# Macroscopic suppression of supersonic quantum transport

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We consider a broad class of strongly interacting quantum lattice gases, including the Fermi-Hubbard and Bose-Hubbard models. We focus on macroscopic particle clusters of size $\theta N$, with $\theta \in (0,1)$ and $N$ the total particle number, and we study the quantum probability that such a cluster is transported across a distance $r$ within time $t$. Conventional effective light cone arguments yield a bound of the form $\exp(vt - r)$. We report a substantially stronger bound $\exp(\theta N(vt - r))$, which provides exponential suppression that scales with system size. Our result establishes a universal dynamical large deviation principle: macroscopic suppression of supersonic macroscopic transport (MASSMAT).

Lieb and Robinson [1] famously discovered that quantum lattice systems exhibit an "effective light cone" reminscient of relativistic systems. Their Lieb-Robinson bound (LRB) controls the probability that quantum information travels a distance $r > 0$ in time $t > 0$ by

$$\exp(C(v_{\mathrm{LR}}t - r)) \tag{1}$$

for constants $C, v_{\mathrm{LR}} > 0$. This establishes an effective light cone $v_{\mathrm{LR}}t = r$ beyond which information propagation is exponentially suppressed. The Lieb-Robinson velocity $v_{\mathrm{LR}}$ is an $\mathcal{O}(1)$ quantity proportional to the strength of local interactions.

As one of the few rigorous tools for analyzing strongly interacting quantum many-body systems, the LRB has proven remarkably powerful. Following breakthroughs of Hastings in the early 2000s [2–4], it was decisive in resolving fundamental problems across condensed matter physics, quantum information theory, and high-energy physics. Applications of LRBs include exponential clustering for gapped systems [2, 5], the definition and stability of topological quantum phases [2, 3, 6–10], the area law for the entanglement entropy [4], the control of dynamical entanglement generation [11, 12], the many-body adiabatic theorem [13, 14], quantum simulation algorithms [15–19], bounds on quantum messaging [20], and the fast scrambling conjecture [21–25]. Given the broad utility of LRBs, a large and continually growing body of research is concerned with extending them and related propagation bounds to new settings, e.g., to long-range interactions [2, 5, 12, 23, 26, 27], open quantum systems [28–31], bosonic lattice gases [16, 19, 32–41], and continuum systems [42–44]. Improved LRB establishing finer control (e.g., slow transport for disordered systems) have also been proved [9, 45–50]. LRBs have also been observed experimentally [51–54], e.g., with ultra-cold atoms in optical lattices. For a comprehensive review of progress up to 2023, see [55].

Ordinarily, it is considered a strength of the standard LRB (1) that it is independent of system size, making it well-suited for analyzing quantum dynamics on microscopic scales, where all relevant parameters are $\mathcal{O}(1)$. However, many physically relevant problems concern the collective transport of *macroscopic numbers of quantum particles*, starting with Ohm's law and ranging to the separation of timescales that is the basis of quantum hydrodynamics [56–58] and prethermalization phenomena [59–61]. Controlling macroscopic particle transport poses unique challenges — particularly in bosonic systems [37, 62–66] which can exhibit large local particles numbers even within regions of $\mathcal{O}(1)$ size.

In this Letter, we establish a new type of dynamical bound on the transport of macroscopic particle clusters in strongly interacting quantum lattice systems. Specifically, we show that such transport is suppressed by an exceptionally rapid and macroscopically large decay rate outside of a light cone: the bound takes the form

$$\exp(CN(vt - r)), \tag{2}$$

where $N$ is the total particle number. Figure 1 compares the standard light cone to the new light cone given by (2). What sets the latter apart is the $N$-factor in the exponent in (2). Consequently, the exponential decay rate outside of the light cone $r > vt$ grows extensively with the system size $N$. We refer to the bound (2) as a manifestation of a new quantum-dynamical large deviation principle: *macroscopic suppression of supersonic macroscopic transport* (MASSMAT). It significantly strengthens

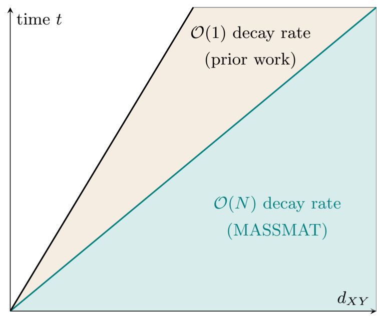


FIG. 1. Our main result establishes the green light cone $\sim vt$, with $v$ given by (6), outside of which the exponential decay rate becomes $\propto N$, i.e., extensive. Since $v$ is larger than the quantity $\kappa$ in (9) that bounded the speed of macroscopic clusters in prior work [37, 63], there is a separation of the new macroscopic MASSMAT light cone and the usual $\mathcal{O}(1)$ light cone (yellow region). Note that we establish the MASSMAT light cone for short-ranged hopping terms, whereas [37, 62–66] considered long-ranged hopping terms as well.

prior bounds on the macroscopic particle transport problem [37, 62–66] for a broad class of quantum many-body Hamiltonians with short-ranged hopping.

A key conceptual consequence of MASSMAT is that the effective light cone $r = vt$ established by (2) becomes extremely sharp already for moderate $N$-values and mathematically exact (meaning free from errors) in the thermodynamic limit $N \to \infty$ — all while $r$ and $t$ are held fixed. This stands in contrast to the standard Lieb-Robinson light cone of the form (1), which is rougher because it allows for $\mathcal{O}(1)$ leakage. Hence, MASSMAT establishes that the transport of macroscopic particle cluster is universally governed by an unforeseen *emergent strict causality.* The well-known analogy connecting LRBs and special relativity through the shared concept of the light cone is thus shown to become an exact correspondence for thermodynamically large particle clusters thanks to MASSMAT. The precise, rigorous statement is given in Theorem 1 below.

Our proof of MASSMAT applies broadly to many strongly interacting quantum lattice gases, including the Fermi-Hubbard and Bose-Hubbard Hamiltonians with short-range hopping [67]. Therefore, MASSMAT is a *universal* dynamical principle that places unforeseen constraints on quantum many-body systems out of equilibrium.

*Example: non-interacting chain.* For illustration, we present a simple situation where MASSMAT obviously holds, while the decay provided by the LRB (1) is far too pessimistic. Consider the dynamics of a chain of free (i.e., non-interacting) bosons with only nearest-neighbor hopping, i.e., the Hamiltonian

$$H_{\text{free}} = \sum_{x=1}^{L-1} (a_x^\dagger a_{x+1} + a_{x+1}^\dagger a_x), \tag{3}$$

where $\{a_x^\dagger, a_x\}_{x\in\Lambda}$ are the bosonic creation and annihilation operators. For simplicity, consider the initial state where all particles are localized at site 1, i.e., $\psi_0 = (a_1^\dagger)^N \Omega$ where $\Omega$ is the vacuum. We capture macroscopic transport via the projection $P_{N_{\{r,\ldots,L\}} \geq \theta N}$ onto the eigenspaces of $N_{\{r,\ldots,L\}} = \sum_{x=r}^{L} n_x$ (the number of particles sitting on sites $\{r, \ldots, L\}$) with eigenvalues at least $\theta N$, $0 < \theta < 1$. Using that the particles are non-interacting, one easily finds that for all $t, r \geq 1$ (see [68] for the details)

$$\langle \psi_t | P_{N_{\{r,\ldots,L\}} \geq \theta N} | \psi_t \rangle \leq e^{\theta N C(vt-r)}, \tag{4}$$

which proves that the MASSMAT principle holds for the non-interacting chain.

For non-interacting particles, the power $N$ in (4) arises because the particles are statistically independent. Of course, this argument breaks down completely for strongly interacting particles. Surprisingly, as we show, MASSMAT holds nonetheless. We are able to achieve this by devising a new way of deriving many-body propagation bounds that we coin *geometric exponential tilting*, which is different from prior approaches to bounding macroscopic transport [37, 62–66]. We explain the core idea of geometric exponential tilting after we present the main result.

*Setup and main result.* We consider a finite graph $(\Lambda, \mathcal{E}_\Lambda)$ with vertex set $\Lambda \subset \mathbb{R}^D$, $D \geq 1$, such that nearest-neighbors all have Euclidean distance $= 1$.

We consider a system of indistinguishable quantum particles living on the graph $\Lambda$ — the result and its proof are identical for fermions and bosons and so we treat both cases in parallel. We take $\{a_x^\dagger, a_x\}_{x\in\Lambda}$ to be a collection of fermionic/bosonic creation and annihilation operators satisfying the usual canonical anticommutation/commutation relations.

*The model.* On the fermionic/bosonic Fock space over the one-body Hilbert space $\ell^2(\Lambda)$, we consider Hamiltonians of the form

$$H = H_0 + V(\{n_x\}_{x\in\Lambda}), \quad H_0 = \sum_{x,y\in\Lambda} J_{xy} a_x^\dagger a_y. \tag{5}$$

Here, $J_{xy}$ represents short-range particle hopping and $V(\{n_x\}_{x\in\Lambda})$ represents a general density-density interaction.

Our assumptions on the Hamiltonian (5) are as follows:

(i) The hopping matrix is Hermitian, i.e., $J_{xy} = \bar{J}_{yx}$ for $x,\, y \in \Lambda$, and satisfies, for a parameter $a > 0$, the *short-range condition* that

$$v = \max_{x\in\Lambda} \sum_{y\in\Lambda} |J_{xy}| \frac{\sinh(a|x-y|)}{a} \tag{6}$$

is bounded independently of $|\Lambda|$.

(ii) $V : \{0, 1, 2, \ldots\}^{|\Lambda|} \to \mathbb{R}$ is a real-valued function of $|\Lambda|$ variables.

Under these assumptions, $H$ is a self-adjoint operator on the Fock space $\mathcal{F}(\ell^2(\Lambda))$; see, e.g., [36].

The class of Hamiltonians of the form (5) satisfying these assumptions is very broad. In particular, it includes the paradigmatic Fermi-Hubbard and Bose-Hubbard Hamiltonians. We call Condition (i) the short-range condition because

$$\sinh(a|x-y|) \sim \frac{1}{2}\exp(a|x-y|), \quad |x-y| \gg 1$$

and so $v$ in (6) is bounded independently of $\Lambda$ precisely when the hopping matrix $J_{xy}$ decays exponentially at large distances $|x-y|$. (We use the sinh in (6) instead of the exponential because it gives the asymptotically sharp value of $v$ in the limit $a \to 0$, as we explain after the main result.) In particular, Condition (i) holds for the physically most important case of nearest-neighbor hopping on the integer lattice $\Lambda \subset \mathbb{Z}^D$, i.e.,

$$J_{xy} = J\delta_{|x-y|=1},$$

in which case the the short-range condition (6) holds for any $a > 0$ with $v = 2DJ\frac{\sinh a}{a}$.

Assumption (ii) on $V$ is extremely weak. In particular, long-range and $k$-body interactions for any $k$ are allowed as long as they are of density-density type. Typical examples of $V(\{n_x\}_{x\in\Lambda})$ are polynomials. E.g., for the paradigmatic Bose-Hubbard Hamiltonian, one has

$$V(\{n_x\}_{x\in\Lambda}) = \sum_{x\in\Lambda} \left(n_x(n_x-1) - \mu n_x\right).$$

We remark that it is easy to include local spin degrees of freedom in our setup and we only refrain from doing so to keep the notation simple. Spin degrees of freedom appear, e.g., in the standard Fermi-Hubbard Hamiltonian. We can also treat time-dependent Hamiltonians, i.e., $J_{xy} = J_{xy}(t)$ and $V = V(t)$. In this case, we simply require that Assumptions (i) and (ii) hold uniformly in $t$.

*The main result.* Since the Hamiltonian (5) preserves the total particle number $N_\Lambda = \sum_{x\in\Lambda} n_x$, we henceforth work on a fixed eigenspace of $N_\Lambda = N$ for a fixed $N \geq 1$. To state our main result, we introduce some notation. For any subset $S \subset \Lambda$, we define

$$N_S = \sum_{x\in S} n_x, \qquad \bar{N}_S = \frac{N_S}{N}. \tag{7}$$

For $0 \leq c \leq 1$, we write $P_{\bar{N}_S \geq c}$ for the associated spectral projector of $\bar{N}_S$. Finally, given two subsets of the lattice $X, Y \subset \Lambda$, we write $d_{XY}$ for their Euclidean distance.

Our main result is the following:

**Theorem 1** (MASSMAT principle)**.** *Consider a Hamiltonian $H$ of the form* (5) *satisfying Assumptions (i)–(ii) with $v$, $a > 0$.*

*Then, for any $0 \leq \alpha < \beta \leq 1$, and any disjoint subsets $X, Y \subset \Lambda$, the following estimate holds on each $N$-particle sector:*

$$\|P_{\bar{N}_X \geq \beta}\, e^{-\mathrm{i}tH} P_{\bar{N}_Y \geq 1-\alpha}\| \leq e^{-aN((\beta-\alpha)d_{XY} - v|t|)}. \tag{8}$$

This theorem is proved in [68]. The bound (8) implies a strong light cone estimate on the quantum probability that a macroscopic cluster comprised of $(\beta-\alpha)N$ particles traverses the distance $d_{XY}$ in time $t$. Indeed, consider two disjoint regions $X, Y \subset \Lambda$ and an initial $N$-particle density operator $\rho_0$ that has at least $(1-\alpha)N$ particles in $Y$ and thus satisfies $\mathrm{Tr}(P_{\bar{N}_Y \geq 1-\alpha}\rho_0) = 1$. Consequently, there are at most $\alpha N < \beta N$ particles in $X$ initially and so $\mathrm{Tr}(P_{\bar{N}_X \geq \beta}\rho_0) = 0$. We denote the time-evolved state by $\rho_t = e^{-\mathrm{i}tH}\rho_0 e^{\mathrm{i}tH}$. Then $\mathrm{Tr}(P_{\bar{N}_X \geq \beta}\rho_t)$ is the probability that after time $t$, at least $(\beta-\alpha)N$ particles are transported from region $Y$ to the region of interest $X$. Thanks to (8), this probability is bounded by

$$\begin{aligned} &\mathrm{Tr}(P_{\bar{N}_X \geq \beta}\rho_t) \\ =&\mathrm{Tr}(P_{\bar{N}_X \geq \beta} e^{-\mathrm{i}tH} P_{\bar{N}_Y \geq 1-\alpha}\rho_0 P_{\bar{N}_Y \geq 1-\alpha} e^{\mathrm{i}tH} P_{\bar{N}_X \geq \beta}) \\ \leq&\|P_{\bar{N}_X \geq \beta} e^{-\mathrm{i}tH} P_{\bar{N}_Y \geq 1-\alpha}\|^2\, \mathrm{Tr}(\rho_0) \\ \leq&e^{-2aN((\beta-\alpha)d_{XY} - v|t|)}. \end{aligned} \tag{*}$$

In words, (8) implies that a macroscopic cluster of $(\beta-\alpha)N$ particles move at most at speed $\frac{v}{\beta-\alpha}$, up to errors that are exponentially small in $N$ and thus effectively completely negligible.

A few remarks about the bound (8) are in order. First, since it is an operator norm bound, it provides state-independent constants. Second, as mentioned above, the propagation speed (i.e., the slope of the MASSMAT light cone) is given by $\frac{v}{\beta-\alpha}$. The constant $v$ is related to previous velocity bounds on particle transport [37, 63] as follows. Since $\frac{\sinh z}{z} \geq 1$

for $z \geq 0$, we have that Assumption (6) implies

$$\kappa = \max_{x \in \Lambda} \sum_{y \in \Lambda} |J_{xy}||x-y| < v. \tag{9}$$

This $\kappa$ is exactly the first moment of the hopping matrix which was used to bound the propagation speed in our prior works [37, 63]. Thus (9) shows that the maximal velocity / light cone slope $v$ for our light cone here is slightly larger than the slope $\kappa$ obtained in [37, 63]. This shows that the macroscopic decay rate outside of the MASSMAT light cone (which has slope $v$) sets on slightly later than the standard $\mathcal{O}(1)$ decay (compare (1)) that was proved in [37, 63]. This is shown in Figure 1. In fact, our choice of $v$ in Condition (i) is sharp in the limit of arbitrarily slow decay $a \to 0$: Using $\sinh(a|x-y|) \sim a|x-y|$, we see that $v$ in (6) converges to $\kappa$ as $a \to 0$. Compared to [37, 63], we see that by slightly increasing the light cone slope, we are able to boost the microscopic error estimate outside of the light cone to an unprecedented, macroscopic one.

For finite-range hopping, Assumption (ii) holds for any $a > 0$, and so it is possible to optimize the choice of $a$ depending on the other parameters. Consider, e.g., nearest neighbor hopping on an integer lattice, i.e, $J_{xy} = J\delta_{|x-y|=1}$. Then, the minimizer is $a_* = \cosh^{-1}\left(\frac{(\beta-\alpha)d_{XY}}{2DJ|t|}\right)$ and it yields an improved bound of the form $\left(\frac{|t|}{(\beta-\alpha)d_{XY}}\right)^{N(\beta-\alpha)d_{XY}}$. This is a MASSMAT strengthening of the refined LRB of the form $(t/d)^d$ which recently played a crucial role in achieving refined control over dynamical entanglement generation [69].

Incorporating physical units in our theorem amounts to replacing $J \to \frac{J}{\hbar}$ and $d_{XY} \to \ell r_0$ with $r_0$ the lattice spacing and $\ell$ an $\mathcal{O}(1)$ number. Let us consider a typical 1D optical lattice experiment realizing the Bose-Hubbard Hamiltonian, e.g., [51, 54], which features $N = 18$ atoms with an effective hopping amplitude $J/\hbar \approx 500\mathrm{s}^{-1}$ between neighboring lattice sites that are spaced $r_0 \approx 500\mathrm{nm}$ apart, observed up to time $t_{\max} \approx 3\hbar/J$. We aim to bound, say, the quantum probability that 1/3 of the $N = 18$ particles are transported across $\ell$ lattice sites in time $t$. We apply our theorem with the dimensionally correct choice $a = 1/r_0$ and use $\sinh(1) \leq 6/5$ to obtain the bound

$$\exp\left(-N\left(\frac{\ell}{3} - \frac{3J}{\hbar}t\right)\right).$$

Experimentally, the interior quantity $\frac{\ell}{3} - \frac{3J}{\hbar}t$ is of order one; e.g., taking $\ell = 6$ and $t = \frac{1}{9}t_{\max} = \frac{1}{3}\hbar/J$, we have $\frac{\ell}{3} - \frac{3J}{\hbar}t = 1$. Then the extra factor of $N = 18$ improves the probability bound from $e^{-1} \approx 0.37$ to $e^{-18} \approx 1.52 \times 10^{-8}$.

*Description of the proof method.* Our proof of Theorem 1 rests on an approach which we call *geometric exponential tilting*. Here, we give a high-level overview and compare the approach to other ones in the literature. The full proof is deferred to [68]. Geometric exponential tilting is completely different to the approaches used in prior works to bound transport of macroscopic boson clusters, namely the second-order adiabatic spacetime localization observables (ASTLO) method [37, 63] and the optimal transport method [65]. The overarching idea of geometric exponential tilting is simple: We introduce a suitably chosen, invertible (but not unitary) many-body similarity transformation $T$, and then we bound the left-hand side of (8) by

$$\begin{aligned} &\|P_{\bar{N}_X \geq \beta}\, e^{-\mathrm{i}tH} P_{\bar{N}_Y \geq 1-\alpha}\| \\ \leq &\|P_{\bar{N}_X \geq \beta} T^{-1}\| \|T e^{-\mathrm{i}tH} T^{-1}\| \|T P_{\bar{N}_Y \geq 1-\alpha}\|. \end{aligned} \tag{10}$$

The first and third norms will produce the spatial decay $e^{-aN(\beta-\alpha)d_{XY}}$. The middle norm of $Te^{-\mathrm{i}tH}T^{-1}$ (which we call the deformed propagator) will produce the growth in time $e^{aNv|t|}$ for $t \in \mathbb{R}$, and so (8) follows.

The crux, of course, lies in choosing the right similarity transformation $T$. We construct a $T$ that exponentially weights the local particle numbers in a site-dependent way, i.e., $T = \exp(\sum_x F(x) n_x)$ for a suitable real-valued function $F(x)$. The function $F(x)$ interpolates continuously between being 1 on the region $X$ and $= -1$ on the region $Y$. Thus, $T$ gives large weight to configurations with many particles in $X$ and small weight to configurations with many particles in $Y$. The bound $\|Te^{-\mathrm{i}tH}T^{-1}\| \leq e^{aNv|t|}$ shows that the exponential weights in $T$ grow at most exponentially in time under the dynamics.

The geometric exponential tilting method is simultaneously conceptually simple (recall (10)), flexible (one can adapt the similarity transform $T$ to the problem at hand) and powerful (it is so far the only method that yields MASSMAT).

The method has various links to prior works. First, it is inspired by a recent complex analysis argument for deriving transport bounds on noninteracting particles in [70]; see also [71]. The connection to complex analysis arises through Paley-Wiener theory [72], which in particular says that it is equivalent to have exponential decay in position space and to have an analytic extension of the Fourier transform to a complex strip [73]. From a broader perspective, using suitable similarity transforms with locally varying exponential weights to adapt the geometry to the question at hand has a long history in mathematical physics, perhaps most famously in Witten's proof of the Morse inequalities [74]. In the context of propagation bounds on quantum many-body systems, related uses of spatially varying weights have recently appeared in Yin-Lucas [35], Osborne-Yin-Lucas [75], and Fresta-Porta-Schlein [76] for different quantum-dynamical problems.

*Conclusions.* In this work, we have identified and rigorously proven a conceptually novel, universal bound on the nonequilibrium dynamics of strongly interacting quantum lattice models: the macroscopic suppression of supersonic macroscopic transport (MASSMAT). MASSMAT is an unforeseen dynamical large deviation principle, which establishes that the quantum probability of supersonic propagation of macroscopic particle numbers actually decays exponentially at a macroscopic rate proportional to the total particle number $N$. This is in stark contrast to what one obtains from Lieb-Robinson bounds, which give an $\mathcal{O}(1)$ decay rate that does not grow with $N$. MASSMAT substantially strengthens the decay rate achieved on macroscopic boson transport in prior works [37, 63–66].

The MASSMAT principle is universal in scope: It applies to both bosons and fermions (as well as mixtures) and holds across general geometries. Our proof is based on a new analytical technique — geometric exponential tilting — that is inspired by complex analysis methods from one-body quantum mechanics and developed here for the first time in a many-body context. We anticipate that this method will find broader applications in macroscopic transport problems, especially in regimes characterized by slow transport of large clusters, such as hydrodynamic limits [56–58] or prethermalization phenomena [59–61].

Our work opens several avenues for future exploration. One key question is the experimental observation of MASSMAT, e.g., in ultracold quantum gases on optical lattices. This requires observation of particle numbers that are large enough so that the improved decay outside of the MASSMAT light cone becomes observable. Another important avenue is to investigate if the MASSMAT principle extends to systems with long-range hopping, as studied in [37, 63–66].

## DATA AVAILABILITY

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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[1] E. H. Lieb and D. W. Robinson, The finite group velocity of quantum spin systems, Communications in Mathematical Physics **28**, 251 (1972).

[2] M. B. Hastings, Lieb-Schultz-Mattis in higher dimensions, Physical Review B **69**, 104431 (2004).

[3] M. B. Hastings and X.-G. Wen, Quasiadiabatic continuation of quantum states: The stability of topological ground-state degeneracy and emergent gauge invariance, Physical Review B **72**, 045141 (2005).

[4] M. B. Hastings, An area law for one-dimensional quantum systems, Journal of Statistical Mechanics: Theory and Experiment **2007**, P08024 (2007).

[5] B. Nachtergaele and R. Sims, Lieb-Robinson bounds and the exponential clustering theorem, Communications in Mathematical Physics **265**, 119 (2006).

[6] S. Bachmann, S. Michalakis, B. Nachtergaele, and R. Sims, Automorphic equivalence within gapped phases of quantum lattice systems, Communications in Mathematical Physics **309**, 835 (2012).

[7] S. Bravyi, M. B. Hastings, and S. Michalakis, Topological quantum order: stability under local perturbations, Journal of mathematical physics **51**, 10.1063/1.3490195 (2010).

[8] B. Nachtergaele, R. Sims, and A. Young, Quasi-locality bounds for quantum lattice systems. part ii. perturbations of frustration-free spin models with gapped ground states, Annales Henri Poincaré **23**, 393 (2022).

[9] C. Yin and A. Lucas, Low-density parity-check codes as stable phases of quantum matter, arXiv preprint arXiv:2411.01002 (2024).

[10] W. De Roeck, V. Khemani, Y. Li, N. O'Dea, and T. Rakovszky, Ldpc stabilizer codes as gapped quantum phases: stability under graph-local perturbations, arXiv preprint arXiv:2411.02384 (2024).
[11] S. Bravyi, M. B. Hastings, and F. Verstraete, Lieb-Robinson bounds and the generation of correlations and topological quantum order, Physical Review Letters **97**, 050401 (2006).
[12] M. C. Tran, A. Y. Guo, A. Deshpande, A. Lucas, and A. V. Gorshkov, Optimal state transfer and entanglement generation in power-law interacting systems, Physical Review X **11**, 031016 (2021).
[13] S. Bachmann, W. De Roeck, and M. Fraas, Adiabatic theorem for quantum spin systems, Physical Review Letters **119**, 060201 (2017).
[14] D. Monaco and S. Teufel, Adiabatic currents for interacting fermions on a lattice, Reviews in Mathematical Physics **31**, 1950009 (2019).
[15] M. Kliesch, C. Gogolin, and J. Eisert, Lieb-Robinson bounds and the simulation of time-evolution of local observables in lattice systems, Many-Electron Approaches in Physics, Chemistry and Mathematics: A Multidisciplinary View , 301 (2014).
[16] M. P. Woods, M. Cramer, and M. B. Plenio, Simulating bosonic baths with error bars, Physical Review Letters **115**, 130401 (2015).
[17] J. Haah, M. B. Hastings, R. Kothari, and G. H. Low, Quantum algorithm for simulating real time evolution of lattice hamiltonians, SIAM Journal on Computing **52**, FOCS18 (2021).
[18] Y. Tong, V. V. Albert, J. R. McClean, J. Preskill, and Y. Su, Provably accurate simulation of gauge theories and bosonic systems, Quantum **6**, 816 (2022).
[19] T. Kuwahara, T. V. Vu, and K. Saito, Effective light cone and digital quantum simulation of interacting bosons, Nature Communications **15**, 2520 (2024).
[20] J. M. Epstein and K. B. Whaley, Quantum speed limits for quantum-information-processing tasks, Physical Review A **95**, 042314 (2017).
[21] N. Lashkari, D. Stanford, M. Hastings, T. Osborne, and P. Hayden, Towards the fast scrambling conjecture, Journal of High Energy Physics **2013**, 1 (2013).
[22] D. A. Roberts and B. Swingle, Lieb-Robinson bound and the butterfly effect in quantum field theories, Physical Review Letters **117**, 091602 (2016).
[23] M. C. Tran, A. Y. Guo, C. L. Baldwin, A. Ehrenberg, A. V. Gorshkov, and A. Lucas, Lieb-Robinson light cone for power-law interactions, Physical Review Letters **127**, 160401 (2021).
[24] S. Xu and B. Swingle, Scrambling dynamics and out-of-time-ordered correlators in quantum many-body systems, PRX quantum **5**, 010201 (2024).
[25] M. Lemm and S. Rademacher, Out-of-time-ordered correlators of mean-field bosons via bogoliubov theory, Quantum **9**, 1587 (2025).
[26] C.-F. Chen and A. Lucas, Finite speed of quantum scrambling with long range interactions, Physical review letters **123**, 250605 (2019).
[27] T. Kuwahara and K. Saito, Strictly linear light cones in long-range interacting systems of arbitrary dimensions, Physical Review X **10**, 031010 (2020).
[28] D. Poulin, Lieb-Robinson bound and locality for general Markovian quantum dynamics, Physical Review Letters **104**, 190401 (2010).
[29] B. Nachtergaele, A. Vershynina, and V. A. Zagrebnov, Lieb-Robinson bounds and existence of the thermodynamic limit for a class of irreversible quantum dynamics, AMS Contemporary Mathematics **552**, 161 (2011).
[30] T. Möbus, A. Bluhm, M. C. Caro, A. H. Werner, and C. Rouzé, Dissipation-enabled bosonic hamiltonian learning via new information-propagation bounds, arXiv preprint arXiv:2307.15026 (2023).
[31] S. Breteaux, J. Faupin, M. Lemm, D. H. Ou Yang, I. M. Sigal, and J. Zhang, Light cones for open quantum systems in the continuum, Reviews in Mathematical Physics **36**, 2460004 (2024).
[32] B. Nachtergaele, H. Raz, B. Schlein, and R. Sims, Lieb-robinson bounds for harmonic and anharmonic lattice systems., Communications in Mathematical Physics **286** (2009).
[33] N. Schuch, S. K. Harrison, T. J. Osborne, and J. Eisert, Information propagation for interacting-particle systems, Physical Review A **84**, 032309 (2011).
[34] Z. Wang and K. R. Hazzard, Tightening the Lieb-Robinson bound in locally interacting systems, PRX Quantum **1**, 010303 (2020).
[35] C. Yin and A. Lucas, Finite speed of quantum information in models of interacting bosons at finite density, Physical Review X **12**, 021039 (2022).
[36] J. Faupin, M. Lemm, and I. M. Sigal, On Lieb–Robinson Bounds for the Bose–Hubbard Model, Communications in Mathematical Physics **394**, 1011 (2022).
[37] M. Lemm, C. Rubiliani, I. M. Sigal, J. Zhang, *et al.*, Information propagation in long-range quantum many-body systems, Physical Review A **108**, L060401 (2023).
[38] M. Lemm, C. Rubiliani, and J. Zhang, On the microscopic propagation speed of long-range quantum many-body systems, arXiv preprint arXiv:2310.14896 (2023).
[39] T. Kuwahara and M. Lemm, Enhanced lieb-robinson bounds for a class of bose-hubbard type hamiltonians, arXiv preprint arXiv:2405.04672 (2024).
[40] M. Lemm and T. Wessel, Enhanced lieb-robinson bounds for commuting long-range interactions, arXiv preprint arXiv:2411.19241 (2024).
[41] M. Lemm, S. Rademacher, and J. Zhang, Local enhancement of the mean-field approximation for bosons, arXiv preprint arXiv:2412.13868 (2024).
[42] M. Gebert, B. Nachtergaele, J. Reschke, and R. Sims, Lieb–robinson bounds and strongly continuous dynamics for a class of many-body fermion systems in r d, Annales Henri Poincaré **21**, 3609 (2020).
[43] B. Hinrichs, M. Lemm, and O. Siebert, On lieb–robinson bounds for a class of continuum fermions,

in *Annales Henri Poincaré* (Springer, 2024) pp. 1–40.

[44] S. Bachmann and G. D. Nittis, Lieb–robinson bounds in the continuum via localized frames, in *Annales Henri Poincaré* (Springer, 2024) pp. 1–40.

[45] E. Hamza, R. Sims, and G. Stolz, Dynamical localization in disordered quantum spin systems, Communications in Mathematical Physics **315**, 215 (2012).

[46] D. Damanik, M. Lemm, M. Lukic, and W. Yessen, New anomalous lieb-robinson bounds in quasiperiodic xy chains, Physical review letters **113**, 127202 (2014).

[47] M. Gebert and M. Lemm, On polynomial lieb–robinson bounds for the xy chain in a decaying random field, Journal of Statistical Physics **164**, 667 (2016).

[48] C. L. Baldwin, A. Ehrenberg, A. Y. Guo, and A. V. Gorshkov, Disordered lieb-robinson bounds in one dimension, PRX Quantum **4**, 020349 (2023).

[49] D. Toniolo and S. Bose, Stability of slow hamiltonian dynamics from lieb-robinson bounds, arXiv preprint arXiv:2405.05958 (2024).

[50] A. Elgart and A. Klein, Slow propagation of information on the random xxz quantum spin chain, Communications in Mathematical Physics **405**, 239 (2024).

[51] M. Cheneau, P. Barmettler, D. Poletti, M. Endres, P. Schauß, T. Fukuhara, C. Gross, I. Bloch, C. Kollath, and S. Kuhr, Light-cone-like spreading of correlations in a quantum many-body system, Nature **481**, 484 (2012).

[52] K. Them, Towards experimental tests and applications of lieb-robinson bounds, Physical Review A **89**, 022126 (2014).

[53] P. Richerme, Z.-X. Gong, A. Lee, C. Senko, J. Smith, M. Foss-Feig, S. Michalakis, A. V. Gorshkov, and C. Monroe, Non-local propagation of correlations in quantum systems with long-range interactions, Nature **511**, 198 (2014).

[54] M. Cheneau, Experimental tests of Lieb-Robinson bounds (EMS Press, Berlin, 2022).

[55] C.-F. Chen, A. Lucas, and C. Yin, Speed limits and locality in many-body quantum dynamics, Reports on Progress in Physics **86**, 116001 (2023).

[56] R. E. Wyatt, *Quantum dynamics with trajectories: introduction to quantum hydrodynamics*, Vol. 28 (Springer Science & Business Media, 2005).

[57] A. Lucas, Hydrodynamic transport in strongly coupled disordered quantum field theories, New Journal of Physics **17**, 113007 (2015).

[58] A. Lucas and K. C. Fong, Hydrodynamics of electrons in graphene, Journal of Physics: Condensed Matter **30**, 053001 (2018).

[59] M. Gring, M. Kuhnert, T. Langen, T. Kitagawa, B. Rauer, M. Schreitl, I. Mazets, D. A. Smith, E. Demler, and J. Schmiedmayer, Relaxation and prethermalization in an isolated quantum system, Science **337**, 1318 (2012).

[60] T. Mori, T. N. Ikeda, E. Kaminishi, and M. Ueda, Thermalization and prethermalization in isolated quantum systems: a theoretical overview, Journal of Physics B: Atomic, Molecular and Optical Physics **51**, 112001 (2018).

[61] K. Mallayya, M. Rigol, and W. De Roeck, Prethermalization and thermalization in isolated quantum systems, Physical Review X **9**, 021027 (2019).

[62] R. Hamazaki, Speed limits for macroscopic transitions, PRX Quantum **3**, 020319 (2022).

[63] J. Faupin, M. Lemm, and I. M. Sigal, Maximal speed for macroscopic particle transport in the Bose-Hubbard model, Physical Review Letters **128**, 150602 (2022).

[64] T. Van Vu and K. Saito, Topological speed limit, Physical review letters **130**, 010402 (2023).

[65] T. Van Vu, T. Kuwahara, and K. Saito, Optimal light cone for macroscopic particle transport in long-range systems: A quantum speed limit approach, Quantum **8**, 1483 (2024).

[66] H. Li, C. Shang, T. Kuwahara, and T. Vu, Macroscopic particle transport in dissipative long-range bosonic systems, arXiv preprint arXiv:2503.13731 (2025).

[67] In fact, the lattice structure is not essential either: what we use abstractly is the short-range interaction and bounded group velocity of the kinetic operator. These requirements also hold, e.g., for semi-relativistic electrons in the continuum which are described by the kinetic operator $\sqrt{-\Delta + m^2}$.

[68] J. Faupin, M. Lemm, I. M. Sigal, and J. Zhang, Supplemental Material: Macroscopic suppression of supersonic quantum transport (2025).

[69] D. Toniolo and S. Bose, The dynamical $\alpha$-reényi entropies of local hamiltonians grow at most linearly in time, arXiv preprint arXiv:2408.00743 (2024).

[70] I. M. Sigal and X. Wu, On propagation of information in quantum mechanics and maximal velocity bounds, Letters in Mathematical Physics **115**, 1 (2025).

[71] C. Cedzich, A. Joye, A. H. Werner, and R. F. Werner, Exponential tail estimates for quantum lattice dynamics, arXiv preprint arXiv:2408.02108 (2024).

[72] R. E. A. C. Paley and N. Wiener, *Fourier transforms in the complex domain*, Vol. 19 (American Mathematical Soc., 1934).

[73] Indeed, Condition (i) is equivalent to saying that the family of transformed Hamiltonians $H_\xi = T_\xi H T_\xi^{-1}$, where $T_\xi = e^{\mathrm{d}\Gamma(\xi\cdot x)}$, has an analytic continuation, $H_\zeta$, into the strip $\mathcal{S}_a^n = \{\zeta \in \mathbb{C}^n : |\mathrm{Im}\zeta_j| < a \;\forall j\}$. This analytic continuation breaks the time-reversal symmetry and one takes $\mathrm{Im}\zeta_j > 0$ or $\mathrm{Im}\zeta_j < 0 \;\forall j$, depending on whether one considers $t > 0$ or $t < 0$.

[74] E. Witten, Supersymmetry and morse theory, Journal of differential geometry **17**, 661 (1982).

[75] A. Osborne, C. Yin, and A. Lucas, Locality bounds for quantum dynamics at low energy, Physical Review B **109**, 094310 (2024).

[76] L. Fresta, M. Porta, and B. Schlein, Effective dynamics of local observables for extended fermi gases in the high-density regime, arXiv preprint arXiv:2409.14841 (2024).

[77] D. G. Luenberger, *Optimization by vector space methods* (John Wiley & Sons, 1997).
[78] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics* (Springer Berlin Heidelberg, 1997).
[79] M. Reed and B. Simon, *IV: Analysis of Operators*, Vol. 4 (Elsevier, 1978).

# Supplemental Material:
# Macroscopic suppression of supersonic quantum transport

Jérémy Faupin, Marius Lemm, Israel Michael Sigal, and Jingxuan Zhang

This appendix has two parts. In Part I, we give the short proof of MASSMAT for the special case of a chain of non-interacting bosons (3), which was displayed in the main text as (4). In Part II, we introduce the geometric exponential tilting method and give the full proof of our main result, Theorem 1.

## I. DIRECT PROOF OF MASSMAT FOR NON-INTERACTING BOSONS

In this appendix, we prove that (4) holds for the non-interacting Hamiltonian (3) by a short calculation. We consider the more general initial state $\psi_0 = (a^\dagger(f))^N\Omega$ where $\Omega$ is the vacuum and $f : \{1, \ldots, L\} \to \mathbb{C}$ is a one-body wave function which is localized around the origin. Since the particles are non-interacting,

$$\psi_t = e^{-\mathrm{i}tH}\psi_0 = (a^\dagger(e^{-\mathrm{i}t\Delta_L}f))^N\Omega$$

and so,

$$\begin{aligned}\langle\psi_t|P_{N_{\{r,\ldots,L\}}\geq\theta N}|\psi_t\rangle =& \sum_{N'=\lceil\theta N\rceil}^{N}\binom{N}{N'}\left(\sum_{x=r}^{L}|\langle e^{-\mathrm{i}t\Delta_L}f,\delta_x\rangle|^2\right)^{N'}\left(1-\sum_{x=r}^{L}|\langle e^{-\mathrm{i}t\Delta_L}f,\delta_x\rangle|^2\right)^{N-N'}\\ \leq& 2^N\left(\sum_{x=r}^{L}|\langle e^{-\mathrm{i}t\Delta_L}f,\delta_x\rangle|^2\right)^{\lceil\theta N\rceil},\end{aligned} \tag{S1}$$

where the last line follows since $\sum_{N'=0}^{N}\binom{N}{N'} = 2^N$. For the one-body Laplacian $\Delta_L$, it is easy to check from Fourier theory that $|\langle e^{-\mathrm{i}t\Delta_L}f,\delta_x\rangle|^2 \leq e^{\tilde{C}(\tilde{v}t-x)}$ for suitable constants $\tilde{C}, \tilde{v} > 0$. Therefore,

$$\langle\psi_t|P_{N_{\{r,\ldots,L\}}\geq\theta N}|\psi_t\rangle \leq e^{\lceil\theta N\rceil C(v't-r)}. \tag{S2}$$

Here we used that, since we assume $t \geq 1$, various time-independent prefactors including $2^N$ can be absorbed in the velocity $v'$.

The derivation can be adapted to include on-site external fields, i.e., to treat Hamiltonians of the form

$$H_{\text{free}} = \sum_{x=1}^{L-1}(a_x^\dagger a_{x+1} + a_{x+1}^\dagger a_x + v_x n_x),$$

with $v_x$ given by a bounded sequence. For this, one uses the one-body propagation bound of the form $|\langle e^{-\mathrm{i}t(\Delta_L+V)}f,\delta_x\rangle|^2 \leq e^{C'(vt-x)}$, which follows, e.g., from [70, 71].

## II. PROOF OF THEOREM 1

The proof of Theorem 1 is organized as follows.

- In Step 1, we render the relative geometry of two disjoint subsets $X$ and $Y$ effectively one-dimensional by constructing a "separation function" $s(x)$ that incorporates the relevant geometry.
- In Step 2, we introduce the exponential tilting operator $T$, which involves a similarity transformation that exponentially weighs the local particle numbers in a site-dependent way. The relative geometry between $X$ and $Y$ is fully taken into account through the function $s(x)$ defined in (S3), resp. (S6).
- In Step 3, we derive the spatial decay from the first and third terms in (10).
- In Step 4, we bound the tilted deformed propagator, i.e., the middle term in (10) and conclude the proof.

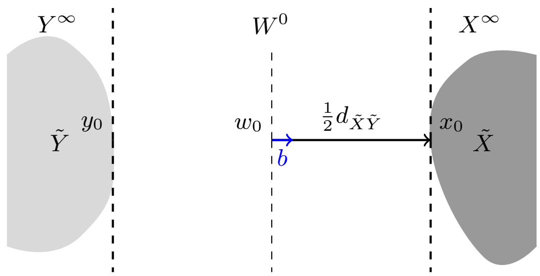


FIG. S1. Schematic diagram for the decomposition (S4).

### Step 1: Separating functions

*Simple geometry: disjoint convex hulls.* To fix ideas, we first consider the simplified scenario of two subsets $X$, $Y \subset \Lambda$ whose convex hulls, $\tilde{X} = \mathrm{conv}(X)$ and $\tilde{Y} = \mathrm{conv}(Y)$, are disjoint. With a slight modification of the argument, the results extend to complete general disjoint subsets; see the construction (S5) and (S6).

Let $x_0 \in \tilde{X}$, $y_0 \in \tilde{Y}$ be such that $d_{\tilde{X}\tilde{Y}} = |x_0 - y_0|$, and introduce the "center of mass" coordinates

$$w_0 = \frac{1}{2}(x_0 + y_0), \qquad b = \frac{x_0 - y_0}{|x_0 - y_0|}.$$

By construction, the hyperplane $\{z \in \mathbb{R}^D : (z - w_0) \cdot b = 0\}$ separates $\tilde{X}$ and $\tilde{Y}$ to two different sides. Below we project the relative geometry of $\tilde{X}$ and $\tilde{Y}$ onto the line joining the points $x_0$ and $y_0$.

We introduce the separating function $s : \mathbb{R}^D \to \mathbb{R}$,

$$s(x) = b \cdot (x - w_0) \tag{S3}$$

and define the following subsets of $\Lambda$ (see Figure S1)

$$\begin{aligned} Y^\infty &= \{x \in \Lambda \,|\, s(x) \le -\frac{1}{2} d_{\tilde{X}\tilde{Y}}\}, \\ W^0 &= \{x \in \Lambda \,|\, -\frac{1}{2} d_{\tilde{X}\tilde{Y}} < s(x) < \frac{1}{2} d_{\tilde{X}\tilde{Y}}\}, \\ X^\infty &= \{x \in \Lambda \,|\, \frac{1}{2} d_{\tilde{X}\tilde{Y}} \le s(x)\}. \end{aligned} \tag{S4}$$

For the simple geometry, we have the following easy lemma.

**Lemma 2.** *Assuming $X$, $Y$ have disjoint convex hulls, we have $\tilde{X} \subset X^\infty$ and $\tilde{Y} \subset Y^\infty$.*

*Proof.* Let $x \in \tilde{X}$. We have

$$\begin{aligned} s(x) =& b \cdot (x - x_0) + b \cdot (x_0 - w_0) \\ =& \frac{x_0 - y_0}{|x_0 - y_0|} \cdot (x - x_0) + \frac{1}{2}|x_0 - y_0|. \end{aligned}$$

The second term equals $\frac{1}{2} d_{\tilde{X}\tilde{Y}}$ by the choice of $x_0$, $y_0$, while the first term is non-negative by the separating plane theorem for disjoint convex sets [77]. This shows that $s(x) \ge \frac{1}{2} d_{\tilde{X}\tilde{Y}}$ and hence that $\tilde{X} \subset X^\infty$. The proof that $\tilde{Y} \subset Y^\infty$ is analogous. □

*Extension to general geometry.* Consider now arbitrary disjoint subsets $X, Y \subset \Lambda$. Let $g(x) = \mathrm{dist}_Y(x) - \mathrm{dist}_X(x)$. The separating hyperplane is now replaced by the separating hypersurface

$$S = \{g(x) = 0\}, \quad \Omega_\pm = \{\pm g(x) > 0\}. \tag{S5}$$

Indeed, the hypersurface $S$ is equidistant to $X$ and $Y$, with $X \subset \Omega_+$, $Y \subset \Omega_-$; see Figure S2.

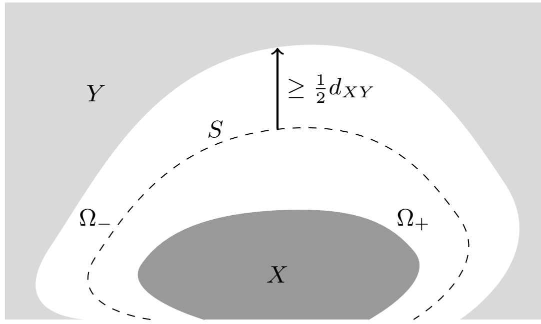


FIG. S2. Schematic diagram for $S$ and $\Omega_{\pm}$.

In this more general case, we take the separating function $s(x)$ to be the signed distance function to $S$ with the sign chosen such that $\pm s(x) > 0$ on $\Omega_{\pm}$. Explicitly,

$$s(x) = \operatorname{sgn}(g(x))\,\operatorname{dist}_S(x). \tag{S6}$$

(This reduces to (S3) in the simplified scenario considered before.) We note for later reference that this function is 1-Lipschitz continuous, i.e.,

$$|s(x) - s(y)| \leq |x - y|\,, \quad x,\, y \in \Lambda. \tag{S7}$$

Indeed, on the same side of $S$, the function $s(x)$ coincides with the distance function up to a sign, which is 1-Lipschitz by the following standard argument. For any $z \in S$ and $x,\, y \in \Lambda$, we have $\operatorname{dist}_S(x) \leq |x - z| \leq |x - y| + |y - z|$ for any $S$. By taking $\inf_{z\in S}$, we obtain $\operatorname{dist}_S(x) \leq |x - y| + \operatorname{dist}_S(y)$. If $x, y$ fall on different sides of $S$, then we join them with a line segment passing $S$ at, say, $z$, and then apply the triangle inequality to $\operatorname{dist}_S(x) \leq |x - z|$ to conclude.

Similarly to (S4), we decompose $\Lambda$ with $s(x)$ from (S6) as follows:

$$\begin{aligned} Y^\infty &= \big\{x \in \Lambda \,|\, s(x) \leq -\frac{1}{2}d_{XY}\big\}, \\ W^0 &= \big\{x \in \Lambda \,|\, -\frac{1}{2}d_{XY} < s(x) < \frac{1}{2}d_{XY}\big\}, \\ X^\infty &= \big\{x \in \Lambda \,|\, \frac{1}{2}d_{XY} \leq s(x)\big\}, \end{aligned} \tag{S8}$$

As in Lemma 2, we have

**Lemma 3.** *We have* $X \subset X^\infty$ *and* $Y \subset Y^\infty$.

*Proof.* Consider any $x \in X$. On the one hand, we have $s(x) = \operatorname{dist}_S(x)$ by Definition (S6) and the fact that $X \subset \Omega_+$. On the other hand, since $S$ is equidistant to $X$ and $Y$, we have $d_{SX} = \frac{1}{2}d_{XY}$. This shows that $s(x) \geq d_{SX} = \frac{1}{2}d_{XY}$ and hence that $X \subset X^\infty$. The proof of $Y \subset Y^\infty$ is analogous. □

**Step 2: Exponential tilting operator**

In this section, we introduce the exponential tilting operator $T$; see (S15) below.

For brevity, we fix disjoint $X,\, Y \subset \Lambda$ throughout and denote

$$d = d_{XY}. \tag{S9}$$

We define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(s) = \mathbf{1}_{s\geq 1/2} - \mathbf{1}_{s\leq -1/2} + 2s\mathbf{1}_{|s|<1/2} \tag{S10}$$

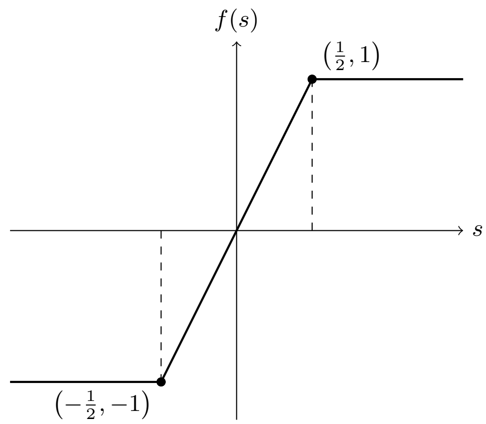


FIG. S3. The function $f(s)$.

as shown in Figure S3. (Here, $\mathbf{1}_{\dots}$ is the indicator function which equals 1 if condition ... is satisfied and which equals zero otherwise.)

We use the function $f$ to truncate the signed distance function $s(x)$ in (S6) to $f(\frac{s(x)}{d})$, with $d>0$ as in (S9). Notice that $f(\frac{s(x)}{d})=1$ on $X$ and $f(\frac{s(x)}{d})=-1$ on $Y$; hence $f$ only acts as a distance in the white region in Figure S2. For this truncated distance function, we introduce, for all $x\in\Lambda$,

$$q_\mu(x) =\exp\Big(\mu f(\tfrac{s(x)}{d})\Big) > 0, \tag{S11}$$

$$\mu =\frac{da}{2}, \tag{S12}$$

where the constant $a$ comes from the short range condition (6). As usual, $q_\mu$ is identified with the corresponding multiplication operator.

To lift $q_\mu$ to the Fock space $\mathcal{F}(\ell^2(\Lambda))$, it is convenient to introduce the following standard notation for the second quantization functor; see, e.g., [78, p. 8] and [79].

**Definition 4** (Second quantization functor)**.** Given a one-body operator $A=(A_{xy})_{x,y\in\Lambda}$, we set

$$\mathrm{d}\Gamma(A)=\sum_{x,y\in\Lambda} A_{xy}a_x^\dagger a_y,$$

and we set

$$\Gamma(e^A)=\exp(\mathrm{d}\Gamma(A)).$$

Note the following special case of this definition: If $A_{xy}=q(x)\delta_{xy}$ is a multiplication operator by a function $q(x)$, then

$$\mathrm{d}\Gamma(q)=\sum_{x\in\Lambda} q(x)n_x.$$

The reason for introducing the second quantization functor is that it allows to state various algebraic properties succinctly. Indeed, we will use the following properties which follow directly from the CAR/CCR [78, 79].

**Proposition 5** (Properties of $\mathrm{d}\Gamma$ and $\Gamma$)**.**

*(i) If $A\leq B$, then $\mathrm{d}\Gamma(A)\leq \mathrm{d}\Gamma(B)$. In particular, on each $N$-particle sector, we have $\mathrm{d}\Gamma(A)\leq\|A\|N$.*

*(ii) For any function $q:\Lambda\to\mathbb{C}$ and any $x\in\Lambda$, we have the pull-through formulas*

$$\Gamma(q)a_x^\dagger = q(x)a_x^\dagger\Gamma(q),\quad a_x\Gamma(q)=\Gamma(q)\bar{q}(x)a_x. \tag{S13}$$

We can now define the central object of the proof.

**Definition 6** (Exponential tilting operator)**.** Set

$$T = \Gamma(q_\mu) = \exp(\mu\, \mathrm{d}\Gamma(f(\tfrac{s}{d}))). \tag{S14}$$

Writing this out explicitly,

$$T = \exp\left(\mu \sum_{x\in\Lambda} f\left(\tfrac{s(x)}{d}\right) n_x\right). \tag{S15}$$

Observe that $T$ is self-adjoint and invertible; see (S11).

We recall the setup of Theorem 1. In particular, we fix the total particle number to be $N$ and we fix two numbers $0 \le \alpha < \beta \le 1$. The central idea in the exponential tilting method is to simply write

$$\begin{aligned} &P_{\bar{N}_X \ge \beta} e^{-\mathrm{i}Ht} P_{\bar{N}_Y \ge 1-\alpha} \\ =& P_{\bar{N}_X \ge \beta} T^{-1} T e^{-\mathrm{i}Ht} T^{-1} T P_{\bar{N}_Y \ge 1-\alpha}, \end{aligned}$$

which leads to the inequality

$$\begin{aligned} &\left\| P_{\bar{N}_X \ge \beta} e^{-\mathrm{i}Ht} P_{\bar{N}_Y \ge 1-\alpha} \right\| \\ \le& \left\| P_{\bar{N}_X \ge \beta} T^{-1} \right\| \left\| T e^{-\mathrm{i}Ht} T^{-1} \right\| \left\| T P_{\bar{N}_Y \ge 1-\alpha} \right\|. \end{aligned} \tag{S16}$$

We will now estimate each term of the right-hand-side separately.

**Bound on $P_{\bar{N}_X \ge \beta} T^{-1}$ and $T P_{\bar{N}_Y \ge 1-\alpha}$**

In this section, we prove the following two bounds.

**Lemma 7.** *On the $N$-particle sector, we have*

$$\left\| P_{\bar{N}_X \ge \beta} T^{-1} \right\| \le e^{\mu(1-2\beta)N}, \tag{S17}$$

$$\left\| T P_{\bar{N}_Y \ge 1-\alpha} \right\| \le e^{\mu(2\alpha-1)N}. \tag{S18}$$

*Proof.* By Definition (S10), the function $f(\frac{s(x)}{d})$ is a regularized version of the map $x \mapsto \frac{2}{d}s(x)$ in the sense that it coincides with $x \mapsto \frac{2}{d}s(x)$ on $W^0$ and continuously becomes constant on $X^\infty \cup Y^\infty = (W^0)^{\mathrm{c}}$. Explicitly,

$$f\left(\frac{s(x)}{d}\right) = \mathbf{1}_{X^\infty}(x) - \mathbf{1}_{Y^\infty}(x) + \frac{2}{d} s(x) \mathbf{1}_{W^0}(x).$$

Hence, by the Definition (S14) of $T$,

$$T = \exp\Big(\mu(N_{X^\infty} - N_{Y^\infty}) + \frac{2\mu}{d} \mathrm{d}\Gamma(s(x)\mathbf{1}_{W^0}(x))\Big).$$

We now aim to prove the first estimate (S17). Using that $-s(x) \le \frac{d}{2}$ for $x \in W^0$, we find

$$-\frac{2\mu}{d} \mathrm{d}\Gamma(s(x)\mathbf{1}_{W^0}(x)) = -\frac{2\mu}{d} \sum_{x\in W^0} s(x) n_x \le \mu N_{W^0}.$$

As both sides of this operator inequality commute (both operators are diagonal in the occupation basis), this implies

$$\begin{aligned} T^{-1} =& \exp\Big(\mu(N_{Y^\infty} - N_{X^\infty}) - \frac{2\mu}{d} \mathrm{d}\Gamma(s(x)\mathbf{1}_{W^0}(x))\Big) \\ \le& e^{\mu(N_{Y^\infty \cup W^0} - N_{X^\infty})}. \end{aligned}$$

Recall from Lemma 3 that $X \subset X^\infty$ and $Y \subset Y^\infty$. Hence, on the subspace $\mathrm{Ran}(P_{\tilde{N}_X \geq \beta})$, we have $N_{X^\infty} \geq N_X \geq \beta N$ and $N_{Y^\infty \cup W^0} \leq N_{X^c} \leq (1-\beta)N$, where $S^c = \Lambda \setminus S$ denotes the complement of $S$ in $\Lambda$. Combining these estimates, we deduce that

$$\begin{aligned}\left\|T^{-1} P_{\tilde{N}_X \geq \beta}\right\| \leq& \left\|e^{\mu(N_{Y^\infty \cup W^0} - N_{X^\infty})} P_{\tilde{N}_X \geq \beta}\right\| \\ \leq& e^{\mu(1-2\beta)N}.\end{aligned}$$

Since $P_{\tilde{N}_X \geq \beta}$ and $T^{-1}$ are self-adjoint, we have $\left\|P_{\tilde{N}_X \geq \beta} T^{-1}\right\| = \left\|T^{-1} P_{\tilde{N}_X \geq \beta}\right\|$ and so (S17) follows.

The second estimate (S18) is proven in the same way, using that $N_{Y^\infty} \geq N_Y \geq (1-\alpha)N$ and $N_{X^\infty \cup W^0} \leq N_{Y^c} \leq \alpha N$ on $\mathrm{Ran}(P_{\tilde{N}_Y \geq 1-\alpha})$, together with $s(x) \leq \frac{d}{2}$ for $x \in W^0$. □

**Bound on the deformed propagator**

To bound (S16), it remains to estimate the norm of the deformed propagator $Te^{-\mathrm{i}Ht}T^{-1}$.

**Lemma 8.** *Suppose that Assumptions (i)–(ii) on the Hamiltonian hold. Then, on the $N$-particle sector, we have, for all $t \in \mathbb{R}$,*

$$\left\|Te^{-\mathrm{i}Ht}T^{-1}\right\| \leq e^{aNv|t|}. \tag{S19}$$

*Proof.* For any bounded operator $A$, we abbreviate $\tilde{A} = TAT^{-1}$. Since $V(\{n_x\})$ commutes with $T$, we have $\tilde{H} = \tilde{H}_0 + V$ and so

$$\tilde{U}_t = Te^{-\mathrm{i}Ht}T^{-1} = e^{-\mathrm{i}t\tilde{H}} = e^{-\mathrm{i}t(\tilde{H}_0 + V)}.$$

For a bounded operator $A$, we also denote $\mathrm{Im}A = \frac{A - A^\dagger}{2\mathrm{i}}$, which is always self-adjoint. Given any state $\psi$ in the $N$-particle sector, we compute

$$\begin{aligned}\partial_t \left\|\tilde{U}_t \psi\right\|^2 =& 2\left\langle \tilde{U}_t \psi, (\mathrm{Im}\tilde{H}_0)\tilde{U}_t \psi\right\rangle \\ \leq& 2 \sup \mathrm{spec}\,(\mathrm{Im}\tilde{H}_0) \left\|\tilde{U}_t \psi\right\|^2.\end{aligned} \tag{S20}$$

Here, for a self-adjoint operator $A$, $\sup\mathrm{spec}(A)$ refers to the supremum over the spectrum of $A$.

Using Gronwall's lemma and taking the supremum over normalized $N$-particle states $\psi$, it follows that

$$\|\tilde{U}_t\| \leq \begin{cases} e^{t \sup \mathrm{spec}\,(\mathrm{Im}\tilde{H}_0)}, & t > 0, \\ e^{-t \inf \mathrm{spec}\,(\mathrm{Im}\tilde{H}_0)}, & t < 0, \end{cases}$$

and so

$$\|\tilde{U}_t\| \leq e^{|t| \|\mathrm{Im}\tilde{H}_0\|}. \tag{S21}$$

It thus remains to bound $\|\mathrm{Im}\tilde{H}_0\|$. We first calculate $\mathrm{Im}\tilde{H}_0$ by using Proposition 5 (ii) with $q = q_\mu$ from (S11). This gives

$$\begin{aligned}\tilde{H}_0 =& TH_0T^{-1} \\ =& \Gamma(q_\mu)\left(\sum_{x,y\in\Lambda} J_{xy} a_x^\dagger a_y\right)\Gamma(q_\mu^{-1}) \\ =& \sum_{x,y\in\Lambda} J_{xy} q_\mu(x) q_\mu^{-1}(y) a_x^\dagger a_y \\ =& \sum_{x,y\in\Lambda} J_{xy} \exp(\mu(f(\tfrac{s(x)}{d}) - f(\tfrac{s(y)}{d}))) a_x^\dagger a_y.\end{aligned}$$

Since $J_{xy} = \bar{J}_{yx}$, we find

$$\begin{aligned} \mathrm{Im}\tilde{H}_0 =& \,\mathrm{d}\Gamma(\tilde{J}), \\ \text{for } \tilde{J}_{xy} =& \frac{1}{\mathrm{i}} J_{xy} \sinh(\mu(f(\tfrac{s(x)}{d}) - f(\tfrac{s(y)}{d}))). \end{aligned}$$

By Proposition 5 (i), we have

$$\|\mathrm{Im}\tilde{H}_0\| = \|\,\mathrm{d}\Gamma(\tilde{J})\| \leq N\|\tilde{J}\| \tag{S22}$$

and so it remains to bound the norm of the deformed hopping matrix, $\|\tilde{J}\|$.

To this end, observe that $f(s)$ satisfies $|f(s) - f(s')| \leq \min(2, 2\,|s - s'|)$ for all $s, s' \in \mathbb{R}$ (see (S10)). Using this and the fact that $s(x)$ is 1-Lipschitz, cf. (S7), we have

$$\left|f(\tfrac{s(x)}{d}) - f(\tfrac{s(y)}{d})\right| \leq \min(2, \tfrac{2}{d}\,|x - y|), \quad x,\, y \in \Lambda.$$

Recalling that $\mu = \frac{da}{2}$, this implies

$$\left|\tilde{J}_{xy}\right| \leq |J_{xy}| \sinh(a \min\{d, |x - y|\}). \tag{S23}$$

By the Schur test for matrix norms and the short-range Assumption (i),

$$\|\tilde{J}\| \leq \max_{x\in\Lambda} \sum_{y\in\Lambda} |J_{xy}| \sinh(a|x - y|) \leq av.$$

Combining this estimate with (S21) and (S22) proves the lemma. □

We now have all the ingredients in place to prove our main result.

*Proof of Theorem 1.* Combining (S16), (S17), (S18) and (S19) and recalling that $\mu = \frac{a}{2} d_{XY}$, we find that

$$\begin{aligned} &\left\|P_{\bar{N}_X \geq \beta} U_t P_{\bar{N}_Y \geq 1-\alpha}\right\| \\ \leq& \exp(2\mu(\beta - \alpha)N) \exp(-|t|vaN) \\ =& \exp\big(-aN\big[(\beta - \alpha)d_{XY} - v|t|\big]\big), \end{aligned}$$

which proves (8). □

We remark that for time-dependent Hamiltonian $H(t)$ satisfying Assumptions (i) and (ii) uniformly for all times, inequalities (S21) – (S22) remain valid, and therefore Lemma 8 generalizes to $H(t)$, upon replacing $e^{-\mathrm{i}Ht}$ by the usual time-ordered propagator

$$U(t, 0) = \mathcal{T} \exp\left(\int_0^t H(s)\mathrm{d}s\right).$$

Since Lemma 8 is the only place where propagator estimate is involved (see (S16)), the conclusion of Theorem 1 extends to $H(t)$, with $e^{-\mathrm{i}Ht}$ replaced by $U(t, 0)$ in eq. (8).