light cones has been rigorously derived in standard QM [4, 21, 37, 40], for non-relativistic QED models [5], and for nonlinear Schrödinger equations [3]. Famously, Lieb and Robinson [27] first derived the existence of light cones in quantum spin systems. Their eponymous Lieb-Robinson bounds have developed into an extremely active research area starting in the early 2000s [18, 19, 20, 29, 30] that continues to grow in scope, e.g., with recent extensions to lattice fermions [17, 31], lattice bosons [13, 14, 26, 36, 39, 43, 44] and long-range interactions [15, 17, 42]. The existence of a maximal speed bound in a quantum theory is a fundamental statement about its non-equilibrium properties which serves as the backbone of many proofs. For instance, it played an essential role in scattering theory [10, 38] and, in quantum information theory Lieb-Robinson bounds were used to prove the celebrated area law for entanglement entropy [18] and bounds on quantum state transfer [11]. They are also central to the notion of quantum phase defined via quasi-adiabatic continuation [20, 32].

In this work we consider quantum particles governed by a continuum Schrödinger operator $H = -\Delta + V$ on $L^2(\mathbb{R}^d)$ that interact with an environment. We show that the corresponding Markovian open quantum dynamics (MOQD), which have an unbounded group velocity, exhibit an energy-dependent light cone, i.e., initially localized states propagate at most with a maximal speed. Previous results about maximal speed bounds of MOQD either concerned lattice systems (where the

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mechanism for maximal speed is different [33, 35], similarly to the Lieb-Robinson bounds) or it excluded the most interesting case when the Hamiltonian H is a standard Schrödinger operator [7]. In this paper, we resolve this question and show that coupling quantum-mechanical particles to an environment cannot lead to acceleration of any finite-energy portion. For this purpose, we develop microlocalization techniques involving functions of noncommuting operators H and x_j. To fix ideas, we work on $L^2(\mathbb{R}^d)$ but we expect that our approach could be extended to abstract Hilbert space with abstract noncommuting self-adjoint operators H and x_j.

1.1. **Setup and main result.** We study the long-time behaviour of solutions to the von Neumann-Lindblad (vNL) equation [9, 28]:

$$\frac{\partial \rho_t}{\partial t} = -i[H, \rho_t] + \frac{1}{2}\sum_{j\geq 1}\big([W_j, \rho_t W_j^*] + [W_j\rho_t, W_j^*]\big). \tag{1.1}$$

Here ρ_t, $t \geq 0$ is a family of density operators (i.e. non-negative-definite operators with unit trace) on a Hilbert space $\mathcal{H}$, H is the quantum Hamiltonian, a self-adjoint operator on $\mathcal{H}$, and the $\{W_j\}$ are bounded operators, arising from interaction with the environment.

We show that, for any E, there exists $\kappa = \kappa(E) > 0$ such that, for any initial condition ρ_0 localized in $X \subset \mathbb{R}^d$ and for any $c > \kappa$, the probability that the system in the state ρ_t is localized in $\mathcal{H}_E \cap X^{\mathrm{c}}_{ct}$ is arbitrarily small, asymptotically as $t \to \infty$, where $\mathcal{H}_E$ is the spectral subspace

$$\mathcal{H}_E := \{H \leq E\} \equiv \mathrm{Ran}(\mathbf{1}_{(-\infty,E]}(H))$$

and $X^{\mathrm{c}}_{ct} = \mathbb{R}^d \setminus X_{ct}$ with

$$X_{ct} \equiv \big\{x \in \mathbb{R}^d : d_X(x) \leq ct\big\} \tag{1.2}$$

the light cone corresponding to a smoothed out distance function $d_X(\cdot)$ defined in (1.11) below. Put differently, there exists an energy-dependent light cone for (1.1) with slope κ.

Throughout this article, we let $\mathcal{H} = L^2(\mathbb{R}^d)$, $d \geq 1$. We make no distinction in our notation between functions and the operators of multiplication defined by those functions. For an operator A on $\mathcal{H}$, denote by $\mathcal{D}(A) \subset \mathcal{H}$ the domain of A.

We now set out the main assumptions in this paper. We take the Hamiltonian H in (1.1) to be the standard Schrödinger operator,

$$H = -\Delta + V(x), \quad V : \mathbb{R}^d \to \mathbb{R}. \tag{1.3}$$

For the potential V,

(H) We assume that $V \in C^\infty(\mathbb{R}^d)$ and that there exists $\sigma > 0$ such that for every multi-index α there exists $C = C(\alpha) > 0$ such that

$$|\partial^\alpha V(x)| \leq C\,\langle x\rangle^{-|\alpha|-\sigma} \quad (x \in \mathbb{R}^d) \tag{1.4}$$

Here and below, we write $\langle\cdot\rangle = \sqrt{1 + |\cdot|^2}$.

Remark 1. If V satisfies **(H)**, then it is bounded and therefore H is self-adjoint on $\mathcal{D}(-\Delta)$ (see e.g. [8]) and bounded from below.

Now, for the operators W_j, $j \geq 1$ in (1.1), we assume that there exists an integer $n \geq 1$ such that the following holds.

(W1) For all integers $j \geq 1$, $W_j \in \mathcal{B}(\mathcal{H})$ and the series $\sum_{j=1}^{\infty} W_j^* W_j$ converges strongly in $\mathcal{B}(\mathcal{H})$ (and consequently, $\sum_{j=1}^{\infty} W_j^* W_j \in \mathcal{B}(\mathcal{H})$).

(W2) Let $C_A = \mathrm{ad}_A : B \to [A, B]$, $p_q = -i\partial_{x_q}$. Then, for every $1 \leq q_1, q_2 \leq d$ and $1 \leq k \leq n+1$, we have

$$\sum_{j=1}^{\infty} \sum_{m=1}^{n+1} \sum_{\substack{k,\ell \geq 0 \\ k+\ell=m}} \|(\langle x \rangle\, C_{p_{q_1}})^k C_{x_{q_2}}^{\ell} W_j\|^2 < \infty. \tag{1.5}$$

Remark 2. Assumptions **(W1)** and **(W2)** can be ensured for example by taking the W_j's to be suitable pseudodifferential operators. See also [7, Section 1.4] and [12, Section 4]. In this case, Assumption **(W2)** can be converted to a regularity assumption on the symbols with respect to both x and p. For working with Assumption **(W2)**, it is convenient to notice that $[C_{p_q}, C_{x_q}] = 0$ by Jacobi's commutator identity.

Remark 3. Let $\mathcal{S}_1$ stand for the Schatten space of trace-class operators. Conditions **(H)** and **(W1)** guarantee global well-posedness for (1.1) in the space

$$\mathcal{D} := \{\rho \in \mathcal{S}_1 \mid \rho\mathcal{D}(H) \subset \mathcal{D}(H) \text{ and } [H, \rho] \in \mathcal{S}_1\}, \tag{1.6}$$

see below.

For each subset $X \subset \mathbb{R}^d$, let $X^{\mathrm{c}} := \mathbb{R}^d \setminus X$ and $\chi_X^\sharp$ stand for the characteristic function of X. The main result of this paper is the following:

Theorem 1.1 (Main result)**.** *Suppose Assumptions **(H)** and **(W1)**–**(W2)** hold for some $n \geq 1$. Let $X \subset \mathbb{R}^d$ be a bounded and closed subset. Suppose $\rho_0 \in \mathcal{D}$ (see* (1.6)*) is supported in X in the sense that*

$$\mathrm{Tr}(\chi_{X^{\mathrm{c}}}^\sharp \rho_0) = 0. \tag{1.7}$$

Then (1.1) *has a unique solution $\rho_t \in \mathcal{D}$, $t \geq 0$, and for any $E \in \sigma(H)$ and $c > \kappa$ with κ as in* (1.17)*, this solution satisfies*

$$\mathrm{Tr}(g(H)\chi_{X_{ct}^{\mathrm{c}}}^\sharp g(H)\rho_t) \leq C_{n,E} t^{-n}, \tag{1.8}$$

for all $t > 0$ and all smooth cutoff functions g with $\mathrm{supp}(g) \subset (-\infty, E]$ and $0 \leq g \leq 1$, where $X_{ct}^{\mathrm{c}} \equiv (X_{ct})^{\mathrm{c}}$ and $C_{n,E}$ is a positive constant depending on n and E.

Remark 4. For the energy-dependent speed κ defined in (1.17), we have the following estimate:

$$\kappa \leq C(1 + |E|)^{1/2} \text{ for some fixed } C > 0 \text{ and all } X \subset \mathbb{R}^d, E \in \mathbb{R}. \tag{1.9}$$

Moreover, the constant $C_{n,E}$ in (1.8) grows polynomially with E.

Theorem 1.1 solves an open problem from [7], namely, to derive a light cone for MOQD when the Hamiltonians is a standard Schrödinger operator $-\Delta + V$ (a situation not covered by the methods in [7]). To our knowledge, it is the first maximal velocity bound for an open quantum system with a Hamiltonian that allows for unbounded group velocity.

Theorem 1.1 is proved in Section 3. Theorem 1.1 implies that "microlocally" the propagation speed for (1.1) is finite, and yields an upper bound for the maximal speed of propagation of initially localized states. Indeed, define the probability

$$\mathrm{Prob}_{\rho_t,E}(Y) := \mathrm{Tr}(g_E(H)\chi_Y^\sharp g_E(H)\rho_t) \tag{1.10}$$

for the system in the state ρ_t to have the part with energy $H \le E$ lie in the region where $x \in Y$; this can be considered a probability to line in the corresponding region of phase space. With notation (1.10) and, recall, $X_{ct}^{\mathrm{c}} \equiv (X_{ct})^{\mathrm{c}}$, the exterior of the light cone X_{ct} in (1.2), Theorem 1.1 says that

$$\mathrm{Prob}_{\rho_t,E}(X_{ct}^{\mathrm{c}}) \le C_{n,E}t^{-n}.$$

The constant $C_{n,E}$ in (1.8) depends on the difference $c-\kappa>0$ (through (2.49) below). For brevity of notation, we do not display the dependence on $c-\kappa$.

In equations (1.16)-(1.17) below, we provide an explicit formula for the number κ in Theorem 1.1. Physically, κ bounds the propagation speed (also called "speed of sound") in the energy-constrained open quantum system. Naturally, κ depends on the system parameters and the energy cutoff.

We introduce some notation. For each closed set $X \subset \mathbb{R}^d$, we define the *smoothed distance function* to X, $d_X \in C^\infty(\mathbb{R}^d)$ in the following way. Let $\epsilon_0 > 0$ be a fixed parameter (the estimate (1.8), in particular, depends on this arbitrary parameter). Let

$$d_X(x) \equiv d_{X,\epsilon_0}(x) \begin{cases} = 0, & \mathrm{dist}_X(x) = 0, \\ \ge 0, & 0 < \mathrm{dist}_X(x) < c_1\epsilon_0, \\ = \delta_X(x) - \epsilon_0, & \mathrm{dist}_X(x) \ge c_1\epsilon_0, \end{cases} \tag{1.11}$$

where $\delta_X \in C^\infty(\mathbb{R}^d)$ satisfies $c_1\,\mathrm{dist}_X(x) \le \delta_X(x) \le c_2\,\mathrm{dist}_X(x)$ for some $c_1, c_2 > 0$, and

$$\mathrm{dist}_X^{|\alpha|-1}(x)\,|\partial^\alpha d_X(x)| \le C_\alpha \quad (x \in \mathbb{R}^d,\, 0 \le |\alpha|), \tag{1.12}$$

for some absolute constants $C_\alpha > 0$. In one-dimension, such functions are easy to construct, see the schematic diagram Figure 1. In any dimension, one can proceed as follows. By the extension theorem of Whitney (see e.g. [41, Theorem 6.2.2]), there exists a function δ_X defined in X^c such that

$$c_1\,\mathrm{dist}_X(x) \le \delta_X(x) \le c_2\,\mathrm{dist}_X(x), \quad \text{for all } x \in X^c$$
$$\delta_X \text{ is } C^\infty \text{ in } X^c \text{ and } \mathrm{dist}_X^{|\alpha|-1}(x)\partial^\alpha\delta_X(x) \le C_\alpha, \quad \text{for all } x \in X^c \text{ and } |\alpha| \ge 0,$$

where c_1, c_2, C_α are positive constants independent of X. Let $f_{\epsilon_0} : \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that $f_{\epsilon_0}(x) = 0$ if $x \le \epsilon_0/2$, and $f_{\epsilon_0}(x) = x - \epsilon_0$ if $x \ge \epsilon_0$. We can then define

$$d_X(x) := f_{\epsilon_0}(\delta_X(x))$$

and verify that it satisfies the conditions above.

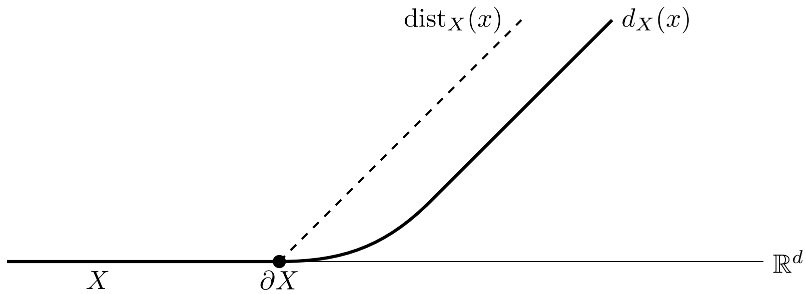

FIGURE 1. Schematic diagram illustrating $d_X \equiv d_{X,\epsilon}$ in (1.11).

We fix $E \in \sigma(H)$ and a function $g \in C^\infty(\mathbb{R})$ satisfying $0 \leq g \leq 1$ and, for some small $\epsilon > 0$,

$$g(\mu) \equiv 1 \text{ for } \mu \leq E - \epsilon, \quad g(\mu) \equiv 0 \text{ for } \mu \geq E, \tag{1.13}$$

and define the *smooth energy cutoff* operator

$$g := g(H). \tag{1.14}$$

Remark 5. Since $g(H) = (g\chi^{\#}_{\sigma(H)})(H)$, the values of g outside of $\sigma(H)$ are irrelevant. Since, moreover, H is bounded from below by **(H)**, one can always take g to have compact support if needed.

Considering the multiplication operator d_X by the smoothed distance function $d_X(x)$, introduced in (1.11) above, we define the spectrally localized distance function

$$d_X^E := g d_X g \qquad \text{defined on} \qquad \{u \in \mathcal{H} : gu \in \mathcal{D}(d_X)\}. \tag{1.15}$$

Now, we define the *energy-dependent velocity operator*

$$\gamma \equiv \gamma(X, E) := i[H, d_X^E] + \frac{1}{2}\sum_{j\geq 1}\left(W_j^*[d_X^E, W_j] + [W_j^*, d_X^E]W_j\right). \tag{1.16}$$

It is shown in Section 4 that γ is bounded on $\mathcal{H}$:

$$\kappa := \|\gamma\| < \infty, \tag{1.17}$$

provided assumptions **(H)** and **(W2)** hold. Notice that the bound on κ is independent of X, see (1.9). Formally, the velocity operator (1.16) has a simple origin:

$$\gamma \equiv \gamma(X, E) = L'(d_X^E), \tag{1.18}$$

where L' is the operator acting on the space of observables $\mathcal{B}(\mathcal{H})$, which is dual to the operator L defined by the r.h.s. of (1.1), see (1.21) below.

Under a different set of assumptions, an estimate similar to (1.8) is shown in [7] with $O(t^{-n})$ remainder for any $n \geq 1$. The assumptions made in [7] exclude in (1.1) the Schrödinger operators (1.3).

It is straightforward to show that under the conditions **(W1)**,

(1.19) $\quad V(x)$ in (1.3) is Δ-bounded with relative bound strictly less than 1, and for any $\rho_0 \in \mathcal{D}$ (see (1.6)), Eq. (1.1) has a solution in $\mathcal{D}$. For more detailed discussions, see Appendix A below and Refs. [9, Section 5.5], [12, Appendix A], [34]. Note that Condition (1.19) holds e.g. for every $V \in L^2(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ and is much weaker than **(H)**.

One can show further (see [1, 9, 12, 24, 25] and Appendix A) that the operator L defines a completely positive, trace-preserving, strongly continuous semigroup of contractions. In particular, for any initial state $\rho_0 \in \mathcal{D}$, the solution ρ_t, $t \geq 0$, to (1.1) satisfies

$$\rho_t \geq 0, \quad \text{if} \quad \rho_0 \geq 0, \quad \text{and} \quad \mathrm{Tr}\rho_t = \mathrm{Tr}\rho_0. \tag{1.20}$$

Finally, we give the explicit expression of the operator L' in (1.18) and its domain. Let L be the operator defined by the r.h.s. of (1.1) on its natural domain $\mathcal{D}$ (see (1.6)), and L' be the operator acting on the space of observables $\mathcal{B}(\mathcal{H})$, which is dual to L with respect to the coupling $(A, \rho) := \mathrm{Tr}(A\rho)$, i.e.,

$$\mathrm{Tr}(AL\rho) = \mathrm{Tr}((L'A)\rho), \tag{1.21}$$

for $\rho \in \mathcal{D}(L)$ and $A \in \mathcal{D}(L') \subset \mathcal{B}(\mathcal{H})$. [1] Explicitly, the dual vNL operator L' defined in (1.21) is given by:

$$L' = L_0' + G', \qquad L_0'A = i[H, A], \tag{1.22}$$

$$G'A := \frac{1}{2}\sum_{j\geq 1}(W_j^*[A, W_j] + [W_j^*, A]W_j), \tag{1.23}$$

with domain

$$\begin{aligned} &\mathcal{D}(L') \equiv \mathcal{D}(L_0') \equiv \big\{A \in \mathcal{B}(\mathcal{H}) \,|\, A\mathcal{D}(H) \subset \mathcal{D}(H) \text{ and} \\ &[H, A] \text{ defined on } \mathcal{D}(A) \cap \mathcal{D}(H) \text{ extends to an operator on } \mathcal{D}(\mathcal{H})\big\}. \end{aligned} \tag{1.24}$$

Notation. In the remainder of this paper, $\|\cdot\|$ stands either for the norm of vectors in $\mathcal{H}$, or for the norm of operators on $\mathcal{H}$, which one is meant is always clear from the context. For two bounded operators A, B, the notation

$$A = O(B) \tag{1.25}$$

means that $\|A\| \leq C_{n,E}\,\|B\|$ for some $C_{n,E} > 0$ independent of $A\,, B\,, t\,, s$. As above, we will write

$$X_a := \big\{x \in \mathbb{R}^d : d_X(x) \leq a\big\} \text{ for } a \geq 0, \quad X_{ct}^{\mathrm{c}} \equiv (X_{ct})^{\mathrm{c}}.$$

In all our estimates, it is understood that, if $n = 1$, the sums $\sum_{k=2}^n(\cdots)$ should be dropped.

2. Recursive monotonicity estimate

In this section, we work in an abstract setting, with H a self-adjoint operator on a Hilbert space $\mathcal{H}$ and, for $\{W_j\}_{j\geq 1}$ bounded operators in $\mathcal{H}$ such that $\sum_{j\geq 1} W_j^* W_j$ strongly converges in $\mathcal{H}$. We consider the vNL operator

$$L\rho = -i[H, \rho] + \frac{1}{2}\sum_{j\geq 1}\big([W_j, \rho W_j^*] + [W_j\rho, W_j^*]\big),$$

[1] L' generates the dual Heisenberg-Lindblad evolution $\partial_t A_t = L'A_t$ of quantum observables.

defined on the domain (1.6), as well as the dual operator L' defined as in (1.21)–(1.24).

We consider in addition a self-adjoint operator Φ on $\mathcal{H}$, semi-bounded from below. We assume that

$$(\Phi + c)^{-1}\mathcal{D}(H) \subset \mathcal{D}(H), \tag{2.1}$$

for some $c \geq 0$ and there is an integer $n \geq 1$ such that, for all $k = 1, \dots, n+1$,

$$M_k := 1 + \left\| \mathrm{ad}_{\Phi}^k(H) \right\|^2 + \| \sum_{j\geq 1} W_j^* W_j \| + \sum_{j \geq 1} \left\| \mathrm{ad}_{\Phi}^k(W_j) \right\|^2 < \infty. \tag{2.2}$$

Hence

$$\mu_n := \max_{2 \leq k \leq n+1} M_k \tag{2.3}$$

is finite.

Later on, H will be the Schrödinger operator (1.3) satisfying **(H)**, W_j will be bounded operators satisfying **(W1)**–**(W2)** and Φ will be taken to be the operator $\Phi \equiv \phi^E = g\phi g$ with $g \equiv g(H)$ described in (1.13) and some $\phi \in C^\infty(\mathbb{R}^d)$, see Section 4.

As in (1.16)–(1.17) we set

$$\kappa_\Phi := \left\| i[H, \Phi] + \frac{1}{2} \sum_{j \geq 1} \left(W_j^* [\Phi, W_j] + [W_j^*, \Phi] W_j \right) \right\|. \tag{2.4}$$

The main result of this section is a key differential inequality, (2.9). The proof of this inequality is *the only place* where the information about equation (1.1) is used.

2.1. **ASTLO and RME.** We construct a class of observables, which we call *adiabatic spacetime localization observables (ASTLOs)*, which play a central role in our analysis.

For a constant $\delta > 0$ specified later on, we define a set of smooth cutoff functions

$$\mathcal{X} \equiv \mathcal{X}_\delta := \left\{ \chi \in C^\infty(\mathbb{R}) \,\middle|\, \begin{array}{l} \operatorname{supp} \chi \subset \mathbb{R}_{\geq 0}, \operatorname{supp} \chi' \subset (0, \delta/2) \\ \chi' \geq 0,\ \sqrt{\chi'} \in C^\infty(\mathbb{R}) \end{array} \right\}. \tag{2.5}$$

See Figure 2 below.

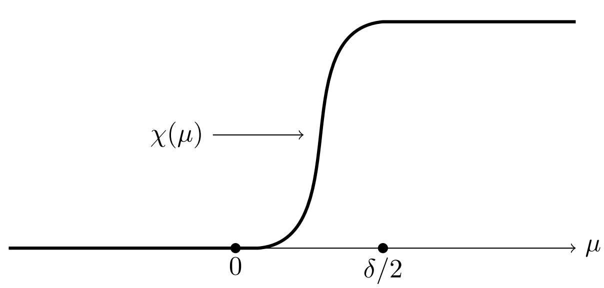

Figure 2. Schematic diagram illustrating $\chi \in \mathcal{X}$.

We note that $\chi \geq 0$ for $\chi \in \mathcal{X}$, and the following two properties hold:

(X1) If $w \in C_c^\infty$ and $\operatorname{supp} w \subset (0, \delta/2)$, then the antiderivative $\int^x w^2 \in \mathcal{X}$.

(X2) If $\xi_1, \dots, \xi_N \in \mathcal{X}$, then $\xi = (\xi_1^{\frac{1}{2}} + \dots + \xi_N^{\frac{1}{2}})^2$ satisfies $\xi \in \mathcal{X}$ and $\xi_1 + \dots + \xi_N \leq \sqrt{N}\xi$.

For a function $\chi \in \mathcal{X}$, a densely defined self-adjoint operator Φ, a constant $v \in (\kappa, c)$ and $s > t \geq 0$, we define a family of self-adjoint operators

$$\chi_{ts} = \chi\left(\frac{\Phi - vt}{s}\right). \tag{2.6}$$

Following [7], Section 3, we use the method of propagation observables. Let β_t' be the Markovian dynamical semigroup generated by the operator L', i.e. $\frac{d}{dt}\beta_t'(\Psi) = \beta_t'(L'\Psi)$ for all observables Ψ in $\mathcal{D}(L') \subset \mathcal{B}(\mathcal{H})$. For a differentiable family of bounded operators $\Psi_t \in \mathcal{D}(L')$, $t \geq 0$, we then have the relation

$$\frac{d}{dt}\beta_t'(\Psi_t) = \beta_t'(D\Psi_t), \tag{2.7}$$

$$D\Psi_t = L'\Psi_t + \partial_t \Psi_t. \tag{2.8}$$

As in [7], we call the operation D the *Heisenberg derivative*.

Note that the condition (2.1) ensures that for all t, s, the bounded observable χ_{ts} belongs to the domain of L' and also that the commutator expansion Lemma C.2 can be applied. The main result of this section is the following:

Theorem 2.1 (recursive monotonicity estimate)**.** *Suppose that* (2.1)–(2.2) *hold. Let $\chi \in \mathcal{X}$ and let χ_{ts} be the operator defined in* (2.6)*. Then there exists $C = C(n, \chi) > 0$ and, if $n \geq 2$, $\xi^k = \xi^k(\chi) \in \mathcal{X}$, $k = 2, \dots, n$, such that as self-adjoint operators,*

$$D\chi_{ts} \leq -\frac{v - \kappa_\Phi}{s}\chi_{ts}' + \sum_{k=2}^n \frac{M_k}{s^k}(\xi^k)_{ts}' + C\frac{\mu_n}{s^{n+1}}, \tag{2.9}$$

where $\kappa_\Phi > 0$ is as in (2.4) *and M_k and μ_n are defined in* (2.2) *and* (2.3)*.*

This theorem is proved in Section 2.2.

Since the second, subleading term on the r.h.s. is of the same form as the leading, negative term, we call (2.9) the *recursive monotonicity estimate (RME)*. It can be bootstrapped as in Proposition 2.2 to obtain an integral inequality with $O(s^{-n})$ remainder. We write, for $r \geq 0$,

$$\chi_{ts}(r) := \beta_r'(\chi_{ts}) \qquad \text{and} \qquad \chi_{ts}'(r) := \beta_r'(\chi_{ts}'). \tag{2.10}$$

Proposition 2.2. *Suppose the assumptions of Theorem 2.1 hold. Then, for all $c > \kappa_\Phi$ and $\chi \in \mathcal{X}$, there exist $C = C(n, \chi) > 0$ and $\xi^k \in \mathcal{X}$, $2 \leq k \leq n$ (dropped for $n = 1$), such that for all $0 \leq t < s$,*

$$\int_0^t \chi_{rs}'(r)dr \leq C\mu_n^n\Big(s\chi_{0s}(0) + \sum_{k=2}^n s^{-k+2}\ \xi_s^k(0) + ts^{-n}\Big), \tag{2.11}$$

where μ_n is given by (2.3)*.*

Remark 6. Instead of the evolution $\chi_{rs}(t)$, we could have used the expectation:

$$\langle \chi_{ts} \rangle_t := \mathrm{Tr}(\chi_{ts}\rho_t) \tag{2.12}$$

of χ_{ts} in the state ρ_t solving (1.1) and instead of (2.7), used the relation

$$\frac{d}{dt}\langle\chi_{ts}\rangle_t = \langle D\chi_{st}\rangle_t\,. \tag{2.13}$$

These two formulations are related as

$$\langle\chi_{ts}\rangle_t = \langle\chi_{ts}(t)\rangle_0\,. \tag{2.14}$$

2.2. **Proof of Theorem 2.1.** To prove the recursive monotonicity estimate, Theorem 2.1, we first need a totally symmetrized commutator expansion. Our next results, Proposition 2.3 and Proposition 2.4, generalize the commutator expansion for bounded operators, first obtained in [37], and subsequently improved in e.g. [16, 22, 23, 40]. We refer to [22] for details and references.

Recall that the dual vNL operator L' satisfies $L' = i[H, A] + G'A$ for all A in $\mathcal{D}(L')$, where G' is given by (1.23).

Proposition 2.3. *Suppose that* (2.1) *and* (2.2) *hold. Let $\chi \in \mathcal{X}$ and let χ_{ts} be the operator defined by* (2.6)*. Then, uniformly in t, for $s > 0$,*

$$i[H, \chi_{ts}] = s^{-1}\sqrt{\chi'_{ts}}\,i[H,\Phi]\sqrt{\chi'_{ts}} + \mathrm{Rem}_H \tag{2.15}$$

where the remainder term Rem_H satisfies the estimate

$$\pm\mathrm{Rem}_H \le \sum_{k=2}^{n}\frac{M_k}{s^k}(\xi^k)'_{ts} + C\frac{M_{n+1}}{s^{n+1}} \tag{2.16}$$

for some $\xi^2, ..., \xi^n \in \mathcal{X}$ depending only on χ, with M_k as in (2.2) *and for some constant $C = C(n, \chi) > 0$.*

Proposition 2.4. *Suppose that* (2.1) *and* (2.2) *hold. Let $\chi \in \mathcal{X}$ and let χ_{ts} be the operator defined by* (2.6)*. Then, uniformly in t, for $s > 0$,*

$$G'(\chi_{ts}) = s^{-1}\sqrt{\chi'_{ts}}\,G'(\Phi)\sqrt{\chi'_{ts}} + \mathrm{Rem}_W, \tag{2.17}$$

where the remainder term Rem_W satisfies the estimate

$$\pm\mathrm{Rem}_W \le \sum_{k=2}^{n}\frac{M_k}{s^k}(\xi^k)'_{ts} + C\frac{\mu_n}{s^{n+1}} \tag{2.18}$$

for some $\xi^2, ..., \xi^n \in \mathcal{X}$ depending only on χ, for some constant $C = C(n, \chi) > 0$, with M_k and μ_n as in (2.2) *and* (2.3)*.*

Remark 7. The estimates above are all uniform in s, t, Φ and, in particular, are valid for the operator $\phi^E = g\phi g$ such as (3.3).

Remark 8. We note that the error term in Theorem 1.1 arises in the symmetrization procedure above, and can be improved as the expansion continues to higher order.

Proof of Proposition 2.3. In this proof, the time t is fixed and is omitted from the notation, so we write χ_s for χ_{ts}. Also, we denote $B_k \equiv i\mathrm{ad}^k_\Phi(H)$ for $k = 1, ..., n+1$. In this case, since H is self-adjoint, we have $B^*_k = (-1)^{k-1}B_k$.

1. By (2.1)–(2.2) and the assumption on χ, the hypotheses of Lemma C.2 are satisfied. Hence, by (C.4)–(C.5), we have

$$i[H, \chi_s] = \frac{1}{2}\sum_{k=1}^{n}\frac{s^{-k}}{k!}\left(\chi_s^{(k)}B_k + B^*_k\chi_s^{(k)}\right) + \frac{1}{2}s^{-(n+1)}\left(R_{n+1} + R^*_{n+1}\right), \tag{2.19}$$

where $\|R_{n+1}\| \le c\|B_{n+1}\|$ for some constant $c > 0$ depending only on χ.

2. Next, we claim that every term on the r.h.s. of (2.19), except for the leading term ($k = 1$), are uniformly bounded by $(\chi_1)'_s$ for some $\chi_1 \in \mathcal{X}$.

To show this, for each k, we choose some smooth function $\theta^k \in C_c^\infty((0, \delta/2))$ that takes value 1 on $\mathrm{supp}(\chi^{(k)})$. Then, we claim that

$$\chi_s^{(k)} B_k = \chi_s^{(k)} B_k \theta_s^k + O(s^{-(n+1-k)}), \tag{2.20}$$

where $\theta_s^k \equiv \theta^k(s^{-1}(\Phi - vt))$. Indeed, using commutator expansion and the fact that $\mathrm{ad}_\Phi^l(B_k) = B_{k+l}$, we have

$$\begin{aligned}\chi_s^{(k)} B_k &= \chi_s^{(k)} \theta_s^k B_k = \chi_s^{(k)} B_k \theta_s^k + \chi_s^{(k)} [\theta_s^k, B_k] \\ &= \chi_s^{(k)} B_k \theta_s^k - \chi_s^{(k)} \sum_{l=1}^{n-k} \frac{(-1)^l s^{-l}}{l!} (\theta^k)_s^{(l)} B_{k+l} \\ &\quad + (-1)^{n+1-k} s^{-(n+1-k)} \chi_s^{(k)} \mathrm{Rem}_{\mathrm{right}}(s),\end{aligned} \tag{2.21}$$

where

$$\mathrm{Rem}_{\mathrm{right}}(s) = \int d\widetilde{\theta^k}(z) R^{n+1-k} B_{n+1} R. \tag{2.22}$$

Since θ^k has compact support, $\mathrm{Rem}_{\mathrm{right}}(s)$ is bounded so that

$$\chi_s^{(k)} B_k = \chi_s^{(k)} B_k \theta_s^k - \chi_s^{(k)} \sum_{l=1}^{n-k} \frac{(-1)^l s^{-l}}{l!} (\theta^k)_s^{(l)} B_{k+l} + O(s^{-(n+1-k)}). \tag{2.23}$$

Next, since $\theta^k \equiv 1$ on $\mathrm{supp}(\chi^{(k)})$, we have $\mathrm{supp}((\theta^k)^{(l)}) \cap \mathrm{supp}(\chi^{(k)}) = \varnothing$ for all $l \ge 1$ so that

$$\chi_s^{(k)} \sum_{l=1}^{n-k} \frac{(-1)^l s^{-l}}{l!} (\theta^k)_s^{(l)} B_{k+l} = 0. \tag{2.24}$$

It follows that

$$\chi_s^{(k)} B_k = \chi_s^{(k)} B_k \theta_s^k + O(s^{-(n+1-k)})$$

so that

$$s^{-k}(\chi_s^k B_k + B_k^* \chi_s^k) = s^{-k}(\chi_s^k B_k \theta_s^k + \theta_s^k B_k^* \chi_s^k) + O(s^{-(n+1)}). \tag{2.25}$$

Now, we apply the following operator inequality

$$\pm(P^*Q + Q^*P) \le P^*P + Q^*Q. \tag{2.26}$$

with $P = \chi_s^{(k)}$ and $Q = B_k \theta_s^k$ on (2.25) to obtain

$$s^{-k}(\chi_s^k B_k + B_k^* \chi_s^k) \le s^{-k} \left((\chi_s^{(k)})^2 + \|B_k\|^2 (\theta_s^k)^2 \right) + O(s^{-(n+1)}). \tag{2.27}$$

Since n is finite, we can choose $\xi^2, ..., \xi^n \in \mathcal{X}$ such that $(\xi^k)'$ majorizes $(\chi^{(k)})_s^2 + \|B_k\|^2 (\theta_s^k)^2$ for each k.

3. Now, we symmetrize the leading order term. Let $u = (\chi')^{1/2}$. Since u is smooth by assumption, we use (C.1) to expand the leading order terms and obtain

$$\begin{aligned}(u_s)^2 B_1 + B_1(u_s)^2 &= 2u_s B_1 u_s + u_s[u_s, B_1] + [B_1, u_s]u_s \\ &= 2u_s B_1 u_s + \sum_{l=1}^{n-1} \frac{s^{-l}}{l!}\left(u_s u_s^{(l)} B_{1+l} + B_{1+l}^* u_s^{(l)} u_s\right) \\ &\quad + s^{-n}(u_s R_n' + R_n'^* u_s),\end{aligned} \tag{2.28}$$

where $\|R_n'\| \le c'\|B_{n+1}\|$ for some constant $c' > 0$ depending only on u.

Again, using operator estimate (2.26), for each $l = 1, ..., n-1$, we have

$$s^{-l}(u_s u_s^{(l)} B_{1+l} + B_{1+l}^* u_s^{(l)} u_s) \le s^{-1}\|B_{1+l}\|^2 (u_s^{(l)})^2 + s^{-2l+1}(u_s)^2, \tag{2.29}$$

and for the remainder term we have

$$s^{-n}(u_s R_n' + R_n'^* u_s) \le s^{-1}(u_s)^2 + s^{-2n+1}\|R_n'\|^2(\tilde\theta_s)^2, \tag{2.30}$$

where $\tilde\theta$ is again some smooth cutoff function supported in $(0, \delta/2)$ that takes value 1 on the support of u and $\tilde\theta_s \equiv \tilde\theta(s^{-1}(\Phi - vt))$. Since u, $u^{(l)}$ and $\tilde\theta$ are supported in $(0, \delta/2)$, we can modify $\xi^2, ..., \xi^n$ in such a way that $\xi^l \in \mathcal{X}$ majorizes u^2, $\tilde\theta^2$ and $(u^{(l)})^2$ for each $l = 1, ..., n-1$.

Collecting all terms except for the leading order ones into the remainder term Rem_H, we obtain (2.15). □

Proof of Proposition 2.4. In this proof, we also fix t and omit it from the notation. Furthermore, we fix $j \ge 1$ and denote $D_{j,k} \equiv \mathrm{ad}_\Phi^k(W_j)$. In particular, we obtain $\mathrm{ad}_\Phi^k(W_j^*) = (-1)^k(\mathrm{ad}_\Phi^k(W_j))^* = (-1)^k D_{j,k}^*$.

1. First, using Lemma C.2 and the boundedness of W_j, we have

$$[\chi_s, W_j] = -\sum_{k=1}^{n} \frac{s^{-k}}{k!}\chi_s^{(k)} D_{j,k} - s^{-(n+1)} R_{j,n+1}^{\mathrm{right}} \tag{2.31}$$

where $R_{j,n+1}^{\mathrm{right}}$ is given in (C.14) and satisfies the estimate

$$\|R_{j,n+1}^{\mathrm{right}}\|^2 \le C\|D_{j,n+1}\|^2, \tag{2.32}$$

for some constant C independent of j. Similarly, we have

$$[W_j^*, \chi_s] = -\sum_{k=1}^{n} \frac{s^{-k}}{k!} D_{j,k}^* \chi_s^{(k)} - (-1)^{n+1} s^{-(n+1)} \widetilde{R}_{j,n+1}^{\mathrm{left}}, \tag{2.33}$$

where $\widetilde{R}_{j,n+1}^{\mathrm{left}} = (-1)^{n+1}(R_{j,n+1}^{\mathrm{right}})^*$. Combining (2.31) and (2.33), we have

$$\begin{aligned}G_j'(\chi_s) &= W_j^*[\chi_s, W_j] + [W_j^*, \chi_s]W_j \\ &= -\sum_{k=1}^{n} \frac{s^{-k}}{k!}\left(W_j^* \chi_s^{(k)} D_{j,k} + D_{j,k}^* \chi_s^{(k)} W_j\right) \\ &\quad - s^{-(n+1)}\left(W_j^* R_{j,n+1}^{\mathrm{right}} + (R_{j,n+1}^{\mathrm{right}})^* W_j\right),\end{aligned} \tag{2.34}$$

where $G_j'(\cdot) = W_j^*[\cdot, W_j] + [W_j^*, \cdot]W_j$.

2. We now verify that the r.h.s. of (2.34) is summable in $j \geq 1$. We begin with the remainder terms. Using the operator estimate (2.26), we obtain

$$\pm \left(W_j^* R_{j,n+1}^{\text{left}} + (R_{j,n+1}^{\text{left}})^* W_j \right) \leq W_j^* W_j + \|R_{j,n+1}^{\text{left}}\|^2, \tag{2.35}$$

which are summable in $j \geq 1$ since $\sum_j W_j^* W_j$ strongly converges in $\mathcal{H}$, and since (2.2) and (2.32) hold.

Next, we estimate the k-th terms in the first two lines of (2.34). Let θ^k be some smooth cutoff function supported in $(0, \delta/2)$ such that $\theta^k \equiv 1$ on $\text{supp}(\chi^{(k)})$. It follows that $\chi_s^{(k)} = \theta_s^k \chi_s^{(k)} \theta_s^k$, where $\theta_s^k \equiv \theta^k(s^{-1}(\Phi - vt))$. Then, we claim that

$$\begin{aligned} &W_j^* \chi_s^{(k)} D_{j,k} + D_{j,k}^* \chi_s^{(k)} W_j \\ &\quad = \theta_s^k \left(W_j^* \chi_s^{(k)} D_{j,k} + D_{j,k}^* \chi_s^{(k)} W_j \right) \theta_s^k + s^{-(n+1-k)} C_k \|D_{j,n+1}\|^2, \end{aligned} \tag{2.36}$$

where C_k is some constant depending only on $\chi^{(k)}$.

If (2.36) holds, then using (2.26), we have

$$\begin{aligned} &\pm \left(W_j^* \chi_s^{(k)} D_{j,k} + D_{j,k}^* \chi_s^{(k)} W_j \right) \\ &\quad \leq \theta_s^k W_j^* W_j \theta_s^k + \|D_{j,k}\|^2 \|\chi^{(k)}\|^2 (\theta_s^k)^2 + C_k s^{-(n+1-k)} \|D_{j,n+1}\|^2, \end{aligned} \tag{2.37}$$

which are also summable in $j \geq 1$ by (2.2).

3. Now, we prove the claim (2.36). By a direct calculation, we have

$$\begin{aligned} &W_j^* \chi_s^{(k)} D_{j,k} - \theta_s^k W_j^* \chi_s^{(k)} D_{j,k} \theta_s^k \\ &\quad = [W_j^*, \theta_s^k] \chi_s^{(k)} D_{j,k} + \theta_s^k W_j^* \chi_s^{(k)} [\theta_s^k, D_{j,k}] \end{aligned} \tag{2.38}$$

and a similar expression for $D_{j,k}^* \chi_s^{(k)} W_j$. Thus, it suffices to show that $[W_j^*, \theta_s^k]$ and $[\theta_s^k, D_{j,k}]$ are $O(s^{-(n-k)})$.

3.1. For the first term, we use (2.33) to obtain

$$[W_j^*, \theta_s^k] = -\sum_{l=1}^{n} \frac{s^{-l}}{l!} D_{j,l}^* (\theta^k)_s^{(l)} - s^{-(n+1)} R_{j,n+1}^*, \tag{2.39}$$

where $R_{j,n+1}$ is given by (C.14) and satisfies the estimate $\|R_{j,n+1}\| \leq C \|D_{j,n+1}\|$. Since $\theta^k \equiv 1$ on $\text{supp}(\chi^{(k)})$, then we have $(\theta^k)_s^{(l)} \chi_s^{(k)} = 0$ for $l \geq 1$ so that

$$[W_j^*, \theta_s^k] \chi_s^{(k)} D_{j,k} = -s^{(n+1)} R_{j,n+1}^* \chi_s^{(k)} D_{j,k}, \tag{2.40}$$

which is $O(s^{-(n+1)})$ and summable in $j \geq 1$, by the Cauchy-Schwarz inequality and (2.2).

3.2. For the second term, we proceed similarly, using (2.31), to obtain

$$[\theta_s^k, D_{j,k}] = -\sum_{l=1}^{n-k} \frac{s^{-l}}{l!} (\theta^k)_s^{(l)} D_{j,k+l} + s^{-(n+1-k)} \widetilde{R}_{j,n+1-k}, \tag{2.41}$$

where $\widetilde{R}_{j,n+1-k}$ is given by (C.13) with n replaced by $n-k$ and satisfies the estimate $\|\widetilde{R}_{j,n+1-k}\| \leq C \|D_{j,n+1-k}\|$ with C only depending on θ^k. Using the same reason as above, since $\chi_s^{(k)} (\theta^k)_s^{(l)} = 0$ for all $l \geq 1$, we conclude that

$$\theta_s^k W_j^* \chi_s^{(k)} [\theta_s^k, D_{j,k}] = s^{-(n+1-k)} \theta_s^k W_j^* \chi_s^{(k)} \widetilde{R}_{j,n+1-k}. \tag{2.42}$$

This completes the proof of the claim (2.36).

4. Now we choose $\xi^2, ..., \xi^n \in \mathcal{X}$ such that

$$\left(\| \sum_{j\geq 1} W_j^* W_j \| + \sum_{j\geq 1} \|D_{j,k}\|^2 \right) (\theta^k)^2 \leq M_k (\xi^k)'.$$

Then, by writing everything as Rem_W in (2.34) except for the leading order terms (obtained for $k = 1$), we obtain, up to some terms coming from the leading order terms which will be dealt with below, the estimate

$$\pm \mathrm{Rem}_W \leq \sum_{k=2}^{n+1} \frac{M_k}{s^k} (\xi^k)'_s + \frac{C\mu_n}{s^{n+1}}, \tag{2.43}$$

where C is a constant depending only on χ and n.

5. Finally, we deal with the leading order terms (obtained for $k = 1$) in (2.34). Following the same lines as in the proof for Proposition 2.3, we define $u = \sqrt{\chi'}$ and use (C.1) to obtain

$$\begin{aligned} &W_j^* \chi_s' D_{j,1} + \text{h.c.} \\ =&u_s W_j^* D_{j,1} u_s + [W_j^*, u_s] u_s D_{j,1} + u_s W_j^* [u_s, D_{j,1}] + \text{h.c.}, \end{aligned} \tag{2.44}$$

where h.c. means the adjoint of the terms before it. Without repeating the same calculation as above, using (C.1) and (2.26), we can show that the commutators are summable in $j \geq 1$. Then, we modify $\xi^k \in \mathcal{X}$ to majorize $(u^{(k)})^2$ and u^2 as well. This completes the proof. □

Now we are ready to prove Theorem 2.1:

Proof of Theorem 2.1. Given Proposition 2.3–2.4, we choose $\xi^2, ..., \xi^n$ depending on χ, in such a way that

$$\mathrm{Rem}_H + \mathrm{Rem}_W \leq \sum_{k=2}^{n} \frac{M_k}{s^k} (\xi^k)'_{ts} + C \frac{\mu_n}{s^{n+1}}, \tag{2.45}$$

where C is some constant which depends only on n and χ.

It remains to calculate $\partial_t \chi_{ts}$. Using the chain rule, we immediately obtain

$$\partial_t \chi_{ts} = -s^{-1} v \chi'_{ts}. \tag{2.46}$$

This completes the proof. □

2.3. **Proof of Proposition 2.2.**

Proof of Proposition 2.2. Within this proof, all constants $C > 0$ depend only on χ and n.

We will use the relation (2.7). First, we observe that, by Condition (2.1) and Definition (2.5), for $\chi \in \mathcal{X}$ and all $0 < t \leq s$, the operator χ_{ts} maps $\mathcal{D}(H)$ into itself. Moreover, (2.19) in the proof of Proposition 2.3 shows that $[H, \chi_{ts}] \in \mathcal{B}(\mathcal{H})$. Hence $\chi_{ts} \in \mathcal{D}(L')$.

Next, for each fixed s, integrating the formula (2.7) with $\Psi_t \equiv \chi_{ts}$ in t gives

$$\chi_{ts}(t) - \int_0^t \beta'_r (D\chi_{rs})\, dr = \chi_{0s}(0). \tag{2.47}$$

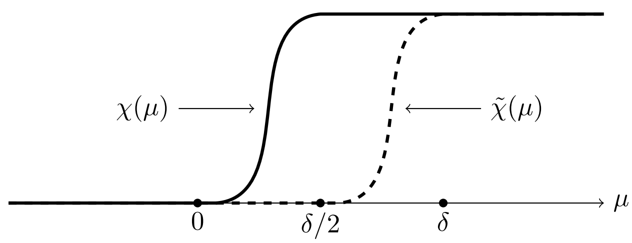

FIGURE 3. Schematic diagram illustrating $\tilde{\chi}$ satisfying (3.1).

The positive-preserving property of the flow (1.1) (see (1.20)) extends by duality to β_r', so that we can apply the inequality (2.9) to the second term on the l.h.s. of (2.47) to obtain

$$\begin{aligned} &\chi_{ts}(t) + (v - \kappa_\Phi)s^{-1}\int_0^t \chi'_{rs}(r)\,dr \\ &\leq \chi_{0s}(0) + C\mu_n\left(\sum_{k=2}^n s^{-k}\int_0^t (\xi^k)'_{rs}(r)\,dr + ts^{-(n+1)}\right), \end{aligned} \tag{2.48}$$

where we recall that the second term in the r.h.s. is dropped for $n = 1$.

Since $\kappa_\Phi < v$ and $t \leq s$, (2.48) implies, after dropping $\chi_{ts}(t) \geq 0$, which is due to the positive-preserving property of the flow (1.1) (see (1.20)), and multiplying by $s(v - \kappa_\Phi)^{-1} \geq 0$, that

$$\int_0^t \chi'_{rs}(r)\,dr \leq C\mu_n\left(s\chi_{0s}(0) + \sum_{k=2}^n s^{-k+1}\int_0^t (\xi^k)'_{rs}(r)\,dr + ts^{-n}\right). \tag{2.49}$$

3. If $n = 1$, then (2.49) gives (2.11). If $n \geq 2$, applying (2.49) to the term $\int_0^t (\xi^k)'_{rs}(r)\,dr$ and using the property (X2), we obtain

$$\int_0^t \chi'_{rs}(r)\,dr \leq C\mu_n^2\Big(s\chi_{0s}(0) + \xi^2_{0s}(0) + \sum_{k=3}^n s^{-k+2}\int_0^t (\eta^k)'_{rs}(r)\,dr + ts^{-n}\Big), \tag{2.50}$$

where the third term in the r.h.s. is dropped for $n = 2$, and $\eta^k = \eta^k(\xi^2, \xi^k) \in \mathcal{X}, k = 3, \dots, n$. Bootstrapping this procedure, we arrive at (2.11). $\square$

3. PROOF OF THEOREM 1.1

We formulate the technical relations mentioned in Theorem 1.1. Given a smooth, non-negative cutoff function g with $\text{supp}(g) \subset (-\infty, E]$ (see also Remark 5)and a smooth function χ from the space (2.5), we choose smooth cutoff functions $\tilde{g}$ and $\tilde{\chi}$ such that $\text{supp}(\tilde{g}) \subset \{g \equiv 1\}$ and $\text{supp}(\tilde{\chi}') \subset (\delta, +\infty) = \{\chi \equiv 1\}$, so that

$$\bar{\chi}(\mu)\tilde{\chi}(\mu) = 0, \tag{3.1}$$

$$\bar{g}(\mu)\tilde{g}(\mu) = 0. \tag{3.2}$$

see Figs. 3–4.

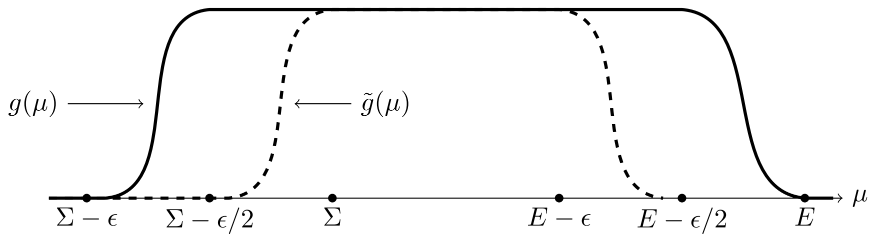

FIGURE 4. Schematic diagram illustrating $\tilde{g}$ satisfying (3.2). Here $\Sigma := \inf \sigma(H)$ (see Remark 5).

We also specify the self-adjoint operator Φ in Theorem 2.1 and Definition (2.6) as

$$\Phi := d_X^E = g(H) d_X g(H), \tag{3.3}$$

where, recall, $X \subset \mathbb{R}^d$ is a bounded subset with smooth boundary and $d_X \in C^\infty(\mathbb{R}^d)$ is the smoothed distance function to X given in (1.11) for some $\epsilon_0 > 0$ and satisfies (1.12).

We introduce the following shorthand notation:

$$\chi_{ts}^E := \chi((d_X^E - vt)/s), \quad \chi_{ts} := \chi((d_X - vt)/s). \tag{3.4}$$

Now, for any $\chi \in \mathcal{X}$ and $\tilde{g}$, $\tilde{\chi}$ as above, we claim that

$$\chi_X^\sharp \chi_{0s}^E \chi_X^\sharp = O(s^{-n}), \tag{3.5}$$

$$\chi_{ts}^E \geq \tilde{g} \tilde{\chi}_{ts} \tilde{g} + O(s^{-n}), \tag{3.6}$$

where we recall that $\chi_X^\sharp$ stands for the characteristic function of X. We postpone the proofs of these claims to Section 5.

Recall that β_t' denotes the evolution generated by the operator L' and that $\chi_{ts}^E(t) := \beta_t'(\chi_{ts}^E)$, $(\chi')_{ts}^E(t) := \beta_t'((\chi')_{ts}^E)$. We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. We want to apply Proposition 2.2 to $H = -\Delta + V(x)$ and W_j satisfying **(H)–(W2)**, with Φ given by (3.3). Hence we need to verify that the abstract conditions (2.1)–(2.2) are satisfied.

First, we fix any $c > 0$ and justify that $(d_X^E + c)^{-1}$ maps $\mathcal{D}(H)$ into itself. Recalling that $d_X^E = g(H) d_X g(H)$ with $\operatorname{supp}(g) \subset (-\infty, E]$, we have

$$\begin{aligned}(d_X^E + c)^{-1} &= \chi^\#_{(-\infty,E]}(H)(d_X^E + c)^{-1} + \chi^\#_{(E,\infty)}(H)(d_X^E + c)^{-1} \\ &= \chi^\#_{(-\infty,E]}(H)(d_X^E + c)^{-1} + c^{-1}\chi^\#_{(E,\infty)}(H).\end{aligned}$$

The first term is a bounded operator from $\mathcal{H}$ to $\mathcal{D}(H)$ while the second term obviously preserves $\mathcal{D}(H)$. This shows that $(d_X^E + c)^{-1}$ maps $\mathcal{D}(H)$ into itself

Next, condition (2.2) is verified in Section 4, see Corollary 4.3. Therefore Proposition 2.2 with $\Phi = d_X^E$ applies.

Now we take $\chi \in \mathcal{X}$ with $\chi(\mu) \equiv 1$ for $\mu \geq \delta/2$. Retaining the first term in the l.h.s. of (2.48) in the proof of Proposition 2.2 and dropping the second one, which is non-negative since $\chi' \geq 0$ and $v > \kappa$, we obtain

$$\chi_{ts}^E(t) \leq \chi_{0s}^E(0) + C_{n,E}\left(\sum_{k=2}^n s^{-k+1}\int_0^t ((\xi^k)')_{rs}^E(r)dr + ts^{-(n+1)}\right).$$

Here we used that the constant $\mu_n = \max_{2\le k\le n+1} M_k$ appearing in the r.h.s. of (2.48) is bounded by $C_{n,E}$ for some positive constant depending on n and E. Applying (2.11) to the second term on the r.h.s., we deduce that, with the notation as in (1.25),

$$\chi^E_{ts}(t) \le \chi^E_{0s}(0) + O(s^{-1}\xi^E_{0s}(0)) + O(s^{-n}), \tag{3.7}$$

for some $\xi \in \mathcal{X}$ and all $s > t$. Taking expectation w.r.t. ρ_0 on both sides of (3.7) and recalling that $\chi_{ts}(t) := \beta'_t(\chi_{ts})$, we find

$$\mathrm{Tr}\left(\beta'_t(\chi^E_{ts})\rho_0\right) \le \mathrm{Tr}\left(\left(\chi^E_{0s} + O(s^{-1}\xi^E_{0s})\right)\rho_0\right) + O(s^{-n}). \tag{3.8}$$

By the localization assumption (1.7) on the initial state, we have $\rho_0 = \chi^\sharp_X \rho_0 \chi^\sharp_X$. By this fact, we find

$$\mathrm{Tr}\left(\left(\chi^E_{0s} + O(s^{-1}\xi^E_{0s})\right)\rho_0\right) = \mathrm{Tr}\left(\chi^\sharp_X\left(\chi^E_{0s} + O(s^{-1}\xi^E_{0s})\right)\chi^\sharp_X\rho_0\right) = O(s^{-n}). \tag{3.9}$$

The relation (3.6) implies

$$\chi^E_{ts} \ge \tilde g \tilde\chi_{ts} \tilde g + O(s^{-n}), \tag{3.10}$$

where we recall that $\tilde g$ is a smooth non-negative cutoff function with $\mathrm{supp}(\tilde g) \subset \{g \equiv 1\}$ and $\tilde\chi$ is a smooth function such that $\tilde\chi \equiv 1$ on $(\delta, +\infty)$. It follows that, by applying the dual evolution β'_t,

$$\beta'_t(\tilde g\tilde\chi_{ts}\tilde g) \le \beta'_t(\chi^E_{ts}) + O(s^{-n}). \tag{3.11}$$

Plugging the estimates (3.9), (3.10) and (3.11) to (3.8) yields

$$\mathrm{Tr}(\tilde g\tilde\chi_{ts}\tilde g\beta_t(\rho_0)) = O(s^{-n}). \tag{3.12}$$

Finally, recalling the Definition (1.11), we find, for all $v \in (\kappa, c)$,

$$\chi^\sharp_{X^c_{ct}} = \theta^+(d_{X_{ct}}) = \theta^+(d_X - ct) \le \tilde\chi_{ts}, \tag{3.13}$$

where θ^+ is the Heaviside function, provided $\delta = c - v$ and $s = t$. See Figure 5.

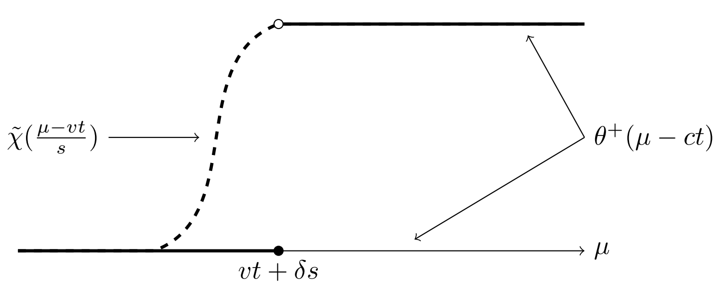

FIGURE 5. Schematic diagram illustrating estimate (3.13).

Hence we obtain estimate (1.8) from (3.12)–(3.13). This completes the proof of Theorem 1.1. □

4. Estimates of multiple commutators

In this section, we establish some key estimates for multiple commutators of the form $\mathrm{ad}^k_{A^E}(B)$. More precisely, we show that the operators $H = -\Delta + V(x)$ and W_j satisfying **(H)**–**(W2)**, with Φ given by (3.3), verify that the abstract conditions (2.2) used to prove the recursive monotonicity estimate in Section 2.

First, we introduce some notation. For an integer k and a function $f \in C^\infty(\mathbb{R}^d)$, we write

$$f \in \mathcal{S}^k \tag{4.1}$$

if there exists $C = C(|\alpha|, f) > 0$ such that for all multi-indices α and $x \in \mathbb{R}^d$,

$$|\partial^\alpha f(x)| \le C \langle x\rangle^{-k-|\alpha|}. \tag{4.2}$$

For any multi-index β, $f \in \mathcal{S}^k$ and $g \in \mathcal{S}^l$, it follows immediately from the definition and Leibniz's rule that

$$\partial^\beta f \in \mathcal{S}^{k+|\beta|}, \tag{4.3}$$

$$fg \in \mathcal{S}^{k+l}. \tag{4.4}$$

We denote by a_k a generic function belonging to $\mathcal{S}^k$.

To further simplify notation, for a fixed operator A on $\mathcal{H}$, define

$$C_A : B \mapsto \mathrm{ad}_A(B) \equiv [A, B]$$

on the set of linear operators on $\mathcal{H}$. We also omit the subindices in x_j and p_j. Restoring these subindices is straightforward.

We we prove results in this section for

$$\phi \in \mathcal{S}^{-1}. \tag{4.5}$$

Later on, we will choose $\phi = d_X$ from (1.11), which satisfies condition (4.5) by (1.12).

4.1. Bounds on multiple commutators with H.

The main result of this section are the following two propositions.

Proposition 4.1. *Let $n \ge 1$. Suppose H satisfies* **(H)** *and let ϕ be as above. Let $\phi^E := g\phi g$, where g is defined in* (1.13)-(1.14). *Then there exists $C = C(n, M, E) > 0$ such that, for all $E \in \mathbb{R}$,*

$$\left\| C^k_{\phi^E}(H) \right\| \le C \quad (k = 1, \ldots, n+1). \tag{4.6}$$

Proof. 1. In the following, we denote the resolvent $(z - A)^{-1}$ of the operator A by $R_A(z)$ and R_A if the argument is not important. For measures, if it is clear from the context, we will also drop the arguments for simplicity.

2. The proof is based on the mapping property of certain derivations. Before we proceed, we define a class of operators

$$\mathcal{F}^{(1)} := \left\{ \text{ polynomials of operators of the form } B^{(1)} \right\}, \tag{4.7}$$

where

$$B^{(1)} = \int d\mu(z_1, \dots, z_\nu) \left(\prod_{j=1}^{\nu} R_H(z_j)^{m_j^1} \right) \left(\prod_{q=1}^{N} \prod_{r=1}^{\nu} a_{k_q} p^{\ell_q} R_H(z_r)^{m_r^q} \right), \tag{4.8}$$

$$\sum_{j=1}^{\nu} m_j^1 \geq 1,\ 0 \leq \ell_q \leq \min(1, \sum_{r=1}^{\nu} m_r^q),\ k_q \geq 0,\ \forall q = 2, \dots N.$$

Here μ is some finite measure on $\mathbb{C}^\nu$, $\nu \geq 2$, N is some finite integer, and, recall, a_k's stand for generic functions belonging to $\mathcal{S}^k$ (see (4.1)). Moreover, here and it what follows, we use the shorthand $a_k p$ for $-i \sum_{j=1}^{d} a_{k,j} \partial_{x_j}$ where all the functions $a_{k,j}$ belong to $\mathcal{S}^k$. Since $\ell_q \leq \sum_{r=1}^{\nu} m_r^q$, $\ell_q \leq 1$, and $k_q \geq 0$ for each q, the second factor in the integrand of (4.8) is bounded, and therefore

$$\mathcal{F}^{(1)} \subset \mathcal{B}(\mathcal{H}). \tag{4.9}$$

Thus our goal is to show that

$$C^k_{\phi^E}(H) \in \mathcal{F}^{(1)} \quad \text{for} \quad 1 \leq k \leq n+1. \tag{4.10}$$

3. We prove (4.10) by induction. For the base case $k = 1$, since $[g, H] = 0$, we find by Leibniz's rule that

$$C_{\phi^E}(H) = g C_\phi(H)\, g. \tag{4.11}$$

Using formula (C.1) for each g, we can rewrite (4.11) using Fubini's theorem as

$$C_{\phi^E}(H) = \iint d\widetilde{g}(z_1) \otimes d\widetilde{g}(z_2) R_H(z_1) C_\phi(H)\, R_H(z_2). \tag{4.12}$$

By Remark 5, we can modify g to have compact support. Thus, we can choose the measure $d\widetilde{g} \otimes d\widetilde{g}$ to have compact support in $\mathbb{C}^2$ (see (B.5) and Appds. B–C for details).

Next, we compute

$$C_\phi(H) = \Delta\phi + 2\nabla\phi \cdot \nabla. \tag{4.13}$$

Using (4.13), together with assumption (4.5) and identity (4.3), we can write

$$C_\phi(H) = a_1 + a_0 p, \quad a_j \in \mathcal{S}^j. \tag{4.14}$$

Plugging expression (4.14) into (4.12) shows that $C_{\phi^E}(H) \in \mathcal{F}^{(1)}$. This completes the proof of the base case of (4.10).

4. Now, assuming $C^k_{\phi^E}(H) \in \mathcal{F}^{(1)}$, we will prove $C^{k+1}_{\phi^E}(H) \in \mathcal{F}^{(1)}$. It is immediately clear that the induction step is equivalent to showing

$$C_{\phi^E}(\mathcal{F}^{(1)}) \subset \mathcal{F}^{(1)}. \tag{4.15}$$

To establish (4.15), we use the crucial fact that the map C_A is a derivation, i.e. a linear operator satisfying the Leibniz rule. In particular, with $A = \phi^E = g\phi g = \phi g^2 + [g, \phi]g$, we have

$$C_{\phi^E} = \phi C_{g^2} + C_\phi(\cdot) g^2 + C_{[g,\phi]g}. \tag{4.16}$$

Also, we note some simple commutator identities

$$C_A R_H = R_H (C_A H) R_H \quad \text{for all operators } A \text{ s.t. } R_H : \mathcal{D}(A) \to \mathcal{D}(A), \tag{4.17}$$

$$C_H p = i\nabla V, \quad C_\phi p = i\nabla\phi, \quad C_\phi H = -C_H \phi = \Delta\phi + 2\nabla\phi \cdot \nabla. \tag{4.18}$$

We will show that each of the three maps in (4.16) maps $\mathcal{F}^{(1)}$ into itself using the relations (4.17)–(4.18).

4.1 First, we show $\phi C_{g^2}(\mathcal{F}^{(1)}) \subset \mathcal{F}^{(1)}$. Since $\phi C_{g^2}(R_H) = 0$, it suffices, by the induction hypothesis, formula (4.8) and Leibniz's rule, to evaluate the operators

$$\phi C_{g^2}(p), \quad \phi C_{g^2}(a_k). \tag{4.19}$$

Using (4.17)–(4.18), together with the relation $C_{g^2} A = -\int d\widetilde{g^2} R_H (C_H A) R_H$ and the fact that $\nabla V \in \mathcal{S}^1$ by Hypothesis **(H)**, we compute, using (4.18) twice,

$$\begin{aligned} \phi C_{g^2}(p) &= \int d\widetilde{g^2} \phi R_H (i\nabla V) R_H \\ &= \int d\widetilde{g^2} R_H a_0 R_H + \int d\widetilde{g^2} R_H (a_1 + a_0 p) R_H a_1 R_H, \end{aligned} \tag{4.20}$$

where in the second equality we commuted ϕ through R_H and used (4.5). Similarly,

$$\begin{aligned} \phi C_{g^2}(a_k) &= \int d\widetilde{g^2} \phi R_H (\Delta a_k + 2\nabla a_k \cdot \nabla) R_H \\ &= \int d\widetilde{g^2} R_H (a_{k+1} + a_k p) R_H \\ &\quad + \int d\widetilde{g^2} R_H (a_1 + a_0 p) R_H (a_{k+2} + a_{k+1} p) R_H, \end{aligned} \tag{4.21}$$

which are indeed of the desired form in order to deduce that $\phi C_{g^2}(\mathcal{F}^{(1)}) \subset \mathcal{F}^{(1)}$.

4.2 Next, we show $C_\phi(\mathcal{F}^{(1)}) g^2 \subset \mathcal{F}^{(1)}$. Since $C_\phi(a_k) = 0$ for all k, it suffices, by induction hypothesis, formula (4.8) and Leibniz's rule, to evaluate the following operators

$$C_\phi(p), \quad C_\phi(R_H), \tag{4.22}$$

where, recall, R_H stands for the resolvent of H. Using the relations (4.17)–(4.18), we compute

$$C_\phi(p) = \nabla\phi \in \mathcal{S}^0, \tag{4.23}$$

$$\begin{aligned} C_\phi(R_H) &= R_H (C_\phi H) R_H = R_H (\Delta\phi + 2\nabla\phi \cdot \nabla) R_H \\ &= R_H (a_1 + a_0 p) R_H, \end{aligned} \tag{4.24}$$

which, inserted into (4.8), allows us to conclude that $C_\phi(\mathcal{F}^{(1)}) g^2 \subset \mathcal{F}^{(1)}$.

4.3 Finally, we show $C_{[g,\phi]g}(\mathcal{F}^{(1)}) \subset \mathcal{F}^{(1)}$. By the induction hypothesis and the Leibniz rule, it suffices to show that $[g,\phi]\, g$ is of the form (4.8). To this end we use (C.1) so that

$$\begin{aligned} [g,\phi]\, g &= \left(\int d\widetilde{g}(z_1)\, [R_H(z), \phi] \right) \left(\int d\widetilde{g}(z_2) R_H(z_2) \right) \\ &= -\left(\int d\widetilde{g}(z_1) R_H(z_1) C_\phi\,(H) R_H(z_1) \right) \left(\int d\widetilde{g}(z_2) R_H(z_2) \right). \end{aligned} \tag{4.25}$$

Inserting identity (4.13) into the first factor in line (4.25), we conclude $C_{[g,\phi]g}(\mathcal{F}^{(1)}) \subset \mathcal{F}^{(1)}$.

This completes the induction step and Proposition 4.1 is proved. □

4.2. **Bounds on multiple commutators with W_j's.**

Proposition 4.2. *Suppose Assumption **(W2)** holds for some $n \geq 1$ and let $\phi \in C^\infty(\mathbb{R}^d)$ satisfy condition* (4.5)*. Let $\phi^E = g\phi g$ where g is defined in* (1.13)–(1.14)*. Then, the following estimates hold:*

$$\sum_j \left\| C^k_{\phi^E}(W_j) \right\|^2 < \infty \quad (k = 0, \dots, n+1). \tag{4.26}$$

Proof. We will adapt the strategy and notations in the proof of Proposition 4.1 to establish certain mapping property for the derivation C_{ϕ^E}. For each $k = 1, \dots, n+1$, we define the following class of operators on $\mathcal{B}(\mathcal{H})$:

$$\begin{aligned} \mathcal{G}^{(2)}_m := \{ \mathcal{L}_A \mathcal{R}_{A'} B^{(2)}_{rs} \mid A, A' \in \mathcal{F}^{(1)} \cup \{\mathbf{1}\}, \\ B^{(2)}_{rs} \equiv (\langle x \rangle C_p)^r C^s_x \text{ with } r, s \geq 0 \text{ and } r + s = m \}. \end{aligned} \tag{4.27}$$

Here $\mathcal{L}, \mathcal{R}$ are left- and right-multiplication operator on $\mathcal{B}(\mathcal{H})$, respectively, and $\mathcal{F}^{(1)}$ is defined in (4.7). Furthermore, we define $\mathcal{F}^{(2)}_k$ acting on $\mathcal{B}(\mathcal{H})$ as *linear combinations* of elements in $\mathcal{G}^{(2)}$,

$$\mathcal{F}^{(2)}_k := \operatorname{span} \bigcup_{m=1}^k \mathcal{G}^{(2)}_m. \tag{4.28}$$

The remainder of the proof proceeds as follows: In Steps 1–2, we prove an abstract algebraic identity for the space $\mathcal{F}^{(2)}_k$ and the derivation C_{ϕ^E}. In Step 3, we apply this identity to each W_j. We can sum the estimates in j thanks to assumption **(W2)** and from there we conclude the desired result (4.26).

Our main claim is the following purely algebraic fact: for any abstract operator W and every $k = 1, \dots, n+1$,

$$C^k_{\phi^E}(W) = \ell_k(W) \quad \text{for some} \quad \ell_k \in \mathcal{F}^{(2)}_k. \tag{4.29}$$

We prove this claim by induction in k.

1. Induction base for (4.29). We begin with the induction base, $k = 1$. First, we note that

$$[\langle x \rangle, p] = C_p \langle x \rangle = i \nabla \langle x \rangle = a_0, \tag{4.30}$$

so that, since $\langle x \rangle^{-1} \in \mathcal{S}^1$,

$$\begin{aligned} \langle x \rangle C_H W &= \langle x \rangle (p C_p W) + \langle x \rangle (C_p W) p + \langle x \rangle C_V W \\ &= p(\langle x \rangle C_p W) + [\langle x \rangle, p] C_p W + (\langle x \rangle C_p W) p + \langle x \rangle C_V W \\ &= p(\langle x \rangle C_p W) + a_0 C_p W + (\langle x \rangle C_p W) p + \langle x \rangle C_V W \\ &= p(\langle x \rangle C_p W) + a_1 \langle x \rangle C_p W + (\langle x \rangle C_p W) p + \langle x \rangle C_V W. \end{aligned} \tag{4.31}$$

The first three terms in (4.31) are all contained in $\mathcal{F}^{(2)}_1(W)$. For the last term $\langle x \rangle C_V W$, using the identity

$$R_x(z) \equiv (z - x)^{-1} = (x - i)^{-1} [\mathbf{1} - (z - i) R_x(z)], \tag{4.32}$$

and noting that $\langle x\rangle(x-i)^{-1}\in\mathcal{S}^0$, we find

$$\begin{aligned}\langle x\rangle C_V W &= \int d\widetilde{V}(z)\langle x\rangle R_x(z)[C_x W]R_x(z)\\ &= a_0\int d\widetilde{V}(z)(\mathbf{1}-(z-i)R_x(z))[C_x W]R_x(z).\end{aligned}\tag{4.33}$$

Note that the integral in (4.33) is convergent, as follows from the fact that $V\in\mathcal{S}^\sigma$ for some $\sigma>0$ (see Hypothesis **(H)**) together with the properties of the almost analytic extension $\widetilde{V}$ described in Appendix B. Hence, combining (4.31) and (4.33), we conclude that $\langle x\rangle C_H W\in F_1^{(2)}$.

Here and below, to simplify the proof we take $d=1$. For $d\ge 1$, we use the Helffer-Sjöstrand representation (C.1) for several variables to write

$$V(x_1,\dots,x_d)=\int d\widetilde{V}(z_1,\dots,z_d)(z-x_1)^{-1}\dots(z-x_d)^{-1},$$

which yields through Leibniz rule that

$$\begin{aligned}\langle x\rangle C_V W = \int d\widetilde{V}(z)\langle x\rangle R_{x_1}(z)[C_{x_1}W]R_{x_1}(z)R_{x_2}(z)\cdots R_{x_d}(z)+\dots\\ +\int d\widetilde{V}(z)\langle x\rangle R_{x_1}(z)\cdots R_{x_d}(z)[C_{x_d}W]R_{x_d}(z).\end{aligned}$$

One can handle each of the d terms on the r.h.s. exactly as in (4.33) and then sum over the results.

Now, using (4.16), we have

$$C_{\phi^E}(W)=\phi C_{g^2}(W)+C_\phi(W)g^2+C_{[g,\phi]g}(W).\tag{4.34}$$

Since $g[\phi,g]\in\mathcal{F}^{(1)}$ (as shown in the proof of Proposition 4.1), the last term in (4.34) is contained in $\mathcal{F}_1^{(2)}(W)$. To deal with the first two terms, we use identities (4.17)–(4.18) and the fact that $\phi\langle x\rangle^{-1}=a_0$, $\Delta\langle x\rangle=a_1$ (because $\langle x\rangle\in\mathcal{S}^{-1}$) to compute

$$\begin{aligned}\phi C_{g^2}(W)=a_0\langle x\rangle C_{g^2}(W) &= a_0\int d\widetilde{g^2}(z)R_H(z)[\langle x\rangle C_H(W)]R_H(z)\\ &\quad + a_0\int d\widetilde{g^2}(z)R_H(z)(a_1+a_0p)R_H(z)a_1[\langle x\rangle C_H W]R_H(z),\end{aligned}\tag{4.35}$$

$$C_\phi(W)g^2=\int d\widetilde{\phi}(z)R_x(z)[C_x W]R_x(z)g^2.\tag{4.36}$$

Since $\langle x\rangle C_H W\in\mathcal{F}_1^{(2)}$ as we proved earlier, expression (4.35) shows $C_{g^2}(W)\in\mathcal{F}_1^{(2)}$. Since $g^2\in\mathcal{F}^{(1)}$ (again, shown in the proof of Proposition 4.1), expression (4.36) shows $C_\phi(W)g^2\in\mathcal{F}_1^{(2)}$. Combining these facts with decomposition (4.34), we conclude that $C_{\phi^E}W\in\mathcal{F}_1^{(2)}(W)$. This completes the proof for the base case.

2. Induction step for (4.29). Now, assuming (4.29) holds for $k=m$, we prove it for $k=m+1$. Since $C_{\phi^E}^{m+1}(W)=C_{\phi^E}(C_{\phi^E}^m(W))$, by inductive hypothesis, it suffices to show that $C_{\phi^E}(AB_m(W)A')\in\mathcal{F}_{m+1}^{(2)}(W)$, where $A,A'\in\mathcal{F}^{(1)}\cup\{\mathbf{1}\}$ and $B_m=(\langle x\rangle C_p)^r C_x^s$ with any pair $r,s\in\mathbb{N}$ such that $r+s\le m$.

Since C_{ϕ^E} is a derivation, we have

$$C_{\phi^E}(AB_m(W)A') = (C_{\phi^E}A)B_m(W)A' + AC_{\phi^E}(B_m(W))A' + AB_m(W)(C_{\phi^E}A'). \tag{4.37}$$

Since $A, A \in \mathcal{F}^{(1)} \cup \{\mathbf{1}\}$, by Proposition 4.1, we have $C_{\phi^E}(A), C_{\phi^E}(A') \in \mathcal{F}^{(1)}$ so that the first and the third term in (4.37) are contained in $\mathcal{F}^{(2)}_{m+1}(W)$. Thus, it suffices to compute the second term in (4.37).

By (4.16), we have

$$C_{\phi^E}(B_m(W)) = C_\phi(B_m(W))g^2 + \phi C_{g^2}(B_m(W)) + C_{g[\phi,g]}(B_m(W)). \tag{4.38}$$

Since $g[\phi, g] \in \mathcal{F}^{(1)}$, the last term in (4.38) is contained in $\mathcal{F}^{(2)}_{m+1}(W)$. Thus it remains to deal with the first two terms.

We first compute the first term in (4.38). Using (C.1), $B_m(W) = (\langle x\rangle C_p)^r C_x^s W$ and Jacobi identity

$$C_pC_x = C_xC_p, \qquad C_x(\langle x\rangle C_p) = \langle x\rangle C_pC_x, \tag{4.39}$$

we have

$$\begin{aligned} C_\phi(B_m(W))g^2 &= \int d\widetilde{\phi}(z)R_x(z)[C_x((\langle x\rangle C_p)^r C_x^s W)]R_x(z)g^2 \\ &= \int d\widetilde{\phi}(z)R_x(z)[(\langle x\rangle C_p)^r C_x^{s+1} W]R_x(z)g^2, \end{aligned} \tag{4.40}$$

which is contained in $\mathcal{F}^{(2)}_{m+1}(W)$, as in (4.36).

Next, we compute the second term in (4.38). Again, using (C.1), (4.16)–(4.18), $\phi\langle x\rangle^{-1} = a_0$, and $\Delta\langle x\rangle = a_1$, we have

$$\begin{aligned} \phi C_{g^2}(B_m(W)) &= a_0\langle x\rangle C_{g^2}(B_m(W)) \\ &= a_0 \int d\widetilde{g^2}(z)\langle x\rangle R_H(z)[C_H(B_m(W))]R_H(z) \\ &= a_0 \int d\widetilde{g^2}(z)R_H(z)[\langle x\rangle C_H(B_m(W))]R_H(z) \\ &\quad + a_0 \int d\widetilde{g^2}(z)R_H(z)[a_1 + a_0 p]R_H(z)a_1[\langle x\rangle C_H(B_m(W))]R_H(z). \end{aligned} \tag{4.41}$$

To finish the proof, it suffices to compute $\langle x\rangle C_H(B_m(W))$. By the definition of H and the facts that $C_p\langle x\rangle = i\nabla\langle x\rangle = a_0$ and $a_0\langle x\rangle^{-1} = a_1$, we have

$$\begin{aligned} \langle x\rangle C_H(B_m(W)) &= \langle x\rangle pC_p(B_m(W)) + (\langle x\rangle C_p(B_m(W)))p + \langle x\rangle C_V(B_m(W)) \\ &= p(\langle x\rangle C_p(B_m(W))) + (\langle x\rangle C_p(B_m(W)))p \\ &\quad + a_0C_p(B_m(W)) + \langle x\rangle C_V(B_m(W)) \\ &= p(\langle x\rangle C_p(B_m(W))) + (\langle x\rangle C_p(B_m(W)))p \\ &\quad + a_1\langle x\rangle C_p(B_m(W)) + \langle x\rangle C_V(B_m(W)). \end{aligned} \tag{4.42}$$

Since $B_m(W) = (\langle x\rangle C_p)^r C_x^s W$, the first three terms in (4.42) give

$$p\langle x\rangle C_p(B_m(W)) = p(\langle x\rangle C_p)(\langle x\rangle C_p)^r C_x^s W = p[(\langle x\rangle C_p)^{r+1} C_x^s W], \tag{4.43}$$

$$(\langle x\rangle C_p(B_m(W)))p = [(\langle x\rangle C_p)^{r+1} C_x^s W]p, \tag{4.44}$$

$$a_1\langle x\rangle C_p(B_m(W)) = a_1(\langle x\rangle C_p)^{r+1} C_x^s W, \tag{4.45}$$

which all belong in $\mathcal{F}^{(2)}_{m+1}(W)$. Finally, for the last term in (4.42), using (4.39), the identity (4.32), and the fact that $\langle x\rangle(x-i)^{-1}\in\mathcal{S}^0$, we have

$$\begin{aligned}\langle x\rangle C_V(B_m(W)) &= \int d\widetilde{V}(z)\langle x\rangle R_x(z)[C_x(B_m(W))]R_x(z)\\ &= a_0\int d\widetilde{V}(z)(\mathbf{1}-(z-i)R_x(z))[(\langle x\rangle C_p)^r C_x^{s+1}W]R_x(z),\end{aligned} \tag{4.46}$$

which again is contained in $\mathcal{F}^{(2)}_{m+1}(W)$. Substituting (4.43)–(4.46) back to (4.42), we conclude that $\langle x\rangle C_H(B_m(W))\in\mathcal{F}^{(2)}_{m+1}(W)$. This fact, together with expression (4.41), implies $\phi C_{g^2}(B_m(W))\in\mathcal{F}^{(2)}_{m+1}(W)$.

Combining the results from Steps 2.1–2 in decomposition (4.38), we find that $C_{\phi^E}(B_m(W))\in\mathcal{F}^{(2)}_{m+1}(W)$ for any $B_m(W)\in\mathcal{F}^{(2)}_{m+1}(W)$. This completes the induction step and so (4.29) follows.

Conclusion. Using (4.29) and **(W2)**, we now prove that $C^k_{\phi^E}(W_j)$ is square-summable in $j\ge 1$ for every $k=1,...,n+1$.

Since $\mathcal{F}^{(2)}_k$ consists of linear combinations of elements in $\bigcup_{m\le k}\mathcal{G}^{(2)}_m$, by (4.29), we have, for some $M\ge 0$, that

$$C^k_{\phi^E}(W_j)=\sum_{q=0}^{M}A_qB^{(2)}_{r_qs_q}(W_j)A'_q,\qquad A_q,\,A'_q\in F^{(1)},\quad 0\le r_q+s_q\le k. \tag{4.47}$$

Note that here M, A_q's and A'_q's are all independent of j, since they arise from the abstract algebraic identity (4.29). Using condition (1.5), the definition $B^{(2)}_{rs}\equiv(\langle x\rangle C_p)^rC_x^s$ (see (4.27)), and the fact that $\mathcal{F}^{(1)}\subset\mathcal{B}(\mathcal{H})$ (proved in (4.9)), we find that for all $m=1,\dots n+1$ and $A,\,A'\in\mathcal{F}^{(1)}$,

$$\sum_j\sum_{\substack{r,s\ge0\\r+s=m}}\left\|AB^{(2)}_{rs}(W_j)A'\right\|^2\le\|A\|^2\,\|A'\|^2\sum_j\sum_{\substack{r,s\ge0\\r+s=m}}\left\|B^{(2)}_{rs}(W_j)\right\|^2<\infty. \tag{4.48}$$

This, together with expression (4.47), implies the claim.

This completes the proof of Proposition 4.2. □

Corollary 4.3. *Suppose that $H=-\Delta+V(x)$ and W_j satisfy **(H)**–**(W2)**. Then, with Φ given by* (3.3)*, condition* (2.2) *holds.*

Proof. Since $d_X(x)$ satisfies condition (4.5) (see (1.11) and (1.12)), it suffices to apply Propositions 4.1–4.2. □

5. Proof of claims (3.5)–(3.6)

5.1. **Proof of Claim** (3.5)**.** Recall that $\chi^\sharp_X$, $X\subset\mathbb{R}^d$, denotes the characteristic functions of X. Recall also that the set of smooth cutoff functions $\mathcal{X}$ is defined in (2.5) and that $d^E_X=gd_Xg$ with $g=g^E(H)$ (see (1.13)–(1.15)) and d_X the smooth distance function defined in (1.11). We reproduce Claim (3.5) below:

Proposition 5.1. *For every $\chi\in\mathcal{X}$ and $\chi_{0s}=\chi(s^{-1}d^E_X)$ (see* (3.4)*),*

$$\chi^\sharp_X\chi_{0s}\chi^\sharp_X=O(s^{-n}). \tag{5.1}$$

Remark 9. This is a semiclassical estimate which physically says that a quantum particle that is essentially localized in phase space inside an energy ball and outside of X (by way of d_X^E) is also localized outside of X in position space up to small errors. A technical challenge here is that the operator d_X is unbounded.

Proof of Proposition 5.1. In the remainder of this proof, we use the following notations: For $z \in \mathbb{C}$ with $\mathrm{Im}(z) \neq 0$, d as in (1.11), and g as in (1.14),

$$\begin{aligned} d \equiv & d_X, \qquad d^E \equiv d_X^E = g d_X g, \qquad R = (d/s - z)^{-1}, \qquad R^E = (d^E/s - z)^{-1}, \\ b = & d - d^E, \qquad \chi^E = \chi(d^E/s), \qquad \chi = \chi(d/s). \end{aligned}$$

We begin with

Lemma 5.2. *The operator Rb is bounded.*

Proof. Since $b = d - d^E$ and Rd is bounded as the multiplication operator by a bounded function, it suffices to show that Rd^E is bounded. For the latter, we have, by (1.15),

$$Rd^E = Rgdg = Rdg^2 + R[g, d]g. \tag{5.2}$$

Since g is bounded and $Rd = s(1 + zR)$ so that $\|Rd\| \leq s(1 + |z|\,|\mathrm{Im}(z)|^{-1})$, it remains to show that $[g, d]$ is bounded. Using the HS representation (C.1) with $k = 0$ and formula (4.13), we have

$$\begin{aligned} [g, d] &= \int d\widetilde{g}(z) \left[(z - H)^{-1}, d\right] \\ &= - \int d\widetilde{g}(z)(z - H)^{-1} \mathrm{ad}_d^1(H)(z - H)^{-1} \\ &= \int d\widetilde{g}(z)(z - H)^{-1} \left(\nabla \cdot (\nabla d) + \nabla d \cdot \nabla\right)(z - H)^{-1}. \end{aligned} \tag{5.3}$$

Next we multiply by i and use the operator Cauchy-Schwarz inequality

$$\begin{aligned} & i\nabla \cdot (\nabla d) + \nabla d \cdot i\nabla \\ \leq & - \langle E \rangle^{-1/2} \Delta + \langle E \rangle^{1/2} |\nabla d|^2 \\ \leq & \frac{H}{\langle E \rangle^{1/2}} + \|V\|_\infty + 1 + \langle E \rangle^{1/2} |\nabla d|^2 =: B_{H,E}. \end{aligned}$$

By (1.12), we have $|\nabla d| \leq C$. This, together with condition (1.4) on V and the HS representation (C.1) with $k = 1$, shows that

$$\left\|(z - H)^{-1} \left(\nabla \cdot (\nabla d) + \nabla d \cdot \nabla\right)(z - H)^{-1}\right\| \tag{5.4}$$

$$\leq \left\|B_{H,E}^{\frac{1}{2}}(\bar{z} - H)^{-1}\right\| \left\|B_{H,E}^{\frac{1}{2}}(z - H)^{-1}\right\| \tag{5.5}$$

$$\leq C\big(\langle E \rangle^{-\frac{1}{2}} |z| + \langle E \rangle^{\frac{1}{2}} |\mathrm{Im}(z)|^{-2}\big). \tag{5.6}$$

Using the properties of the almost analytic extension $\tilde{g}$ (in particular the fact that it is compactly supported, see (B.5) and Remark 5), this shows that the integral in (5.3) is norm convergent, which completes the proof. □

Now, using the Helffer-Sjöstrand representation (C.1) and omitting the measure, we write

$$\chi^E = \int R^E. \tag{5.7}$$

Using that the operator Rb is bounded and expanding $R^E = (d^E/s - z)^{-1} = (d/s - z - b/s)^{-1}$ in powers of Rb/s up to the order $n-1$, we obtain

$$R^E = (d/s - z - b/s)^{-1} = \sum_{k=0}^{n-1} s^{-k}(Rb)^k R + s^{-n}(Rb)^n R^E. \tag{5.8}$$

Plugging this expansion into (5.7) yields

$$\chi^E = \sum_{k=0}^{n-1} \chi_k + s^{-n}\mathrm{Rem}_1, \tag{5.9}$$

where

$$\chi_k = \int (Rb/s)^k R \quad \text{and} \quad \mathrm{Rem}_1 = \int (Rb)^n R^E. \tag{5.10}$$

Our goal is to move the R's in the first integrand to the right. Using the relations $Rb = bR + [R, b]$ and $[R, b] = -s^{-1}R\mathrm{ad}_d(b)R$, we would like to obtain an expansion of the form

$$(Rb)^k R = \sum_l s^{-i_l}\tilde{B}_l R^{l+1} + s^{-n}\tilde{M}_k, \tag{5.11}$$

where the operators $\tilde{B}_l$ and $\tilde{M}_k$ are polynomials of operators $\mathrm{ad}_d^k(b), k = 0, 1, \dots,$ (and R for $\tilde{M}_k$), and then use $\int R^{l+1} = (-1)^{l+1}\chi^{(l)}$ (see (C.1)) and $\chi^{(l)}\chi_X^{\#} = 0$ for all $l \geq 0$. The problem here is that the operators $\mathrm{ad}_d^k(b)$ are not bounded, so $\tilde{B}_l$ and $\tilde{M}_k$ are not guaranteed to be bounded operators. Hence, we proceed differently.

We transform the product $(Rb/s)^k$ as follows. We use the relation

$$b = gd\bar{g} + \bar{g}d = dh - \mathrm{ad}_d(\bar{g})g, \tag{5.12}$$

$$\text{where } \bar{g} = 1 - g \text{ and } h := \bar{g}(1 + g), \tag{5.13}$$

and the definition $R = (d/s - z)^{-1}$ to write

$$Rb/s = d_s Rh + Rc_s, \text{ where} \tag{5.14}$$

$$d_s := d/s, \quad c := \mathrm{ad}_d(g)g, \quad c_s = c/s. \tag{5.15}$$

Notice that the operators c_s, h and $d_s R$ are bounded and

$$d_s R = \mathbf{1} + zR. \tag{5.16}$$

The last two relations imply

$$Rb/s = h + Rc_s + zRh. \tag{5.17}$$

Our goal is to move the R's to the extreme right to obtain the following:

Lemma 5.3. *The operator $(Rb/s)^k$ has the following expansion:*

$$(Rb/s)^k = h^k + \sum_{q=0}^{k}\sum_{l=0}^{n-1} s^{-l}B_{q,l}R^{l+1}p_{q,l}(z) + s^{-n}\sum_{q=0}^{k} M_{q,n}p_{q,n}(z), \tag{5.18}$$

where

(a) $k = 1, ..., n-1,$

(b) *the operators* $B_{q,l}$ *are polynomials of bounded operators* $\mathrm{ad}_d^m(h)$ *and* $\mathrm{ad}_d^m(c_s)$, *with* $0 \leq m \leq l$,

(c) *the operators* $M_{q,n}$ *are polynomials of bounded operators* R, $\mathrm{ad}_d^m(h)$ *and* $\mathrm{ad}_d^m(c_s)$, *with* $0 \leq m \leq n$ *and*

$$\deg_R(M_{q,n}) := \text{powers of } R \text{ in } M_{q,n} \in [n+1, n+k]. \tag{5.19}$$

(d) $p_{q,l}(z)$ *are polynomials in* z *of the degree* $\leq q$.

We call the operators described in (b) as *l-operators*. Note that if B_l is an l-operator, then it is also an $(l+m)$-operator for $m \geq 1$.

Remark 10. The negative powers of s come from the commutator identity

$$[R, B] = -s^{-1} R \,\mathrm{ad}_d(B) R, \tag{5.20}$$

valid for any bounded operator B and $\mathrm{Im}(z) \neq 0$.

Proof of Lemma 5.3. We prove (5.18) by induction on k.

For the base case $k = 1$, we use the commutator expansion

$$RB = \sum_{r=0}^{p-1} (-1)^r s^{-r} \,\mathrm{ad}_d^r(B) R^{r+1} + (-1)^p s^{-p} R \,\mathrm{ad}_d^p(B) R^p, \tag{5.21}$$

valid for any bounded operators B and integer $p \geq 1$. Applying (5.21) to $B = h$ and c_s (see (5.14)), we find

$$\begin{aligned} Rb/s =& h + Rc_s + zRh \\ =& h + \sum_{r=0}^{n-1} (-1)^r s^{-r} \,\mathrm{ad}_d^r(c_s) R^{r+1} + (-1)^n s^{-n} R \,\mathrm{ad}_d^n(c_s) R^n \\ & + z \left(\sum_{r=0}^{n-1} (-1)^r s^{-r} \,\mathrm{ad}_d^r(h) R^{r+1} + (-1)^n s^{-n} R \,\mathrm{ad}_d^n(h) R^n \right). \end{aligned} \tag{5.22}$$

This is of the form (5.18) with

$$B_{0,r} := (-1)^r \,\mathrm{ad}_d^r(c_s), \quad M_{0,n} := (-1)^n R \,\mathrm{ad}_d^n(c_s) R^n, \tag{5.23}$$

$$B_{1,r} := (-1)^r \,\mathrm{ad}_d^r(h), \quad M_{1,n} := (-1)^n R \,\mathrm{ad}_d^n(h) R^n, \tag{5.24}$$

where

$$\deg_R(M_{0,n}) = \deg_R(M_{1,n}) = n + 1 \tag{5.25}$$

satisfies (5.19).

Now we assume (5.18) for a given $k \geq 1$ and prove it for $k \to k+1$. We use (5.17) to write

$$\begin{aligned} (Rb/s)^{k+1} &= (zRh + Rc_s + h)^{k+1} \\ &= zRh(Rb/s)^k + Rc_s(Rb/s)^k + h(Rb/s)^k \\ &=: A + B + C. \end{aligned} \tag{5.26}$$

Using the induction hypothesis, we see that the third term on the r.h.s. of (5.26) is already in the desired form (notice that the term h^{k+1} in (5.18) comes from this contribution). The first two terms on the r.h.s. of (5.26) are treated similarly, so we only consider the first term.

We transform the term A in line (5.26) as

$$A = A_1 + A_2 + A_3, \tag{5.27}$$

where

$$A_1 := zRh^{k+1}, \tag{5.28}$$

$$A_2 := zRh \sum_{q=0}^{k} \sum_{l=0}^{n-1} s^{-l} B_{q,l} R^{l+1} p_{q,l}(z), \tag{5.29}$$

$$A_3 := s^{-n} zRh \sum_{q=0}^{k} M_{q,n} p_{q,n}(z). \tag{5.30}$$

The term A_1 can be handled using expansion (5.21) as

$$A_1 = z \left(\sum_{l=0}^{n-1} (-1)^l s^{-l} \operatorname{ad}_d^l(h^{k+1}) R^{l+1} + (-1)^n s^{-n} R \operatorname{ad}_d^n(h^{k+1}) R^n \right). \tag{5.31}$$

By Leibniz's rule, for each l, $\operatorname{ad}_d^l(h^{k+1})$ is an l-operator as defined in part (b) of Lemma 5.3, and so A_1 is of the form (5.18) with

$$B_{1,l}^{(1)} := (-1)^l s^{-l} \operatorname{ad}_d^l(h^{k+1}), \quad p_{q,l}^{(1)} := \delta_{1q} z, \tag{5.32}$$

$$M_{1,n}^{(1)} := (-1)^n s^{-n} R \operatorname{ad}_d^n(h^{k+1}) R^n \text{ satisfying } \deg_R(M_{1,n}^{(1)}) = n+1. \tag{5.33}$$

The term A_3 can be written as

$$A_3 = \sum_{q=0}^{k} (RhM_{q,n})(zp_{q,n}(z)) = \sum_{q=1}^{k+1} M_{q,n}^{(2)} p_{q,n}^{(2)}, \tag{5.34}$$

where

$$M_{q,n}^{(2)} := RhM_{q-1,n}, \quad p_{q,n}^{(2)} := zp_{q-1,n}(z), \tag{5.35}$$

with notations as in parts (c)-(d) of Lemma 5.3. Since $\deg_R M_{q,n} \leq n+k$, we have

$$\deg_R M_{q,n}^{(2)} \in [n+2, n+k+1], \tag{5.36}$$

which satisfies the bound (5.19) with $k \to k+1$. Thus A_3 is of the form (5.18).

To bring the term A_2 into the desired form, we commute R's in (5.28) to the right using expansion (5.21). For each $q = 0, \dots, k$, we consider the sum

$$A_2(q) := \sum_{l=0}^{n-1} s^{-l} zRhB_{q,l} R^{l+1} p_{q,l}(z), \tag{5.37}$$

so that

$$A_2 = \sum_{q=0}^{k} A_2(q). \tag{5.38}$$

Let $B'_{q,l} = hB_{q,l}$. Using (5.21), we have, for each $l = 0, \dots, n-1$,

(5.39)

$$\begin{aligned} RhB_{q,l}R^l &= RB'_{q,l}R^l \\ &= \sum_{r=0}^{n-l-1} (-1)^r s^{-r} \operatorname{ad}_d^r(B'_{q,l}) R^{l+r+1} + (-1)^{n-l} s^{-(n-l)} R \operatorname{ad}_d^{n-l}(B'_{q,l}) R^n. \end{aligned}$$

Using Leibniz rule for commutators and the structure of $B_{q,l}$, we conclude that the operators $\operatorname{ad}_d^r(B'_{q,l})$ are polynomials of $\operatorname{ad}_d^m(h)$ and $\operatorname{ad}_d^m(c_s)$, $m = 0, 1, ..., l + r$, and therefore are $(l+r)$-operators as defined above. So, setting $B''_{q,l+r} = (-1)^r \operatorname{ad}_d^r(B'_{q,l})$, expansion (5.39) becomes

$$RhB_{q,l}R^l = \sum_{r=0}^{n-l-1} s^{-r} B''_{q,l+r} R^{l+r+1} + s^{-(n-l)} RB''_{q,n} R^n. \tag{5.40}$$

Substituting (5.40) into (5.37) and setting $p'_{q+1,l}(z) := zp_{q,l}(z)$ for $l = 0, \dots, n-1$, we obtain

$$\begin{aligned} A_2(q) = &\sum_{l=0}^{n-1} \sum_{r=0}^{n-l-1} s^{-(l+r)} B''_{q,l+r} R^{l+r+1} p'_{q+1,l}(z) \\ &+ s^{-n} \sum_{l=0}^{n-1} RB''_{q,n} R^n p'_{q+1,l}(z). \end{aligned} \tag{5.41}$$

Changing the summation index $(l + r, l) \to (l', r')$, the r.h.s. in line (5.41) can be written as

$$\sum_{l=0}^{n-1} \sum_{r=0}^{n-l-1} s^{-(l+r)} B''_{q,l+r} R^{l+r+1} p'_{q+1,l}(z) = \sum_{l'=0}^{n-1} \sum_{r'=0}^{l'} s^{-l'} B''_{q,l'} R^{l'+1} p'_{q+1,r'}(z). \tag{5.42}$$

Setting $p''_{q+1,n} := \sum_{l=0}^{n-1} p'_{q+1,l}(z)$ in (5.41) and $p''_{q+1,l'} := \sum_{r'=0}^{l'} p'_{q+1,r'}$ for each $l' = 0, \dots, n-1$ in (5.42), we conclude that

$$\begin{aligned} A_2(q) = &\sum_{l=0}^{n-1} s^{-l} B''_{q,l} R^{l+1} p''_{q+1,l}(z) \\ &+ s^{-n} RB''_{q,n} R^n p''_{q+1,n}(z). \end{aligned} \tag{5.43}$$

Plugging (5.43) into (5.38) yields

$$\begin{aligned} A_2 = &\sum_{q=0}^{k} \sum_{l=0}^{n-1} s^{-l} B''_{q,l} R^{l+1} p''_{q+1,l}(z) \\ &+ s^{-n} \sum_{q=0}^{k} RB''_{q,n} R^n p''_{q+1,n}(z) \end{aligned} \tag{5.44}$$

Shifting the dummy index $q \to q+1$ and setting

$$B^{(3)}_{q,l} := B''_{q-1,l+1}, \quad p^{(3)}_{q,n}(z) := p''_{q+1,n}(z), \tag{5.45}$$

$$M^{(3)}_{q,n} := RB''_{q-1,n} R^n \text{ with } \deg_R(M^{(3)}_{q,n}) = n+1, \tag{5.46}$$

we conclude that A_2 is of the form (5.18).

This completes the proof of Lemma 5.3. $\square$

Corollary 5.4. *For any $\chi \in C^\infty(\mathbb{R})$ with compactly supported derivative and $\chi_k = \int (Rb/s)^k R\, d\tilde{\chi}(z)$,*

$$\chi_k = h^k \chi(d_s) + \sum_{q=0}^{k} \sum_{l=0}^{n-1} s^{-l} B_{q,l} (\chi p_{q,l})^{(l+1)}(d_s) + s^{-n} \mathrm{Rem}_{2,k}, \tag{5.47}$$

where $B_{q,l}$ are as in Lemma 5.3 and $\mathrm{Rem}_{2,k} = O(1)$.

Proof. We have by the Heffler-Sjörstrand representation (C.1) that $\int R^{l+1} p_l(z) = (-1)^{l+1} (\chi p_l)^{(l)}(d_s)$ (see (C.1)).

This, together with the definition $\chi_k = \int (Rb/s)^k R$ and expansion (5.18), implies

$$\begin{aligned} \chi_k =& h^k \chi(d_s) + \sum_{q=0}^{k} \sum_{l=0}^{n-1} s^{-l} B_{q,l} (\chi p_{q,l})^{(l+1)}(d_s) \\ &+ s^{-n} \sum_{q=0}^{k} \int M_{q,n} R p_{q,n}(z)\, d\tilde{\chi}(z). \end{aligned} \tag{5.48}$$

Thus it remains to show the integral on line (5.48) is $O(1)$.

Using the estimate $\|R\| \le |\mathrm{Im}(z)|^{-1}$ and the degree bound (5.19) and that $k \le n-1$, we have

$$\|M_{q,n}\| \le C \sum_{j=n}^{2n} |\mathrm{Im}(z)|^{-(j+1)} \text{ for all } q. \tag{5.49}$$

Since $p_{q,n}$ has degree at most n and $\tilde{\chi}$ has compactly supported derivatives, we find by expression (5.49) and Corollary B.5 with $(p,l) = (n+1, n), \dots, (2n+1, n)$ that

$$\left\| \int M_{q,n} R p_{q,n}\, d\widetilde{\chi}(z) \right\| \le C \int \sum_{j=n}^{2n} |\mathrm{Im}(z)|^{-(j+2)}\ |p_{q,n}(z)|\ d\widetilde{\chi}(z) \le C. \tag{5.50}$$

Summing (5.50) over q shows that the integral on line (5.48) is $O(1)$. This completes the proof of Corollary 5.4. $\square$

Since $\chi^{(l)}(d_s)\chi_X^{\#} = 0$ for all $l \ge 0$, expansion (5.47) gives

$$\chi_k \chi_X^{\#} = s^{-n} \mathrm{Rem}_{2,k} \chi_X^{\sharp} = O(s^{-n}). \tag{5.51}$$

Next, we deal with the Rem_1 term in (5.9). We use the splitting

$$Rb = Rc + R_2 h, \quad c := \mathrm{ad}_d(g) g, \quad R_2 := dR, \tag{5.52}$$

which follows from (5.14). We prove:

Lemma 5.5. *For $k \ge 1$, the operator $(Rb)^k$ has the following expansion:*

$$(Rb)^k = \sum_{l=0}^{k} R_2^l N_{k-l}, \tag{5.53}$$

where the operators N_j are polynomials of bounded operators R, $\mathrm{ad}_d^m(h)$ and $\mathrm{ad}_d^m(c)$ with $0 \le m \le k-1$ and

$$\deg_R(N_j) := \text{powers of } R \text{ in } N_j \in [j, j+2k]. \tag{5.54}$$

Proof. We prove this by induction on $k = 1, 2, \ldots$. For the base case $k = 1$, we use expansion (5.52), which is of the form (5.53) with $N_1 = Rc$ and $N_0 = h$, satisfying degree bound (5.54).

Suppose now (5.53) holds with some $k \geq 1$, and we prove it for $k \to k+1$. Using (5.52) and the induction hypothesis, we write

$$\begin{aligned}(Rb)^{k+1} =&(Rc + R_2h)(Rb)^k\\ =&\sum_{l=0}^{k} RcR_2^l N_{k-l} + \sum_{l=0}^{k} R_2hR_2^l N_{k-l}\\ =:&A + B. \end{aligned} \tag{5.55}$$

The goal now is to commute the bounded operator R_2 successively to the left. Using the relation

$$R_2 = s(1 + zR) \tag{5.56}$$

and identity (5.20), we find

$$\mathrm{ad}_{R_2}(D) = (s^{-1}R_2 - 1)\,\mathrm{ad}_d(D)R, \tag{5.57}$$

for any operator D allowed by the domain consideration. Iterating identity (5.57) for $p \geq 1$ times shows that there exist absolute constants $c_0, \ldots, c_p$ s.t.

$$\mathrm{ad}_{R_2}^p(D) = \sum_{q=0}^{p} c_q s^{-q} R_2^q \mathrm{ad}_d^p(D)R^p. \tag{5.58}$$

Moreover, for any bounded operators D, E and integers $l \geq 1$, we have

$$DE^l = E^l D + \sum_{p=1}^{l} (-1)^p \binom{l}{p} E^{l-p} \mathrm{ad}_E^p(D). \tag{5.59}$$

Applying (5.58)–(5.59) to term A in (5.55) with $D = c$ and $E = R_2$, and using that $[R_2, R] = 0$, we find

$$\begin{aligned} A \equiv&RcN_k + \sum_{l=1}^{k} RcR_2^l N_{k-l}\\ =&RcN_k + \sum_{l=1}^{k} R_2^l RcN_{k-l}\\ &+ \sum_{l=1}^{k}\sum_{p=1}^{l} (-1)^p \binom{l}{p} R_2^{l-p} R\ \mathrm{ad}_{R_2}^p(c)N_{k-l}\\ =&RcN_k + \sum_{l=1}^{k} R_2^l RcN_{k-l}\\ &+ \sum_{l=1}^{k}\sum_{p=1}^{l}\sum_{q=0}^{p} (-1)^p c_q s^{-q} \binom{l}{p} R_2^{l-p+q} R\ \mathrm{ad}_d^p(c)R^p N_{k-l}. \end{aligned} \tag{5.60}$$

Regrouping (5.60) according to the power in R_2 shows that

$$A = \sum_{l=0}^{k} R_2^l N_{k+1-l}^{(1)}, \tag{5.61}$$

$$N_{k+1}^{(1)} := RcN_k, \tag{5.62}$$

$$\begin{aligned} N_{k+1-l}^{(1)} =& RcN_{k-l} \\ &+ \sum_{l'=l}^{k} \sum_{\substack{p=1,\dots,l' \\ q=0,\dots,p, \\ q-p=l-l'}} (-1)^p c_q s^{-q} \binom{l'}{p} R \ \mathrm{ad}_d^p(c) R^p N_{k-l'}, \quad l = 1, \dots, k. \end{aligned} \tag{5.63}$$

Since $\deg_R N_j \in [j, j+2k]$, we derive from expressions (5.62)–(5.63) that

$$\deg_R(N_j^{(1)}) \in [j+1, j+2k+1], \quad j = 0, \dots k. \tag{5.64}$$

Similarly, applying (5.58)–(5.59) to term B in (5.55) with $D = h$ and $E = R_2$ yields

$$\begin{aligned} B \equiv& R_2 h N_k + \sum_{l=1}^{k} R_2 h R_2^l N_{k-l} \\ =& R_2 h N_k + \sum_{l=1}^{k} R_2^{l+1} h N_{k-l} \\ &+ \sum_{l=1}^{k} \sum_{p=1}^{l} (-1)^p \binom{l}{p} R_2^{l-p+1} \ \mathrm{ad}_{R_2}^p(h) N_{k-l} \\ =& R_2 h N_k + \sum_{l=1}^{k} R_2^{l+1} h N_{k-l} \\ &+ \sum_{l=1}^{k} \sum_{p=1}^{l} \sum_{q=0}^{p} (-1)^p c_q s^{-q} \binom{l}{p} R_2^{l-p+q+1} \ \mathrm{ad}_d^p(h) R^p N_{k-l}. \end{aligned} \tag{5.65}$$

Regrouping (5.65) according to the power in R_2 shows that

$$B = \sum_{l=1}^{k+1} R_2^l N_{k+1-l}^{(2)}, \tag{5.66}$$

$$N_k^{(2)} := hN_k, \tag{5.67}$$

$$\begin{aligned} N_{k+1-l}^{(2)} =& hN_{k+1-l} \\ &+ \sum_{l'=l}^{k+1} \sum_{\substack{p=1,\dots,l' \\ q=0,\dots,p, \\ q-p=l-l'}} (-1)^p c_q s^{-q} \binom{l'-1}{p} \ \mathrm{ad}_d^p(h) R^p N_{k+1-l'}, \end{aligned} \tag{5.68}$$

with $l = 2, \dots, k+1$ and

$$\deg_R(N_j^{(2)}) \in [j, j+2k+1], \quad j = 1, \dots k+1. \tag{5.69}$$

Combining expansions (5.61), (5.66) in line (5.55) yields

$$(Rb)^{k+1} = N_{k+1}^{(1)} + \sum_{l=1}^{k} R_2^l (N_{k+1-l}^{(1)} + N_{k+1-l}^{(2)}) + R_2^{k+1} N_0^{(2)}, \tag{5.70}$$

which is of the form (5.53) with $k \to k+1$. This completes the induction and the proof of Lemma 5.5. □

Next, we have the following lemma

Lemma 5.6. *Let* Rem_1 *be as in* (5.10). *If* K *is any bounded operator with* $\operatorname{ran} d \subset \ker K$ *then*

$$K\mathrm{Rem}_1 = O(\|K\|). \tag{5.71}$$

Proof. We use expansion (5.53). Since $\operatorname{ran} d \subset \ker K$, we have $KR_2 = (Kd)R = 0$ by definition (5.52). Thus only the leading term in (5.53) survives left multiplication by K, yielding

$$K\mathrm{Rem}_1 = \int K(Rb)^n R^E = \int KN_n R^E. \tag{5.72}$$

By the definition of N_n (see Lemma 5.5), we have

$$\|N_n\| \le C \sum_{j=n}^{3n} |\mathrm{Im}(z)|^{-j}. \tag{5.73}$$

Thus, by (5.72),

$$\|K\mathrm{Rem}_1\| \le \|K\| \sum_{j=n}^{3n} \int |\mathrm{Im}(z)|^{-(j+1)}. \tag{5.74}$$

This, together with estimate (B.17) with $(p,l) = (n,0), \ldots, (3n,0)$ (recall $n \ge 1$ to begin with), implies the desired result, (5.71). □

Applying (5.71) with $K = \chi_X^\sharp$, whose kernel contains $\operatorname{ran} d$ due to (1.11), we obtain

$$\chi_X^\sharp \mathrm{Rem}_1 = O(1). \tag{5.75}$$

Finally, plugging (5.51) and (5.75) back to expansion (5.9) yields the desired estimate (5.1). This completes the proof of (5.1). □

Remark 11. We mention the following alternative proof of Proposition 5.1. Recalling that $\chi_{0s} = \chi(s^{-1} d_X^E)$ with χ supported on $[c_\delta, \infty)$ for some positive c_δ, we write

$$\big\|\chi_{0s}\chi_X^\sharp\big\| = \big\|\chi_{0s}\big(d_X^E\big)^{-n}\big(d_X^E\big)^n\chi_X^\sharp\big\| \le (c_\delta s)^{-n}\big\|\big(d_X^E\big)^n\chi_X^\sharp\big\|.$$

Now, with the convention $\prod_{i=2}^n A_i = A_2 \ldots A_n$, we have

$$\begin{aligned}\big(d_X^E\big)^n &= g(H)d_X\Big(\prod_{i=2}^n g^2(H)d_X\Big)g(H)\\ &= g(H)d_X\langle x\rangle^{-1}\langle x\rangle\Big(\prod_{i=2}^n g^2(H)\langle x\rangle^{-i+1}d_X\langle x\rangle^{-1}\langle x\rangle^i\Big)g(H)\langle x\rangle^{-n}\langle x\rangle^n.\end{aligned}$$

A standard induction argument shows that $\langle x\rangle^i g^2(H)\langle x\rangle^{-i}$ is a bounded operator for any positive integer i (since H is the Schrödinger operator $H=-\Delta+V$), and likewise with g instead of g^2. Since in addition $d_X\langle x\rangle^{-1}$ is bounded, we deduce that

$$\left\|\left(d_X^E\right)^n\chi_X^\sharp\right\|\le C_n\left\|\langle x\rangle^n\chi_X^\sharp\right\|\le C_n',$$

since X is bounded. This establishes Proposition 5.1. (Note that if X is unbounded, the same holds, replacing $\langle x\rangle$ by $\langle d_X\rangle$ in the argument above.)

The proof we gave in Section 5.1 has the advantage of being more robust. Moreover the arguments we used are also crucial in our proof of (3.6) given in the next section.

5.2. **Proof of claim** (3.6)**.** Recall χ, $\tilde g$, and $\tilde\chi$ are smooth cutoff functions such that $\mathrm{supp}(\tilde g)\subset\{g=1\}$ and $\mathrm{supp}(\tilde\chi')\subset(\delta,+\infty)=\{\chi=1\}$ (see Figs. 3–4). Let $\bar g=1-g$ and $\bar\chi=1-\chi$. It follows that

$$\bar g(\mu)\tilde g(\mu)=0, \tag{5.76}$$

$$\bar\chi(\mu)\tilde\chi(\mu)=0. \tag{5.77}$$

In the remainder of this section, we use the following notations: For s, v, t as in (3.4) and $z\in\mathbb{C}$, $\mathrm{Im}(z)\neq0$,

$$\begin{aligned} d_t\equiv&d_X-vt, \qquad d_t^E\equiv d_X^E-vt=gd_Xg-vt,\\ R\equiv&(d_t/s-z)^{-1}, \qquad R^E\equiv(d_t^E/s-z)^{-1}, \end{aligned}$$

and

$$\xi(\mu):=\sqrt{\chi(\mu)},\quad \bar\xi(\mu)=1-\xi(\mu), \tag{5.78}$$

$$\phi=\phi(d_t/s),\quad \phi^E=\phi(d_t^E/s)\text{ for }\phi\in\mathcal{X}, \tag{5.79}$$

$$g=g(H),\quad \tilde g=\tilde g(H)\text{ for }g,\,\tilde g\text{ from (5.76).} \tag{5.80}$$

Using these notations, we reproduce Claim 3.6 as follows:

Proposition 5.7. *For every $\chi\in\mathcal{X}$ and $\tilde g$, $\tilde\chi$ as in* (5.76)–(5.77)*,*

$$\chi^E\ge\tilde g\tilde\chi\tilde g+O(s^{-n}). \tag{5.81}$$

Proof. Since $\|\tilde g\tilde\chi\tilde g\|\le1$, we have

$$\chi^E\ge\xi^E\tilde g\tilde\chi\tilde g\xi^E=\tilde g\tilde\chi\tilde g-\bar\xi^E\tilde g\tilde\chi\tilde g-\tilde g\tilde\chi\tilde g\bar\xi^E+\bar\xi^E\tilde g\tilde\chi\tilde g\bar\xi^E. \tag{5.82}$$

We now claim

$$\bar\xi^E\tilde g\tilde\chi=O(s^{-n}). \tag{5.83}$$

If (5.83) holds, then the last three terms on the r.h.s. of (5.82) are $O(s^{-n})$ and we are done.

Since the operator $b\equiv d-d^E=d_t-d_t^E$ as in the proof of Proposition 5.1, proceeding as in (5.9)–(5.10), we find the expansion

$$\bar\xi^E=\sum_{k=0}^{n-1}\bar\xi_k+s^{-n}\mathrm{Rem}_1, \tag{5.84}$$

where

$$\bar\xi_k=\int(Rb/s)^kR\,d\tilde{\bar\xi}(z)\quad\text{and}\quad\mathrm{Rem}_1=\int(Rb)^nR^E\,d\tilde{\bar\xi}(z), \tag{5.85}$$

where $\tilde{\bar{\xi}}(z)$ is an almost analytic extension of the function $\bar{\xi}(\mu)$. (Below we will omit the measure $d\tilde{\bar{\xi}}(z)$ when no confusion arises.) By expansion (5.84), Claim (5.83) is equivalent to the relations

$$\bar{\xi}_k\tilde{g}\tilde{\chi} = O(s^{-n}), \tag{5.86}$$

$$\mathrm{Rem}_1\tilde{g}\tilde{\chi} = O(1). \tag{5.87}$$

We first prove (5.86). We write the l.h.s. of (5.86) as

$$\bar{\xi}_k\tilde{g}\tilde{\chi} = \bar{\xi}_k\tilde{\chi}\tilde{g} + \bar{\xi}_k[\tilde{g},\tilde{\chi}]. \tag{5.88}$$

Since $\mathrm{ad}^k_{d_t/s}(\tilde{g}) = s^{-k}\mathrm{ad}^k_d(\tilde{g})$ is bounded for $0 \le k \le n$, we have by expansion (C.5) that

$$[\tilde{g},\tilde{\chi}] = \sum_{k=1}^{n-1}(-1)^k\frac{s^{-k}}{k!}\tilde{\chi}^{(k)}(d_t/s)\mathrm{ad}^k_d(\tilde{g}) + s^{-n}\mathrm{Rem}_3, \tag{5.89}$$

where $\mathrm{Rem}_3 = O(1)$. Plugging (5.89) into (5.88) yields

$$\begin{aligned}\bar{\xi}_k\tilde{g}\tilde{\chi} =&\bar{\xi}_k\tilde{\chi}\tilde{g} + \sum_{k=1}^{n-1}\frac{s^{-k}}{k!}\bar{\xi}_k\tilde{\chi}^{(k)}(d_t/s)\mathrm{ad}^k_d(\tilde{g}) + s^{-n}\bar{\xi}_k\mathrm{Rem}_3\\ =:&A+B+C.\end{aligned} \tag{5.90}$$

We apply Corollary 5.4 to the function $\bar{\xi}$ to obtain the expansion

$$\bar{\xi}_k =h^k\bar{\xi}(d_t/s) + \sum_{q=0}^{k}\sum_{l=0}^{n-1}s^{-l}B_{q,l}(\bar{\xi}p_{q,l})^{(l+1)}(d_t/s) + s^{-n}\mathrm{Rem}_{2,k}, \tag{5.91}$$

where $\|h\| \le 2$, $B_{q,l} = O(1)$ are defined in Lemma 5.3, part (b), and $\mathrm{Rem}_{2,k} = O(1)$. Thus $\bar{\xi}_k = O(1)$ and so the term C in line (5.90) is $O(s^{-n})$. By Definition (5.78), we have

$$\bar{\xi}^{(l)}(\mu)\tilde{\chi}^{(m)}(\mu) = 0 \quad \text{for any integers } l,m \ge 0, \tag{5.92}$$

see Figure 3 . Thus, inserting (5.91) to (5.90) and using (5.92), we find

$$A =s^{-n}\sum_{k=0}^{n-1}\mathrm{Rem}_{2,k}\tilde{\chi}\tilde{g} = O(s^{-n}), \tag{5.93}$$

$$B =s^{-n}\sum_{k=1}^{n-1}\sum_{l=0}^{n-1}\frac{s^{-k}}{k!}\mathrm{Rem}_{2,l}\tilde{\chi}^{(k)}\mathrm{ad}^k_d(\tilde{g}) = O(s^{-n}). \tag{5.94}$$

Thus we have proved (5.86).

Next, we prove (5.87) by the following lemma:

Lemma 5.8. *For $k = 1,\dots,n$ and $\mathrm{Rem}_1(k) := \int(Rb)^kR^E$,*

$$\mathrm{Rem}_1(k)\tilde{g}\tilde{\chi} = O(1). \tag{5.95}$$

Proof. We prove this by induction on k. We have by expansion (5.52) that $Rb = Rc + R_2h$. For the base case $k = 1$, we write

$$\begin{aligned}RbR^E =&RcR^E + R_2R^Eh + R_2[h,R^E]\\ =&RcR^E + R_2R^Eh + s^{-1}R_2R^E\mathrm{ad}_{d^E}(h)R^E,\end{aligned} \tag{5.96}$$

where we use the relation (c.f. (5.20))

$$[B, R^E] = s^{-1} R^E \operatorname{ad}_{d^E}(B) R^E, \tag{5.97}$$

valid for any operator B allowed by the domain consideration.

The second term (5.96) is a priori large $O(s)$ but it is removed by $\tilde{g}$. Indeed, since $h\tilde{g} = 0$ by (5.13) and the relation (5.77) (c.f. Figure 4), and $s^{-1}R_2 = 1 + zR$ by (5.56), we have

$$\begin{aligned}\operatorname{Rem}_1(1)\,\tilde{g} &= \int RcR^E\tilde{g} + \int s^{-1}R_2R^E\operatorname{ad}_{d^E}(h)R^E\tilde{g} \\ &= \int RcR^E\tilde{g} + \int R^E\operatorname{ad}_{d^E}(h)R^E\tilde{g} + \int zRR^E\operatorname{ad}_{d^E}(h)R^E\tilde{g}.\end{aligned} \tag{5.98}$$

For $f \in C_c^\infty(\mathbb{R})$, the operators $\operatorname{ad}^k_{d^E}(f)$ are $O(1)$ by results from Section 4, see (4.6) and [22, eqn. (B.20)]. Thus the three integrals in line (5.98) are $O(1)$ by the estimates $\|\tilde{g}\| \le 1$, $\|c\|, \|\operatorname{ad}_{d^E}(h)\| = O(1)$, $\|R\|,\ \left\|R^E\right\| \le |\operatorname{Im}(z)|^{-1}$, and Corollary B.5 with $(p, l) = (1, 0)$, $(2, 1)$. This shows (5.95) with $k = 1$.

Suppose now (5.95) holds with some $k \ge 1$, and we prove it for $k \to k+1$. First, we note the relation $R^E - R = RbR^E$ and so

$$\begin{aligned}\operatorname{Rem}_1(k) &= \int (Rb)^kR + \int (Rb)^k(R^E - R) \\ &= \int (Rb)^kR + \int (Rb)^{k+1}R^E = s^k\bar{\xi}_k + \operatorname{Rem}_1(k+1),\end{aligned} \tag{5.99}$$

where $\bar{\xi}_k$ is defined by (5.85). Right-multiplying $\tilde{g}\tilde{\chi}$ on both sides of (5.99) and rearranging, we find

$$\operatorname{Rem}_1(k+1)\tilde{g}\tilde{\chi} = \operatorname{Rem}_1(k)\tilde{g}\tilde{\chi} - s^k\bar{\xi}_k\tilde{g}\tilde{\chi}. \tag{5.100}$$

The first term on the r.h.s. is $O(1)$ by induction hypothesis. The second term is $O(s^{k-n})$ by (5.86) proved earlier. Since $k \le n$, this completes the induction and the proof of Lemma 5.8. □

Since $\operatorname{Rem}_1 \equiv \operatorname{Rem}_1(n)$ in Lemma 5.8, estimate (5.95) implies (5.87). This, together with (5.86), implies the claim (5.83). This completes the proof of Proposition 5.7. □

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- Data availability: Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Appendix A. Existence of unique solution to vNL equation

In this section, we prove existence of unique mild solution to (1.1) in the Schatten space S^1 of trace-class operators. Throughout the section, we assume **(W1)**, i.e. $\sum_{j\geq 1} W_j^* W_j$ with W_j in (1.1) converges strongly.

The main mechanism is the following theorem (see e.g. [6, Theorem 3.1.33]):

Theorem A.1. *Let U be a strongly continuous semigroup on the Banach space X with generator S and let P be a bounded operator on X. Then, $S + P$ generates a strongly continuous semigroup U^P.*

In our case, X is the Schatten space $\mathcal{S}^1$ with trace-norm $\|\cdot\|_1$, the strongly continuous semigroup U is the unitary semigroup generated by $-i[H, \cdot]$ and the perturbation P is the Lindblad operator G (see (1.1)).

In the next lemma, we show that G is norm closed and bounded, so that Theorem A.1 indeed applies.

Lemma A.2. *The Lindblad operator G defined in* (1.1) *is bounded on $\mathcal{S}_1$.*

Proof. Without loss of generality, we assume $\rho \in \mathcal{S}^1$ is positive. Let $G_j(\cdot) = W_j\,(\cdot)\,W_j^* - \frac{1}{2}\{W_j^* W_j, (\cdot)\}$. For a positive ρ, it is clear the operators $W_j \rho W_j^*$ and $\{W_j^* W_j, \rho\}$ are positive for all j. Then, by cyclicity of the trace, we have

$$\begin{aligned} \|G_j(\rho)\|_1 &\leq \|W_j \rho W_j^*\|_1 + \frac{1}{2}\|\{W_j^* W_j, \rho\}\|_1 \\ &\leq \operatorname{Tr}|W_j \rho W_j^*| + \frac{1}{2}\operatorname{Tr}|\{W_j^* W_j, \rho\}| \\ &= \operatorname{Tr}(W_j \rho W_j^*) + \frac{1}{2}\operatorname{Tr}(\{W_j^* W_j, \rho\}) \\ &= 2\operatorname{Tr}(W_j^* W_j \rho). \end{aligned} \tag{A.1}$$

Thus,

$$\|G(\rho)\|_1 = \Big\|\sum_{j\geq 1} G_j(\rho)\Big\|_1 \leq 2\sum_{j\geq 1}\operatorname{Tr}(W_j^* W_j \rho) \leq 2\Big\|\sum_{j\geq 1} W_j^* W_j\Big\|\,\|\rho\|_1. \tag{A.2}$$

Since $\sum_{j\geq 1} W_j^* W_j$ is bounded by the uniform boundedness theorem, this proves G is bounded on $\mathcal{S}_1$, which completes the proof. ∎

Theorem A.1 shows that (1.1) has a unique strong solution in $\mathcal{D}(L)$ and a unique mild solution in $\mathcal{S}_1$. We denote the semigroup generated by vNL operator L by β_t as before. Note that since $e^{L_0 t}$ is a group (defined on $\mathbb{R}$), then so is $\beta_t = e^{Lt}$.

The positivity preserving property of β_t follows from [9, Theorem 5.2]; see also [28]. We summarize the key result in the following lemma:

Lemma A.3. *The semigroup β_t is positive for all $t \geq 0$.*

Proof. First, recall that β_t is generated by the vNL operator L which we can rewrite as

$$L(\rho) = K_{-iH-P}(\rho) + F(\rho), \tag{A.3}$$

where $P = P^* = \frac{1}{2}\sum_{j\geq 1} W_j^* W_j \geq 0$, $K_A(\rho) = A\rho + \rho A^*$ and $F(\rho) = \sum_{j\geq 1} W_j \rho W_j^*$. Since P is bounded on $\mathcal{H}$, we have $\mathcal{D}(K_{-iH-P}) = \mathcal{D}(K_{-iH}) = \mathcal{D}(L_0)$.

Since H is self-adjoint and P is bounded, the operator $-iH-P$ is well-defined and generates a semigroup $B_t = e^{-iHt-Pt}$ of contractions on $\mathcal{H}$. Then, we define a semigroup S_t of contractions on $\mathcal{S}_1$ by

$$S_t(\rho) = B_t \rho B_t^*, \tag{A.4}$$

which is obviously positive for each $t \geq 0$. For each $\rho \in \mathcal{D}(L_0)$, by differentiating in t, we obtain

$$\begin{aligned} \partial_t S_t(\rho) &= (\partial_t B_t)\rho B_t^* + B_t \rho (\partial_t B_t)^* \\ &= B_t(K_{-iH-P}(\rho))B_t^*. \end{aligned} \tag{A.5}$$

Thus, S_t is generated by K_{-iH-P}.

On the other hand, since F is bounded and

$$F(\rho) = \sum_{j\geq 1} W_j \rho W_j^* \geq 0$$

for all $\rho \geq 0$, the semigroup

$$e^{Ft} = \sum_{k\geq 0} \frac{t^k}{k!} F^k, \tag{A.6}$$

which is generated by F, is also positive for each $t \geq 0$. Hence, for each $n \geq 0$ and $t \geq 0$, the map

$$\rho \mapsto (S_{t/n} e^{Ft/n})^n(\rho) \tag{A.7}$$

is positive on $\mathcal{S}_1$.

Finally, by the Trotter-Lie formula, for each $t \geq 0$, we have

$$\beta_t(\rho) = e^{(K_{-iH-P}+F)t}\rho = \lim_{n\to\infty} (S_{t/n} e^{Ft/n})^n(\rho), \tag{A.8}$$

where the limit is taken in the trace-norm. Since $(S_{t/n}e^{Ft/n})^n$ is positive on $\mathcal{S}_1$ for each $t \geq 0$ and $n \geq 0$, we therefore have that β_t is positive for all $t \geq 0$.

We remark that, alternatively, one can prove the positivity of β_t by using the Duhamel principle $\beta_t(\rho) = S_t(\rho) + \int_0^t S_{t-r}(F(\beta_r(\rho)))\mathrm{d}r$ for small t and extending to general t via the semigroup property. $\square$

Appendix B. Remainder estimates

In this appendix and the next one, we present some estimates and commutator expansions, first derived in [37] and then improved in [16, 22, 23, 40]. We adapt some of the arguments from [22] and refer to this paper for details and references.

Throughout this section we fix an integer $\nu \geq 0$. For integers $p \geq 0$ and smooth functions $f \in C^{\nu+2}(\mathbb{R})$, we define a weighted norm

$$\mathcal{N}(f,p) := \sum_{m=0}^{\nu+2} \int_{\mathbb{R}} \langle x \rangle^{m-p-1} \left| f^{(m)}(x) \right| dx. \tag{B.1}$$

Note that

$$p \leq p' \implies \mathcal{N}(f,p') \leq \mathcal{N}(f,p), \tag{B.2}$$

and we have the following property:

Lemma B.1. *Let $p \geq 0$ be an integer. Suppose $f \in C^{\nu+2}$ and there exist C_0, $\sigma > 0$ such that, for $m = 0, \dots, \nu+2$,*

$$\left\| \langle x \rangle^{m-p+\sigma} f^{(m)}(x) \right\|_{L^\infty} \leq C_0. \tag{B.3}$$

Then there exists $C > 0$ depending only on σ, C_0, ν such that

$$\mathcal{N}(f, p) \leq C. \tag{B.4}$$

Proof. We have

$$\begin{aligned}\mathcal{N}(f,p) \leq & \sum_{m=0}^{\nu+2} \left\| \langle x \rangle^{m-p+\sigma} f^{(m)}(x) \right\| \int_{\mathbb{R}} \langle x \rangle^{-1-\sigma} \, dx \\ \leq & (\nu+3) C_0 \int_{\mathbb{R}} \langle x \rangle^{-1-\sigma} \, dx,\end{aligned}$$

and the integral converges for $\sigma > 0$. $\square$

Corollary B.2. *Let p and l be two integers with $p > l \geq 0$. If $f \in C^\infty(\mathbb{R})$ and $f^{(l+1)}$ has compact support, then* (B.4) *holds.*

Proof. It suffices to verify condition (B.3) for the function f, whence (B.4) follows from Lemma B.1. For $m \geq l+1$, (B.3) holds since $f^{(m)} \in C_c^\infty$. For $m \leq l$, integrating $f^{(l+1)}$ shows that $\left| f^{(m)}(x) \right| \leq C \langle x \rangle^{l-m}$. Since $p \geq l+1$, we have (B.3) with $\sigma = 1$. $\square$

Write $z = x + iy \in \mathbb{C}$. In what follows, as in [22, Eq. (B.5)], for $f \in C^{\nu+2}(\mathbb{R})$, we take $\tilde{f}(z)$ to be an almost analytic extension of f defined by

$$\tilde{f}(z) := \eta\left(\frac{y}{\langle x \rangle} \right) \sum_{k=0}^{\nu+1} f^{(k)}(x) \frac{(iy)^k}{k!}, \tag{B.5}$$

where $\eta \in C_c^\infty(\mathbb{R})$ is a cutoff function with $\eta(\mu) \equiv 1$ for $|\mu| \leq 1$, $\eta(\mu) \equiv 0$ for $|\mu| \geq 2$, and $|\eta'(\mu)| \leq 1$ for all μ. This $\tilde{f}(z)$ induces a measure on $\mathbb{C}$ as

$$d\tilde{f}(z) := -\frac{1}{2\pi} \partial_{\bar{z}} \tilde{f}(z) dx \, dy. \tag{B.6}$$

In the remainder of this appendix, we derive integral estimate for various functions against the measure (B.6).

The next result is obtained by adapting the argument in [22, Lem. B.1]:

Lemma B.3 (Remainder estimate)**.** *Let $0 \leq p \leq \nu$. Let $f \in C^{\nu+2}(\mathbb{R})$ satisfy* (B.4)*. Then the extension $\tilde{f}$ from* (B.5) *satisfies the following estimate for some $C = C(f, \nu, p) > 0$:*

$$\int \left| d\widetilde{f}(z) \right| |\mathrm{Im}(z)|^{-(p+1)} \leq C. \tag{B.7}$$

Proof. Differentiating formula (B.5), we obtain the estimate

$$\left| \partial_{\bar{z}} \tilde{f}(z) \right| \leq \eta\left(\frac{y}{\langle x \rangle} \right) \frac{|y|^{\nu+1}}{(\nu+1)!} \left| f^{(\nu+2)}(x) \right| + \sum_{k=0}^{\nu+1} \varphi\left(\frac{y}{\langle x \rangle} \right) \frac{|y|^k}{k!} \left| \frac{1}{\langle x \rangle} f^{(k)}(x) \right|, \tag{B.8}$$

where

$$\varphi(\mu) := |\eta'(\mu)| \langle \mu \rangle \tag{B.9}$$

is supported on $1 < |\mu| < 2$.

For each fixed x, we define

$$G(x) := p.v. \int |\partial_{\bar{z}} f(z)| \, |y|^{-(p+1)} \, dy \tag{B.10}$$

by integrating (B.8) against $|y|^{-(p+1)}$. Using that $\eta(y/\langle x\rangle) \equiv 0$ for $|y| > \langle x\rangle$ and $\varphi(y/\langle x\rangle) \equiv 0$ for $|y| \le \langle x\rangle$ or $|y| \ge 2\langle x\rangle$, we find

$$G(x) \le \int_{|y|\le\langle x\rangle} \frac{|y|^{\nu-p}}{(\nu+1)!}\eta\left(\frac{y}{\langle x\rangle}\right) dy \left|f^{(\nu+2)}(x)\right| \tag{B.11}$$

$$+ \sum_{k=0}^{\nu+1} \int_{\langle x\rangle<|y|<2\langle x\rangle} \varphi\left(\frac{y}{\langle x\rangle}\right) \frac{|y|^{k-p-1}}{k!} \, dy \left|\frac{1}{\langle x\rangle} f^{(k)}(x)\right|. \tag{B.12}$$

Since $0 \le \eta(\mu) \le 1$ and $\nu \ge p$, the integral in line (B.11) converges and can be bounded as

$$\int_{|y|\le\langle x\rangle} \frac{|y|^{\nu-p}}{(p+1)!}\eta\left(\frac{y}{\langle x\rangle}\right) dy \left|f^{(p+2)}(x)\right| \le \frac{2\langle x\rangle^{\nu-p+1}}{(p+1)!}\left|f^{(p+2)}(x)\right|. \tag{B.13}$$

To bound line (B.12), we use that $\varphi(y/\langle x\rangle) < \sqrt{5}$ and $|y|^{k-p-1} \le \langle x\rangle^{k-p-1}$ for $\langle x\rangle < |y| < 2\langle x\rangle$, $0 \le k \le p+1$ (see (B.9)). Thus each integral in line (B.12) can be bounded as

$$\begin{aligned}
&\sum_{k=0}^{\nu+1} \int_{\langle x\rangle<|y|<2\langle x\rangle} \varphi\left(\frac{y}{\langle x\rangle}\right) \frac{|y|^{k-p-1}}{k!} \, dy \left|\frac{1}{\langle x\rangle} f^{(k)}(x)\right| \\
\le& \sum_{k=0}^{p+1} \frac{4\sqrt{5}\langle x\rangle^{k-p-1}}{k!}\left|f^{(k)}(x)\right| + \sum_{k=p+1}^{\nu+1} \frac{\sqrt{5}\cdot 2^{k-p+1}\langle x\rangle^{k-p-1}}{k!}\left|f^{(k)}(x)\right|.
\end{aligned} \tag{B.14}$$

Combining (B.13)–(B.14) in (B.12), we conclude that

$$|G(x)| \le CF(x), \quad F(x) := \sum_{m=0}^{\nu+2} \langle x\rangle^{m-p-1}\left|f^{(m)}(x)\right|. \tag{B.15}$$

Let $G_\lambda(x) := \mathbf{1}_{[-\lambda,\lambda]}G(x)$ with $\lambda > 0$. Then $G_\lambda \in L^1$ and $|G_\lambda(x)| \le CF(x)$ for any λ. By assumption (B.4) and definition(B.1), we have $\|F\|_{L^1} = \mathcal{N}(f,p) < \infty$ and so $F \in L^1$. Therefore, sending $\lambda \to \infty$ and using the dominated convergence theorem yields $G \in L^1$ with

$$\|G\|_{L^1} \le C\|F\|_{L^1}. \tag{B.16}$$

Recalling Definition (B.10), we find $(2\pi)^{-1}\|G\|_{L^1}$ =l.h.s. of (B.7). Thus we conclude (B.7) from (B.16). $\square$

Lemma B.3 and Corollary B.2 together imply the following results:

Corollary B.4. *Let p and l be two integers with $\nu \ge p > l \ge 0$. If $f \in C^\infty(\mathbb{R})$ and $f^{(l+1)}$ has compact support, then there exists $C > 0$ such that the extension $\tilde{f}$ from* (B.5) *satisfies the remainder estimate* (B.7).

Corollary B.5. *Let p and l be two integers with $\nu \geq p > l \geq 0$. Let $P_l(x)$ be a polynomial with* $\deg \leq l$. *Let $f \in C^\infty(\mathbb{R})$ have compactly supported derivatives. Then there exists $C > 0$ such that the extension $\tilde{f}$ from* (B.5) *satisfies*

$$\int \left| d\widetilde{f}(z) P_l(z) \right| |\mathrm{Im}(z)|^{-(p+1)} \leq C. \tag{B.17}$$

Proof. Let $f_l(x) := P_l(x)\chi(x)$. Observe that since $\partial_{\bar{z}} P_l(z) = 0$, we have by (B.6) that

$$P_l(z) d\widetilde{f}(z) = d\widetilde{f_l}(z). \tag{B.18}$$

We compute

$$f_l^{(l+1)} = P_l^{(l+1)} f + \sum_{k=0}^{l} \binom{l+1}{k} P_l^{(k)} f^{(l+1-k)}. \tag{B.19}$$

The term leading term on the r.h.s. vanishes since $\deg p \leq l$. Each term in the sum lies in C_c^∞ since $f^{(q)} \in C_c^\infty$ for $q \geq 1$. Thus f_l verifies the condition of Corollary B.4 and so (B.17) follows. □

Appendix C. Commutator expansions

In this appendix, we take $\tilde{f}(z)$, $d\widetilde{f}(z)$ to be as in (B.5)–(B.6).

We frequently use the following result, taken from [22, Lemma B.2]:

Lemma C.1. *Let $f \in C^{\nu+2}(\mathbb{R})$ satisfy* (B.4) *for some $p \geq 0$. Then for any self-adjoint operator A on $\mathcal{H}$,*

$$\frac{1}{p!} f^{(p)}(A) = \int_{\mathbb{C}} d\tilde{f}(z)(z - A)^{-(p+1)}, \tag{C.1}$$

where the integral converges absolutely in operator norm and is uniformly bounded in A.

Remark 12. Note that (B.4) ensures $f^{(p)}$ is bounded independent of A and the remainder estimate in Lemma B.3 ensures the norm convergence of the r.h.s. of (C.1).

We call Equation (C.1) the *Helffer-Sjöstrand (HS) representation.* It is possible to obtain stronger results with less regularity assumption on f using some technical estimates from [2, Sec. 5]. We do not pursue this generality here, as the assumption (B.4) already suffices for our purposes.

The HS representation (C.1), together with the remainder estimate (B.7), implies the following commutator expansion:

Lemma C.2. *Let $n \geq 1$. Let $f \in C^{n+3}(\mathbb{R})$ satisfy* (B.4) *with $p = 1$. Let A be an operator on $\mathcal{H}$. Let Φ be a lower semi-bounded self-adjoint operator on $\mathcal{H}$. Let $f_s := f(s^{-1}(\Phi - \alpha))$ for some fixed α and all $s > 0$. Suppose there exists $c \geq 0$ such that*

$$(\Phi + c)^{-1} \mathcal{D}(A) \subset \mathcal{D}(A), \tag{C.2}$$

and

$$B_k := \mathrm{ad}_\Phi^k(A) \in \mathcal{B}(\mathcal{H}) \quad (1 \leq k \leq n+1). \tag{C.3}$$

Then $[A, f_s] \in \mathcal{B}(\mathcal{H})$, and we have the expansion

$$[A, f_s] = -\sum_{k=1}^{n} \frac{s^{-k}}{k!} B_k f_s^{(k)} - s^{-(n+1)} \mathrm{Rem}_{\mathrm{left}}(s) \tag{C.4}$$

$$= \sum_{k=1}^{n} (-1)^k \frac{s^{-k}}{k!} f_s^{(k)} B_k + (-1)^{n+1} s^{-(n+1)} \mathrm{Rem}_{\mathrm{right}}(s), \tag{C.5}$$

where the remainders are defined by these relations and given explicitly by (C.13)–(C.14)*. Moreover, there exists $c > 0$ depending only on n and $\mathcal{N}(f, n+1)$, such that*

$$\|\mathrm{Rem}_{\mathrm{left}}(s)\|_{\mathrm{op}} + \|\mathrm{Rem}_{\mathrm{right}}(s)\|_{\mathrm{op}} \leq c \|B_{n+1}\|. \tag{C.6}$$

Proof. Within this proof we write $R = (z - x_s)^{-1}$ with $x_s = s^{-1}(\Phi - \alpha)$. Hypothesis (C.2) shows that

$$R = (\Phi + c)^{-1}(z(\Phi + c)^{-1} - x_s(\Phi + c)^{-1})^{-1}$$

maps $\mathcal{D}(A)$ into itself for z with large $|\mathrm{Im}(z)|$ and therefore for all z with $\mathrm{Im}(z) \neq 0$.

It follows that

$$\big[A, R\big] = -R[x_s, A]R = -s^{-1} R \,\mathrm{ad}_\Phi(A) R \tag{C.7}$$

holds in the sense of quadratic forms on $\mathcal{D}(A)$. Since R is bounded and $\mathrm{ad}_\Phi(A)$ is bounded by assumption, the r.h.s. of (C.7) is bounded and so $[A, R]$ extends to a bounded operator on $\mathcal{H}$.

Using (C.7), we proceed by commuting successively the commutators $B_k := \mathrm{ad}_\Phi^k(A)$ to the left to obtain

$$\begin{aligned} [A, R] &= -\, s^{-1} \,\mathrm{ad}_\Phi(A) R^2 - s^{-2} R \,\mathrm{ad}_\Phi(A) R^2 \\ &= -\, s^{-1} B_1 R^2 - s^{-2} R B_2 R^2 \\ &= -\sum_{k=1}^{n} s^{-k} B_k R^{k+1} - s^{-(n+1)} R B_{n+1} R^{n+1} \end{aligned} \tag{C.8}$$

Similarly, commuting B_k's to the right or taking adjoint of (C.8), we find

$$[A, R] = \sum_{k=1}^{n} (-1)^k s^{-k} R^{k+1} B_k + (-1)^{n+1} s^{-(n+1)} R^{n+1} B_{n+1} R, \tag{C.9}$$

which hold on all of $\mathcal{H}$ since B_k's are bounded operators by assumption (C.3).

Since f may not decay at ∞, we cannot directly express $f_s = f(s^{-1}(\Phi - \alpha))$ using the HS representation C.1. We therefore introduce a cutoff as follows. Let $\eta^\lambda \in C_c^\infty(\mathbb{R})$, $\lambda > 0$ be cutoff functions with $\eta^\lambda(x) \equiv 1$ for $|x| \leq \lambda$, $\eta(x) \equiv 0$ for $|\mu| \geq \lambda + 1$, and $\|\eta^\lambda\|_{C^{n+3}} \leq C$ for all λ. Set $f^\lambda := \eta^\lambda f$. Since $f^\lambda \in C_c^{n+3}$, it satisfies (B.4) for all $p \geq 0$. Thus the HS representation C.1 holds with $p = 0$ and so

$$[A, f_s^\lambda] = \int d\widetilde{f^\lambda}(z) \big[A, R\big], \tag{C.10}$$

which holds a priori on $\mathcal{D}(A)$.

Plugging expansions (C.8)–(C.9) into (C.10) yields

$$[A, f_s^\lambda]$$

$$= -\sum_{k=1}^{n} \frac{s^{-k}}{k!} B_k \int d\widetilde{f^\lambda}(z) R^{k+1} - s^{-(n+1)} \mathrm{Rem}_{\mathrm{left}}^\lambda(s), \tag{C.11}$$

$$= \sum_{k=1}^{n} (-1)^k \frac{s^{-k}}{k!} \int d\widetilde{f^\lambda}(z) R^{k+1} B_k + (-1)^{n+1} s^{-(n+1)} \mathrm{Rem}_{\mathrm{right}}^\lambda(s), \tag{C.12}$$

where

$$\mathrm{Rem}_{\mathrm{left}}^\lambda(s) = \int d\widetilde{f^\lambda}(z) R B_{n+1} R^{(n+1)}, \tag{C.13}$$

$$\mathrm{Rem}_{\mathrm{right}}^\lambda(s) = \int d\widetilde{f^\lambda}(z) R^{(n+1)} B_{n+1} R. \tag{C.14}$$

Since the operator B_{n+1} is bounded independent of λ, z, and $\|R\| \leq |\mathrm{Im}(z)|^{-1}$, we have

$$\begin{aligned} &\left\| \mathrm{Rem}_{\mathrm{left}}^\lambda(s) \right\|_{\mathrm{op}} + \left\| \mathrm{Rem}_{\mathrm{right}}^\lambda(s) \right\|_{\mathrm{op}} \\ \leq &2\|B_{n+1}\| \int |d\widetilde{f^\lambda}(z)| R^{n+2} \\ \leq &2\|B_{n+1}\| \int |d\widetilde{f^\lambda}(z)| |\mathrm{Im}(z)|^{-(n+2)}. \end{aligned} \tag{C.15}$$

Similarly we could bound the sums in (C.11)–(C.12). Thus we see $[A, f_s^\lambda]$ extends to a bounded operator on $\mathcal{H}$ for each λ.

By (B.2) and the assumption $\mathcal{N}(f, 1) \leq C$, f satisfies condition (B.4) with $p = 1, \ldots, n+1$. Hence, sending $\lambda \to \infty$ in (C.11)–(C.14) and using (C.1) for $p = 1, \ldots, n$ and the remainder estimate (B.7) for $p = n+1$, we conclude that $[A, f_s] \in \mathcal{B}(\mathcal{H})$ and expansions (C.4)–(C.5) and estimate (C.6) hold. □

References

[1] R. Alicki and K. Lendi. *Quantum dynamical semigroups and applications*, volume 286 of Lecture Notes in Physics. Springer-Verlag, Berlin, 1987.

[2] W. Amrein, A. Boutet de Monvel, V. Georgescu, *C_0-groups, commutator methods and spectral theory of N-body Hamiltonians*, Progress in Mathematics, Birkhäuser Verlag, Basel (1996)

[3] J. Arbunich, J. Faupin, F. Pusateri, I.M. Sigal, *Maximal Speed of Quantum Propagation for the Hartree equation*, To appear in Communications in Partial Differential Equations

[4] J. Arbunich, F. Pusateri, I.M. Sigal, A. Soffer, *Maximal speed of quantum propagation*. Lett. Math. Phys. **111** (2021), No. 62.

[5] J.-F. Bony, J. Faupin, I.M. Sigal, *Maximal velocity of photons in non-relativistic QED*, Adv. Math. **231** (2012), 3054–3078.

[6] O. Brattelli, D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics 1, 2nd Edition*, Springer Berlin, Heidelberg.

[7] S. Breteaux, J. Faupin, M. Lemm, and I. M. Sigal, *Maximal Propagation Speed in Open Quantum Systems*, In: The Physics and Mathematics of Elliott Lieb, The 90th Anniversary Volume I, Rupert L. Frank et al. (eds.), 109–130, EMS Press, Berlin, 2022

[8] H. Cycon, R. Froese, W. Kirsch and B. Simon, *Schrödinger operators*, Texts and Monographs in Physics, Springer Verlag (1987).

[9] E. B. Davies, *Quantum Theory of Open Systems*, Academic Press 1976.

[10] J. Dereziński: *Asymptotic completeness of long-range N-body quantum systems*, Ann. of Math. **138**, 427–476 (1993)

[11] J. Epstein, and K.B. Whaley, *Quantum speed limits for quantum-information-processing tasks* Phys. Rev. A, **95**,(4) (2017), 042314

[12] M. Falconi, J. Faupin, J. Fröhlich, and B. Schubnel. *Scattering theory for Lindblad master equations.* Commun. Math. Phys., **350**(3):1185–1218, 2017.

[13] J. Faupin, M. Lemm, and I. M. Sigal, *Maximal speed for macroscopic particle transport in the Bose-Hubbard model*, Phys. Rev. Lett. **128** (2022), 150602

[14] J. Faupin, M. Lemm, and I. M. Sigal, *On Lieb-Robinson bounds for the Bose-Hubbard model*, Commun. Math. Phys. **394** (2022), 1011–1037

[15] M. Foss-Feig, Z.-X. Gong, C.W. Clark, and A.V. Gorshkov, *Nearly-linear light cones in long-range interacting quantum systems*, Phys. Rev. Lett. **114** (2015), 157201

[16] S. Golénia and T. Jecko, *A new look at Mourre's commutator theory*, Complex Anal. Oper. Theory **1**, No. 3, 399–422 (2007).

[17] Z. Gong, T. Guaita, and J.I. Cirac, *Long-Range Free Fermions: Lieb-Robinson Bound, Clustering Properties, and Topological Phases* , Phys. Rev. Lett. **130**, 070401 (2023)

[18] M.B. Hastings, *Lieb-Schultz-Mattis in higher dimensions* Phys. Rev. B **69** (2004), no. 10, 104431.

[19] M.B. Hastings, *An area law for one-dimensional quantum systems*, J. Stat. Mech.: Theor. Exper. **2007** (2007), P08024

[20] M.B. Hastings and X.G. Wen, *Quasi-adiabatic continuation of quantum states: The stability of topological groundstate degeneracy and emergent gauge invariance*, Phys. Rev. B. **72** (2005), 045141

[21] I. Herbst and E. Skibsted, *Free channel Fourier transform in the long-range N-body problem* Journal d'Analyse Mathematique, 65(1) (1995), 297-332

[22] W. Hunziker and I.M. Sigal, *Time-dependent scattering theory of n-body quantum systems*, Rev. Math. Phys., **12** (2000) 1033–1084

[23] W. Hunziker, I.M. Sigal and A. Soffer, *Minimal escape velocities.* Communications in Partial Differential Equations, **24** (1999), 2279–2295

[24] R. S. Ingarden and A. Kossakowski, *On the connection of nonequilibrium information thermodynamics with non-Hamiltonian quantum mechanics of open systems*, *Ann. Physics*, **89**:451–485, 1975

[25] A. Kossakowski, *On necessary and sufficient conditions for a generator of a quantum dynamical semi-group*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., **20**:1021–1025, 1972.

[26] T. Kuwahara, T. Van Vu, and K. Saito, *Optimal light cone and digital quantum simulation of interacting bosons*, arXiv preprint, arXiv:2206.14736.

[27] E.H. Lieb, and D.W. Robinson, *The finite group velocity of quantum spin systems*, In Statistical mechanics, 425-431. Springer, Berlin, 1972 (see also E.H. Lieb: Statistical Mechanics: Selecta of Elliott H. Lieb, Bruno Nachtergaele, Jan Philip Solovej and J. Yngvason, Editors, Springer, 2014).

[28] G. Lindblad, *On the generators of quantum dynamical semigroups*, Comm. Math. Phys. **48** (1976), 119–130.

[29] B. Nachtergaele, H. Raz, B. Schlein, and R. Sims, *Lieb-Robinson bounds for harmonic and anharmonic lattice systems*, Commun. Math. Phys. **286** (2009), no. 3, 1073–1098.

[30] B. Nachtergaele and R. Sims, *Lieb-Robinson bounds and the exponential clustering theorem*, Commun. Math. Phys. **265** (2006), no. 1, 119–130.

[31] B. Nachtergaele, R. Sims, and A. Young, *Lieb-Robinson bounds, the spectral flow, and stability of the spectral gap for lattice fermion systems*, Mathematical Problems in Quantum Physics **717** (2018).

[32] B. Nachtergaele, R. Sims, and A. Young, *Quasi-locality bounds for quantum lattice systems. I. Lieb-Robinson bounds, quasi-local maps, and spectral flow automorphisms* J. Math. Phys. **60**, no. 6, 061101.

[33] B. Nachtergaele, A. Vershynina, V. Zagrebnov, *Lieb-Robinson bounds and existence of the thermodynamic limit for a class of irreversible quantum dynamics*, AMS Contemporary Mathematics **552** (2011) 161–175.

[34] D. Ou Yang and I.M. Sigal, *Approach to equilibrium in the von Neumann-Lindblad equation* (2024), Submitted.

[35] D. Poulin, *Lieb-Robinson bound and locality for general Markovian quantum dynamics*, Phys. Rev. Lett. **104** (2010), no. 19, 190401.

[36] N. Schuch, S.K. Harrison, T.J. Osborne, and J. Eisert, *Information propagation for interacting-particle systems*, Phys. Rev. A **84** (2011), 032309.

[37] I.M. Sigal and A. Soffer, *Local decay and propagation estimates for time-dependent and time-independent Hamiltonians* Preprint, Princeton Univ. (1988) http://www.math.toronto.edu/sigal/publications/SigSofVelBnd.pdf.

[38] I.M. Sigal and A. Soffer, *Long-range many-body scattering*, Invent. Math. **99** (1990) 115–143.

[39] I.M. Sigal and J. Zhang, On propagation of information in quantum many-body systems. ArXiv 2023.

[40] E. Skibsted, *Propagation estimates for N-body Schrödinger operators*, Commun. Math. Phys. **142** (1992) 67–98.

[41] E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, 1970.

[42] M.C. Tran, A.Y. Guo, C.L. Baldwin, A. Ehrenberg, A.V. Gorshkov, and A. Lucas, *The Lieb-Robinson light cone for power-law interactions*, Phys. Rev. Lett. **127** (2021), no. 16, 160401.

[43] Z. Wang and K.R. Hazzard, *Tightening the Lieb-Robinson bound in locally interacting systems*, PRX Quantum **1** (2020), 010303.

[44] C. Yin and A. Lucas, *Finite speed of quantum information in models of interacting bosons at finite density*, Phys. Rev. X 12, 021039

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