

RIEMANNIAN FLOWS AND ADIABATIC LIMITS

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ABSTRACT. We show the convergence properties of the eigenvalues of the Dirac operator on a spin manifold with a Riemannian flow when the metric is collapsed along the flow.

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1. INTRODUCTION

Many researchers have studied the spectrum of the Laplacian and Dirac-type operators on families of manifolds where the metric is collapsed. We point out in particular the references [9], [12], [18], where the behavior of the spectrum of Laplacians on Riemannian submersions are noted under collapse of the fiber metrics. In [22], R. R. Mazzeo and R. B. Melrose related the properties of the Laplace eigenvalues under adiabatic limits in a Riemannian fiber bundle to the Leray spectral sequence, and J. A. Álvarez-López and Y. Kordyukov extended this analysis in [2] to the more general case of Riemannian foliations; see [20] for an exposition and further references. Adiabatic limits of the eta invariants of Dirac operators have also been considered, as in [27], [6], and [10].

In [4], B. Ammann and C. Bär examined the eigenvalues of the Dirac operator of circle bundles over a closed Riemannian manifold M/S^1 , such that the bundle projection is a Riemannian submersion. They found that as the metric is changed such that the lengths of the circles collapse to zero, the eigenvalues separate into two categories: those that converge to the eigenvalues of the base (quotient) manifold which correspond to the *projectable spinors* — for which the Lie derivative is zero in the direction of the fibers — and those eigenvalues that go to infinity, corresponding to *non-projectable spinors*. The main idea is to decompose the Lie derivative of any spinor field on M into finite-dimensional eigenspaces V_k ($k \in \mathbb{Z}$), and such a decomposition is preserved by the Dirac operator. This comes from the representation of the Lie group S^1 on the spinor bundle on M . In a second step, they decompose the Dirac operator of the whole manifold M into a horizontal and vertical Dirac operator and a zeroth order term. It turns out that the horizontal Dirac operator commutes with the Lie derivative, while the vertical part anticommutes. This allows the researchers to compute explicitly the eigenvalues of the Dirac operator on M on each eigenspace V_k in terms of k . Here the zeroth order term does not contribute in the adiabatic limit, since it is a bounded operator and tends to zero with the length of the fibers. In [3], B. Ammann extended the result above to the case where the circles form a more general Riemannian submersion with projectable spin structures over a base manifold. Also, in [24], F. Pfäffle studied the degeneration of Dirac eigenvalues in a sequence of compact spin hyperbolic manifolds in the case the limit has discrete Dirac spectrum. We also mention

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the work of J. Lott in [21], where the limit of a general Dirac-type operator is studied under a collapse for which the diameter and sectional curvature are bounded. In this case, the spectrum of the Dirac operator converges to the spectrum of a limiting first order operator.

In this paper, we consider a particular case of foliations, namely *Riemannian flows*. On a Riemannian manifold (M, g) , a Riemannian flow is a foliation of 1-dimensional leaves given by the integral curves of a unit vector field ξ such that g is a bundle-like metric. This means the Lie derivative of the transverse metric in the direction of ξ vanishes. Examples of such flows are those given by Killing vector fields and Sasakian manifolds. Those are called taut (meaning the mean curvature form is exact), but examples of nontaut Riemannian flows exist (see, for example, [8]).

We now take the adiabatic limit of the Riemannian flow, and in our situation it is often not the case that the limit is a manifold. This means we consider the bundle-like metric

$$g_f = f^2 \xi^* \otimes \xi^* + g_{\xi^\perp},$$

where f is a positive basic function on M , and we prove that the eigenvalues of the Dirac operator on (M, g_f) corresponding to basic sections tend to those of the basic Dirac operator D_b , which is morally the Dirac operator of the local quotients in the foliation charts; see the next section for details. We point out that our case does not require the leaves to be circles, unlike the situation in [4] or in [3]. Also, we prove that when the flow is taut, the eigenvalues from the L^2 -orthogonal complement of the space of basic sections of the spinor bundle go to $\pm\infty$. The main difference between our case and the one in [4] is that there is not necessarily a circle action on the manifold M , which mainly means that the L^2 -decomposition of the Lie derivative in the direction of the flow cannot carry over. Moreover, the leafwise Dirac operator could fail to have discrete spectrum. We also mention the work of P. Jammes in [19], where he considered adiabatic limits of Riemannian flows, similar to our setting, and examined their effect on the eigenvalues of the Laplacian.

In Section 2, we provide preliminary details on spin Riemannian flows and in particular define the leafwise Dirac operator $D_{\mathcal{F}}$ and the symmetric transversal Dirac operator D_Q ($Q = \xi^\perp = N\mathcal{F}$). In Lemma 2.8, we express the anticommutator of these operators in terms of the mean curvature. We show the operator $D_{\mathcal{F}}$ is symmetric, and its kernel is the L^2 -closure of the space of basic sections (see Proposition 2.7). In Corollary 2.9, we prove that when the flow is minimal, the spectrum of $D_{\mathcal{F}}$ contains a countable number of real eigenvalues, and there exists a complete orthonormal basis of the L^2 spinors consisting of smooth eigensections of $D_{\mathcal{F}}$.

Our main result is Theorem 3.2, where we show that the eigenvalues behave as stated above in the adiabatic limit. In Section 4, we exhibit examples which show interesting behavior of the operators $D_{\mathcal{F}}$ and D_Q . In these examples, which are not fibrations, the operator $D_{\mathcal{F}}$ does not have discrete spectrum, but nonetheless the conclusion of the main theorem is made clear.

2. DIRAC OPERATORS ON RIEMANNIAN FLOWS

Let (M, g) be a closed $(n + 1)$ -dimensional Riemannian manifold, endowed with an oriented Riemannian flow. This means that there exists a unit vector field ξ on M such that the Lie derivative of the transverse metric vanishes: $\mathcal{L}_\xi (g|_{\xi^\perp}) = 0$ (see [25], [8], [26]). Suppose in addition that M is spin, and let D_M be the Dirac operator associated to the spin structure acting on sections of the spinor bundle ΣM , which has a given hermitian metric and metric spin connection.

We wish to construct the basic Dirac operator associated to the induced spin structure on the normal bundle. Since $TM = \mathbb{R}\xi \oplus \xi^\perp$, the pullback of the spin structure on M induces a spin structure on the normal bundle $Q = \xi^\perp = N\mathcal{F}$. In this case, the spinor bundle ΣM is canonically identified with the spinor bundle ΣQ of Q , for n even, and with the direct sum $\Sigma Q \oplus \Sigma Q$ for n odd. The metric on ΣM induces a metric on ΣQ . When n is even, then $i\xi \cdot$ is taken to be the chirality operator, as $(i\xi \cdot)^2 = \text{id}$, and we let $(\Sigma Q)^\pm$ be the eigenspaces associated to the ± 1 eigenvalues, with Clifford multiplication \cdot_Q defined by $Z \cdot_Q \varphi = Z \cdot_M \varphi$ for $Z \in \Gamma(Q)$, $\varphi \in \Gamma(\Sigma Q)$. When n is odd, the Clifford multiplications \cdot_M on ΣM and \cdot_Q on $\Sigma Q := \Sigma M^+$ are related by $Z \cdot_Q \varphi = Z \cdot_M \xi \cdot_M \varphi$ (as in [5]). Therefore, by using the above identification, the spinor

connections $\nabla^{\Sigma M}$ and $\nabla^{\Sigma Q}$ are related by the following relations (see [13, formula 4.8]). For all $Z \in \Gamma(Q)$,

$$\begin{aligned}\nabla_{\xi}^{\Sigma M} \varphi &= \nabla_{\xi}^{\Sigma Q} \varphi + \frac{1}{2} \Omega \cdot_M \varphi + \frac{1}{2} \xi \cdot_M \kappa \cdot_M \varphi, \\ \nabla_Z^{\Sigma M} \varphi &= \nabla_Z^{\Sigma Q} \varphi + \frac{1}{2} \xi \cdot_M (\nabla_Z^M \xi) \cdot_M \varphi,\end{aligned}\tag{2.1}$$

where the Euler form Ω is the 2-form given for all $Y, Z \in \Gamma(Q)$ by $\Omega(Y, Z) = g(\nabla_Y^M \xi, Z)$ and $\kappa^\# = \nabla_{\xi}^M \xi \in \Gamma(Q)$ is the mean curvature vector field of the flow. The one-form κ is also identified with the corresponding Clifford algebra element. We identify Ω with the associated element of the Clifford algebra by $\Omega = \frac{1}{2} \sum_{j=1}^n e^j \wedge (\nabla_{e_j}^M \xi)^\flat = \frac{1}{2} \sum_{j=1}^n e_j \cdot_M (\nabla_{e_j}^M \xi) \cdot_M$ where here and in the following $\{e_j\}_{j=1}^n$ is a local orthonormal frame of $\Gamma(Q)$.

Lemma 2.1. (in [13]) *If $K(X, Y) = X \cdot Y \cdot (\nabla_X^{\Sigma Q} \nabla_Y^{\Sigma Q} - \nabla_Y^{\Sigma Q} \nabla_X^{\Sigma Q} + \nabla_{[X_i, Y]}^{\Sigma Q})$ is the Clifford curvature of ΣQ , then $K(X, Y) = 0$ if $X = \xi$.*

Lemma 2.2. *The transverse connection commutes with the Clifford action of ξ ; that is, $\nabla_X^{\Sigma Q} (\xi \cdot_M \varphi) = \xi \cdot_M \nabla_X^{\Sigma Q} \varphi$ for any spinor field $\varphi \in \Gamma(\Sigma Q)$ and $X \in \Gamma(TM)$. In particular, this means that the spinor field $\xi \cdot_M \varphi$ is basic if and only if φ is basic.*

Proof. We use (2.1). For $Z \in \Gamma(Q)$,

$$\begin{aligned}\nabla_Z^{\Sigma Q} (\xi \cdot_M \varphi) &= \nabla_Z^{\Sigma M} (\xi \cdot_M \varphi) - \frac{1}{2} \xi \cdot_M \nabla_Z^M \xi \cdot_M \xi \cdot_M \varphi \\ &= (\nabla_Z^{\Sigma M} \xi) \cdot_M \varphi + \xi \cdot_M \nabla_Z^{\Sigma M} \varphi - \frac{1}{2} (\nabla_Z^M \xi) \cdot_M \varphi \\ &= \xi \cdot_M \nabla_Z^{\Sigma M} \varphi - \frac{1}{2} \xi \cdot_M \xi \cdot_M (\nabla_Z^M \xi) \cdot_M \varphi \\ &= \xi \cdot_M \nabla_Z^{\Sigma Q} \varphi,\end{aligned}$$

since $\nabla_Z^M \xi$ is orthogonal to ξ . Next,

$$\begin{aligned}\nabla_{\xi}^{\Sigma Q} (\xi \cdot_M \varphi) &= \nabla_{\xi}^{\Sigma M} (\xi \cdot_M \varphi) - \frac{1}{2} \Omega \cdot_M \xi \cdot_M \varphi - \frac{1}{2} \xi \cdot_M \kappa \cdot_M \xi \cdot_M \varphi \\ &= H \cdot_M \varphi + \xi \cdot_M \nabla_{\xi}^{\Sigma M} \varphi - \frac{1}{2} \xi \cdot_M \Omega \cdot_M \varphi - \frac{1}{2} \kappa \cdot_M \varphi \\ &= \xi \cdot_M \nabla_{\xi}^{\Sigma M} \varphi - \frac{1}{2} \xi \cdot_M \Omega \cdot_M \varphi - \frac{1}{2} \xi \cdot_M \xi \cdot_M \kappa \cdot_M \varphi \\ &= \xi \cdot_M \nabla_{\xi}^{\Sigma Q} \varphi.\end{aligned}$$

□

We define the transversal Dirac operator D_Q on $\Gamma(\Sigma Q)$ as

$$D_Q = \sum_{i=1}^n e_i \cdot_Q \nabla_{e_i}^{\Sigma Q} - \frac{1}{2} \kappa \cdot_Q.$$

This differential operator is first-order and transversally elliptic. Using the metric on ΣQ induced from the metric on ΣM , we obtain the L^2 metric on $\Gamma(\Sigma Q)$.

Lemma 2.3. (From [13, p. 31]) *The operator D_Q is self-adjoint on $L^2(\Gamma(\Sigma Q))$.*

The basic Dirac operator D_b is the restriction of

$$PD_Q = \sum_{i=1}^n e_i \cdot_Q \nabla_{e_i}^{\Sigma Q} - \frac{1}{2} \kappa \cdot_Q$$

to the set $\Gamma_b(\Sigma Q)$ of basic sections (sections φ in $\Gamma(\Sigma Q)$ satisfying $\nabla_{\xi}^{\Sigma Q} \varphi = 0$):

$$D_b = PD_Q|_{\Gamma_b(\Sigma Q)}.$$

In the above, $P : L^2(\Gamma(\Sigma Q)) \rightarrow L^2(\Gamma_b(\Sigma Q))$ is the orthogonal projection onto basic sections, and $\kappa_b = P_b \kappa$ where $P_b : L^2(\Omega^*(M)) \rightarrow L^2(\Omega_b^*(M))$ (see [1], [23], [7]). It is always true that P preserves the smooth sections and that κ_b is a closed one-form. Recall that the basic Dirac operator preserves the set of basic sections and is transversally elliptic and essentially self-adjoint (on the basic sections). Therefore, by the spectral theory of transversally elliptic operators, it is a Fredholm operator and has discrete spectrum ([17], [16]). Observe that when κ is a basic form,

$$\kappa_b = \kappa, D_b = D_Q|_{\Gamma_b(\Sigma Q)}.$$

If the mean curvature is not necessarily basic, then

$$\begin{aligned} D_Q &= \sum_{i=1}^n e_i \cdot_Q \nabla_{e_i}^{\Sigma Q} - \frac{1}{2} \kappa \cdot_Q \\ &= \sum_{i=1}^n e_i \cdot_Q \nabla_{e_i}^{\Sigma Q} - \frac{1}{2} \kappa_b \cdot_Q + \frac{1}{2} (\kappa_b - \kappa) \cdot_Q, \\ D_Q|_{\Gamma_b(\Sigma Q)} &= D_b + \frac{1}{2} (\kappa_b - \kappa) \cdot_Q. \end{aligned}$$

Next, we give the relationship between D_M and D_Q on $\Gamma(\Sigma M)$. By (2.1) we have

$$\begin{aligned} D_M &= D_Q - \frac{1}{2} \xi \cdot_M \Omega \cdot_M + \xi \cdot_M \nabla_{\xi}^{\Sigma Q} \text{ for } n \text{ even,} \\ D_M &= \xi \cdot_M (D_Q \oplus (-D_Q)) - \frac{1}{2} \xi \cdot_M \Omega \cdot_M + \xi \cdot_M \left(\nabla_{\xi}^{\Sigma Q \oplus \Sigma Q} \right) \text{ for } n \text{ odd.} \end{aligned} \quad (2.2)$$

Using the formulas above, the restrictions of the Dirac operators D_M and D_b to basic sections are related by

$$\begin{aligned} D_M|_{\Gamma_b(\Sigma Q)} &= D_b + \frac{1}{2} (\kappa_b - \kappa) \cdot_Q - \frac{1}{2} \xi \cdot_M \Omega \cdot_M \text{ for } n \text{ even,} \\ D_M|_{\Gamma_b(\Sigma Q)} &= \xi \cdot_M D_b + \frac{1}{2} \xi \cdot_M (\kappa_b - \kappa) \cdot_Q - \frac{1}{2} \xi \cdot_M \Omega \cdot_M \text{ for } n \text{ odd.} \end{aligned} \quad (2.3)$$

For n even, respectively n odd, and for any basic spinor field φ , we have that $D_b(\xi \cdot_M \varphi) = -\xi \cdot_M D_b(\varphi)$, respectively $D_b(\xi \cdot_M \varphi) = \xi \cdot_M D_b(\varphi)$. Hence, the spectrum of D_b is symmetric about 0 for n even.

Observe that Rummmler's formula is

$$\begin{aligned} d(\xi^*) &= -\kappa \wedge \xi^* + \varphi_0 \\ &= \xi^* \wedge \nabla_{\xi}^M \xi^* + \sum_{j=1}^n e^j \wedge \nabla_{e_j}^M \xi^* \\ &= -\kappa \wedge \xi^* + 2\Omega, \end{aligned}$$

so that $\varphi_0 = 2\Omega$. Since φ_0 is always of type $(2, 0)$ in $\Lambda^* Q \wedge \Lambda^* T\mathcal{F}$ for flows, we see $\Omega \in \Gamma(M, \Lambda^2 Q)$.

Lemma 2.4. *If κ is a basic form, then Ω is basic.*

Proof. We see that

$$i_{\xi} \Omega = \frac{1}{2} i_{\xi} (d(\xi^*) + \kappa \wedge \xi^*) = 0,$$

which is clear since $\varphi_0 = 2\Omega$ is of type $(2, 0)$ in $\Lambda^* Q \wedge \Lambda^* T\mathcal{F}$. Next, since κ is a basic closed form,

$$\begin{aligned} i_{\xi} d\Omega &= \frac{1}{2} i_{\xi} ((d\kappa) \wedge \xi^* - \kappa \wedge d(\xi^*)) \\ &= \frac{1}{2} i_{\xi} (-\kappa \wedge (-\kappa \wedge \xi^* + 2\Omega)) \\ &= \frac{1}{2} i_{\xi} (-\kappa \wedge (2\Omega)) = 0. \end{aligned}$$

□

Remark 2.5. *The calculation above also shows that in the case where κ is not necessarily basic,*

$$i_\xi d\Omega = \frac{1}{2}d_{1,0}\kappa = \frac{1}{2}d_{1,0}(\kappa - \kappa_b).$$

For the case when $\kappa = \kappa_b$, by the equations above for D_M when n is even, we see that D_M preserves the basic sections of $\Sigma M = \Sigma Q$, and since D_M is orthogonally diagonalizable over $L^2(\Sigma M) = L^2(\Sigma Q)$, there exists an orthonormal basis of $L^2(\Gamma_b(\Sigma Q))$ consisting of eigensections of D_M . Similarly, there exists an orthonormal basis of $L^2(\Gamma_b(\Sigma Q))^\perp$ consisting of eigensections of D_M . The analogous facts are true for n odd and $D_M|_{\Gamma_b(\Sigma Q \oplus \Sigma Q)}$ and $D_M|_{(\Gamma_b(\Sigma Q \oplus \Sigma Q))^\perp}$. We have shown the following.

Lemma 2.6. *Suppose that κ is basic. Then the operator D_M decomposes as $D_M|_{\Gamma_b(\Sigma Q)} \oplus D_M|_{\Gamma_b(\Sigma Q)^\perp}$ as an L^2 -orthogonal direct sum, when n is even. It decomposes as $D_M|_{\Gamma_b(\Sigma Q \oplus \Sigma Q)} \oplus D_M|_{(\Gamma_b(\Sigma Q \oplus \Sigma Q))^\perp}$ when n is odd.*

We call the operator $D_{\mathcal{F}} := \xi \cdot_M \nabla_\xi^{\Sigma Q}$ acting on $\Gamma(\Sigma Q)$ the **tangential Dirac operator**.

Proposition 2.7. *The operator $D_{\mathcal{F}}$ is symmetric, and $\ker D_{\mathcal{F}} = L^2(\Gamma_b(\Sigma Q))$.*

Proof. For any (smooth) spinor fields ψ and φ , letting (\bullet, \bullet) be the pointwise inner product,

$$\begin{aligned} (D_{\mathcal{F}}\psi, \varphi) &= \left(\xi \cdot_M \nabla_\xi^{\Sigma Q} \psi, \varphi \right) \\ &= \left(\nabla_\xi^{\Sigma Q} (\xi \cdot_M \psi), \varphi \right) \end{aligned}$$

by Lemma 2.2. Then

$$\begin{aligned} (D_{\mathcal{F}}\psi, \varphi) &= \xi (\xi \cdot_M \psi, \varphi) - \left(\xi \cdot_M \psi, \nabla_\xi^{\Sigma Q} \varphi \right) \\ &= \xi (\xi \cdot_M \psi, \varphi) + \left(\psi, \xi \cdot_M \nabla_\xi^{\Sigma Q} \varphi \right) \\ &= \xi (\xi \cdot_M \psi, \varphi) + (\psi, D_{\mathcal{F}}\varphi). \end{aligned}$$

Observe that, letting f be the function $f = (\xi \cdot_M \psi, \varphi)$,

$$\int_M \xi(f) = - \int_M f \operatorname{div}(\xi) = 0,$$

since ξ generates a Riemannian flow and thus is divergence-free. Thus, by integrating $\langle D_{\mathcal{F}}\psi, \varphi \rangle = \langle \psi, D_{\mathcal{F}}\varphi \rangle$. Next, if $D_{\mathcal{F}}(\varphi) = 0$ for some section $\varphi \in \Gamma(\Sigma Q)$, then

$$\xi \cdot_M 0 = \xi \cdot_M \xi \cdot_M \nabla_\xi^{\Sigma Q} \varphi = -\nabla_\xi^{\Sigma Q} \varphi,$$

so φ is basic. □

Lemma 2.8. *We have $D_Q D_{\mathcal{F}} = -D_{\mathcal{F}} D_Q + \kappa \cdot_M D_{\mathcal{F}} = -D_{\mathcal{F}}(D_Q + \kappa \cdot_M)$.*

Proof. We see that, letting e_1, \dots, e_n be a local orthonormal frame for Q ,

$$\begin{aligned} D_Q \left(\xi \cdot_M \nabla_\xi^{\Sigma Q} \right) &= \left(\sum_{i=1}^n e_i \cdot_Q \nabla_{e_i}^{\Sigma Q} - \frac{1}{2} \kappa \cdot_Q \right) \left(\xi \cdot_M \nabla_\xi^{\Sigma Q} \right) \\ &= \sum_{i=1}^n \left(e_i \cdot_M \xi \cdot_M \nabla_{e_i}^{\Sigma Q} \nabla_\xi^{\Sigma Q} \right) - \frac{1}{2} \kappa \cdot_M \xi \cdot_M \nabla_\xi^{\Sigma Q}, \end{aligned}$$

by Lemma 2.2. Then

$$D_Q \left(\xi \cdot_M \nabla_\xi^{\Sigma Q} \right) = \sum_{i=1}^n \left(K(e_i, \xi) + e_i \cdot_M \xi \cdot_M \left(\nabla_\xi^{\Sigma Q} \nabla_{e_i}^{\Sigma Q} + \nabla_{[e_i, \xi]}^{\Sigma Q} \right) \right) - \frac{1}{2} \kappa \cdot_M \xi \cdot_M \nabla_\xi^{\Sigma Q}.$$

By Lemma 2.1, $K(e_i, \xi) = 0$ for every i . Note that $[e_i, \xi] \in T\mathcal{F}$ so that

$$\begin{aligned} [e_i, \xi] &= \langle [e_i, \xi], \xi \rangle \xi = \langle \nabla_{e_i} \xi - \nabla_\xi e_i, \xi \rangle \xi \\ &= \frac{1}{2} e_i \langle \xi, \xi \rangle - \langle \nabla_\xi e_i, \xi \rangle \xi \\ &= \langle e_i, \nabla_\xi \xi \rangle \xi = \kappa(e_i) \xi. \end{aligned}$$

Thus,

$$\begin{aligned}
D_Q \left(\xi \cdot_M \nabla_\xi^{\Sigma Q} \right) &= \sum_{i=1}^n e_i \cdot_M \xi \cdot_M \left(\nabla_\xi^{\Sigma Q} \nabla_{e_i}^{\Sigma Q} + \kappa(e_i) \nabla_\xi^{\Sigma Q} \right) - \frac{1}{2} \kappa \cdot_M \xi \cdot_M \nabla_\xi^{\Sigma Q} \\
&= \sum_{i=1}^n e_i \cdot_M \xi \cdot_M \nabla_\xi^{\Sigma Q} \nabla_{e_i}^{\Sigma Q} + \sum_{i=1}^n \kappa(e_i) e_i \cdot_M \xi \cdot_M \nabla_\xi^{\Sigma Q} - \frac{1}{2} \kappa \cdot_M \xi \cdot_M \nabla_\xi^{\Sigma Q} \\
&= \sum_{i=1}^n e_i \cdot_M \xi \cdot_M \nabla_\xi^{\Sigma Q} \nabla_{e_i}^{\Sigma Q} + \frac{1}{2} \kappa \cdot_M \xi \cdot_M \nabla_\xi^{\Sigma Q} \\
&= - \left(\xi \cdot_M \nabla_\xi^{\Sigma Q} \right) \left(e_i \cdot_Q \nabla_{e_i}^{\Sigma Q} \right) - \left(\xi \cdot_M \nabla_\xi^{\Sigma Q} \right) \frac{1}{2} \kappa \cdot_M \\
&= - \left(\xi \cdot_M \nabla_\xi^{\Sigma Q} \right) D_Q - \left(\xi \cdot_M \nabla_\xi^{\Sigma Q} \right) \kappa \cdot_M \\
&= - \left(\xi \cdot_M \nabla_\xi^{\Sigma Q} \right) D_Q + \kappa \cdot_M \left(\xi \cdot_M \nabla_\xi^{\Sigma Q} \right).
\end{aligned}$$

□

Corollary 2.9. *If $\kappa = 0$, then the spectrum of $D_{\mathcal{F}}$ contains a countable number of real eigenvalues, and there exists a complete orthonormal basis of $L^2(\Sigma Q)$ consisting of smooth eigensections of $D_{\mathcal{F}}$.*

Proof. If $\kappa = 0$, we consider the essentially self-adjoint, elliptic operator

$$L = D_Q + D_{\mathcal{F}}.$$

There exists a complete orthonormal basis of $L^2(\Sigma Q)$ consisting of smooth eigensections of L , and each eigenspace is finite-dimensional. By Lemma 2.8, $D_Q D_{\mathcal{F}} + D_{\mathcal{F}} D_Q = 0$, so $L^2 = D_Q^2 + D_{\mathcal{F}}^2$, and $D_{\mathcal{F}}$ commutes with L^2 . Then $D_{\mathcal{F}}$ restricts to a self-adjoint operator on the finite-dimensional eigenspaces of L^2 and thus has pure real eigenvalue spectrum restricted to those subspaces. The result follows. □

Remark 2.10. *As shown in Example 4.1, it is possible that the spectrum of $D_{\mathcal{F}}$ is \mathbb{R} but also contains a countable number of real eigenvalues, whose smooth eigensections form a complete orthonormal basis of $L^2(\Sigma Q)$.*

Remark 2.11. *Suppose instead that $\kappa = df$. Note that this means that f is a basic function, since otherwise κ would have ξ^* components. Then we modify the metric on M so that $\langle \xi, \xi \rangle' = e^{2f}$ but otherwise keep everything the same. Then the leafwise volume form is*

$$\chi' = e^f \xi^*,$$

and

$$d\chi' = -(\kappa - df) \wedge \chi + \varphi_0 = \varphi_0,$$

so that $\kappa' = 0$. Then in the new metric $L' = D'_Q + D'_{\mathcal{F}}$ has the same properties, and $D'_{\mathcal{F}}$ commutes with $(L')^2$. But observe that $D'_{\mathcal{F}} = e^{-f} D_{\mathcal{F}}$ because for all $\psi \in \Gamma(\Sigma Q)$, $\xi' \cdot'_M \psi = \xi \cdot_M \psi$, and $\xi' = e^{-f} \xi$. In examples it appears that $D_{\mathcal{F}}$ does not have a complete basis of eigenvectors, even though $D'_{\mathcal{F}}$ does.

3. ADIABATIC LIMITS

In this section, given the bundle-like metric g on (M, \mathcal{F}) , we consider the family of metrics

$$g_f = f^2 \xi^* \otimes \xi^* + g_{\xi^\perp},$$

where f is a positive basic function on M . This metric is bundle-like for the foliation and has the same transverse metric as the original metric, and $\xi_f = \frac{1}{f} \xi$ is the corresponding unit tangent vector field of the foliation.

Lemma 3.1. *The spaces $L^2(\Gamma_b(\Sigma Q))$ and $L^2(\Gamma_b(\Sigma Q))^\perp$ are the same for any such metric g_f .*

Proof. The space $\Gamma_b(\Sigma Q)$ does not depend on the metric and thus is independent of f . Since f is a smooth positive function, we see easily that $L^2(\Gamma_b(\Sigma Q))$ is also independent of f . Next, suppose that α is orthogonal to any given $\beta \in \Gamma_b(\Sigma Q)$ with respect to the old metric. Then if we let (\bullet, \bullet) denote the original pointwise metric on ΣQ , we have that (α, β) is independent of f since β has no components with ξ^* . Also, $\nu \wedge \xi^*$ is the original volume form on M with ν the transverse volume form. In the new metric, $f\nu \wedge \xi^*$ is the volume form. Then

$$\langle \alpha, \beta \rangle_f = \int (\alpha, \beta) f\nu \wedge \xi^* = \int (\alpha, f\beta) \nu \wedge \xi^* = 0$$

since $f\beta$ is also a basic form. Therefore, we also have that the space $L^2(\Gamma_b(\Sigma Q))^\perp$ is independent of f . \square

Recall that the basic component κ_b of the mean curvature form κ is always a closed form and defines a class $[\kappa_b]$ in basic cohomology $H_b^1(M, \mathcal{F})$ that is invariant of the transverse Riemannian foliation structure and bundle-like metric (see [1]). Such a Riemannian foliation is taut if and only if $[\kappa_b] = 0$. Also, recall from [11]: given any Riemannian foliation (M, \mathcal{F}) with bundle-like metric, there exists another bundle-like metric on M with identical transverse metric such that the mean curvature is basic.

Theorem 3.2. *Let M be a closed Riemannian spin manifold, endowed with an oriented Riemannian flow given by the unit vector field ξ . Suppose that the mean curvature form κ is basic. Let $D_{M,f}$ be the Dirac operator associated to the metric g_f and spin structure. The eigenvalues of $D_{M,f}$ are $\{\lambda_j(f)\}_{j=1}^\infty \cup \{\mu_k(f)\}_{k=1}^\infty$, corresponding to the restrictions of $D_{M,f}$ to $L^2(\Gamma_b(\Sigma Q))$ and $L^2(\Gamma_b(\Sigma Q))^\perp$, respectively. Then these eigenvalues can be indexed such that*

- (1) (a) (n even) as $f \rightarrow 0$, $\lambda_j(f)$ converges to eigenvalues of the basic Dirac operator D_b .
 - (b) (n odd) as $f \rightarrow 0$, $\lambda_j(f)$ converges to the eigenvalues of the basic Dirac operators $\pm D_b$.
- In the cases above, the convergence is uniform in j .*
- (2) *If \mathcal{F} is taut (i.e. $\kappa = dh$ for a function h), the nonzero eigenvalues in $\{\mu_k(f)\}$ approach $\pm\infty$ as $f \rightarrow 0$ uniformly with $\frac{df}{f}$ uniformly bounded.*

Proof. (1a) Observe that $\xi_f = \frac{1}{f}\xi$, $\xi_f^* = f\xi$, $\kappa_f = \kappa - \frac{df}{f}$ and $\Omega_f = f\Omega$. For the case where n is even, from (2.2),

$$\begin{aligned} D_{M,f} &= D_{Q,f} - \frac{1}{2}\xi_f \cdot_{M,f} \Omega_f \cdot_{M,f} + \xi_f \cdot_{M,f} \nabla_{\xi_f}^{\Sigma Q} \\ &= D_{Q,f} - \frac{f}{2}\xi_f \cdot_{M,f} \Omega \cdot_{M,f} + \frac{1}{f}\xi_f \cdot_{M,f} \nabla_{\xi}^{\Sigma Q}. \end{aligned}$$

Then for any basic spinor ψ ,

$$D_{M,f}(\psi) = D_{b,f}\psi - \frac{f}{2}\xi_f \cdot_{M,f} \Omega \cdot_M \psi, \quad (3.1)$$

since κ is basic. Thus,

$$\begin{aligned} \frac{\|(D_{M,f} - D_{b,f})\psi\|_{L^2}}{\|\psi\|_{L^2}} &= \frac{\left\| \frac{f}{2}\xi_f \cdot_{M,f} \Omega \cdot_M \psi \right\|_{L^2}}{\|\psi\|_{L^2}} \leq \frac{\left\| \frac{f}{2}\Omega \cdot_M \psi \right\|_{L^2}}{\|\psi\|_{L^2}} \\ &\leq \frac{\max|f|}{2}C, \end{aligned}$$

where C is the operator norm of $(\Omega \cdot_M)$. Thus

$$\frac{\|(D_{M,f} - D_{b,f})\psi\|_{L^2}}{\|\psi\|_{L^2}} \rightarrow 0$$

uniformly in f and ψ , hence

$$\|(D_{M,f} - D_{b,f})\|_{Op} \leq \frac{\max|f|}{2}C \rightarrow 0$$

as $f \rightarrow 0$ uniformly. Since the eigenvalues of $D_{b,f}$ are constant in f and are those of D_b (see [14]), the eigenvalues of $D_{M,f}$ converge to those of D_b , because the spectrum is continuous as a function of the operator norm (see Lemma 5.1 in the appendix).

(1b) For the case where n is odd, from (2.2),

$$\begin{aligned} D_{M,f} &= \xi \cdot_{M,f} (D_{Q,f} \oplus (-D_{Q,f})) - \frac{1}{2} \xi_f \cdot_{M,f} \Omega_f \cdot_{M,f} + \xi_f \cdot_{M,f} \nabla_{\xi_f}^{\Sigma Q \oplus \Sigma Q} \\ &= \xi \cdot_{M,f} (D_{Q,f} \oplus (-D_{Q,f})) - \frac{f}{2} \xi_f \cdot_{M,f} \Omega \cdot_{M,f} + \frac{1}{f} \xi_f \cdot_{M,f} \nabla_{\xi}^{\Sigma Q \oplus \Sigma Q}. \end{aligned}$$

Then, since κ is basic, for any basic spinor $\psi = (\psi_1, \psi_2) \in \Gamma_b(\Sigma Q \oplus \Sigma Q)$,

$$\begin{aligned} D_{M,f}(\psi) &= \xi \cdot_{M,f} (D_{Q,f} \psi_1 \oplus (-D_{Q,f} \psi_2)) - \frac{f}{2} \xi_f \cdot_{M,f} \Omega \cdot_{M,f} (\psi_1, \psi_2) + \frac{1}{f} \xi_f \cdot_{M,f} \nabla_{\xi}^{\Sigma Q} (\psi_1, \psi_2) \\ &= \xi \cdot_{M,f} (D_{b,f} \psi_1, -D_{b,f} \psi_2) - \frac{f}{2} \xi_f \cdot_{M,f} \Omega \cdot_M (\psi_1, \psi_2). \end{aligned}$$

Thus,

$$\begin{aligned} \frac{\|D_{M,f}(\psi_1, \psi_2) - \xi \cdot_{M,f} (D_{b,f} \psi_1, -D_{b,f} \psi_2)\|_{L^2}}{\|\psi\|_{L^2}} &= \frac{\left\| \frac{f}{2} \xi_f \cdot_{M,f} \Omega \cdot_M \psi \right\|_{L^2}}{\|\psi\|_{L^2}} \leq \frac{\left\| \frac{f}{2} \Omega \cdot_M \psi \right\|_{L^2}}{\|\psi\|_{L^2}} \\ &\leq \frac{\max |f|}{2} C, \end{aligned}$$

where C is the operator norm of $(\Omega \cdot_M)$. The same conclusions follow.

(2') Now we suppose the particular case that $\kappa = 0$. Then $\kappa_f = -\frac{df}{f}$. For the case where n is even,

$$\begin{aligned} D_{M,f} &= D_{Q,f} - \frac{f}{2} \xi \cdot_M \Omega \cdot_M + \frac{1}{f} \xi \cdot_M \nabla_{\xi}^{\Sigma Q} \\ &= \sum_{i=1}^n e_i \cdot_Q \nabla_{e_i}^{\Sigma Q} + \frac{1}{2} \left(\frac{df}{f} \right) \cdot_Q - \frac{f}{2} \xi \cdot_M \Omega \cdot_M + \frac{1}{f} D_{\mathcal{F}}. \end{aligned}$$

We consider the elliptic operator $L_f = D_{\text{tr}} + \frac{1}{f} D_{\mathcal{F}} = D_Q + \frac{1}{f} D_{\mathcal{F}}$, which is self-adjoint with respect to the original metric and therefore has discrete real spectrum. Then if $*$ is used as the adjoint with respect to the $L^2(M, g_f)$ metric,

$$\begin{aligned} L_f^* L_f &= \left(D_{\text{tr}} + \frac{1}{f} D_{\mathcal{F}} \right)^* \left(D_{\text{tr}} + \frac{1}{f} D_{\mathcal{F}} \right) \\ &= \left(D_{\text{tr}}^* + \frac{1}{f} D_{\mathcal{F}}^* \right) \left(D_{\text{tr}} + \frac{1}{f} D_{\mathcal{F}} \right) \\ &= \left(D_{\text{tr}} + \frac{df}{f} \cdot_Q + \frac{1}{f} D_{\mathcal{F}} \right) \left(D_{\text{tr}} + \frac{1}{f} D_{\mathcal{F}} \right) \\ &= D_{\text{tr}}^2 + \frac{df}{f} \cdot_Q D_{\text{tr}} + \frac{1}{f} D_{\mathcal{F}} D_{\text{tr}} + D_{\text{tr}} \circ \frac{1}{f} D_{\mathcal{F}} + \frac{df}{f^2} \cdot_Q D_{\mathcal{F}} + \frac{1}{f^2} D_{\mathcal{F}}^2 \\ &= D_{\text{tr}}^2 + \frac{df}{f} \cdot_Q D_{\text{tr}} + \frac{1}{f} D_{\mathcal{F}} D_{\text{tr}} - \frac{df}{f^2} \cdot_Q D_{\mathcal{F}} + \frac{1}{f} D_{\text{tr}} D_{\mathcal{F}} + \frac{df}{f^2} \cdot_Q D_{\mathcal{F}} + \frac{1}{f^2} D_{\mathcal{F}}^2 \\ &= D_{\text{tr}}^2 + \frac{df}{f} \cdot_Q D_{\text{tr}} + \frac{1}{f^2} D_{\mathcal{F}}^2 \end{aligned}$$

where $D_{\text{tr}} = \sum_{i=1}^n e_i \cdot_Q \nabla_{e_i}^{\Sigma Q}$, which is self-adjoint with respect to the original metric. Clearly $L_f^* L_f$ is nonnegative, elliptic, and self-adjoint with respect to the new metric and thus has discrete spectrum. The operator $D_{\mathcal{F}}$ restricts to the eigenspaces of $L_f^* L_f$ since they commute. Indeed, $D_{\mathcal{F}}$ anticommutes with D_{tr} and with $\frac{df}{f^2} \cdot_Q$ and commutes with $\frac{1}{f}$. By Corollary 2.9, we may restrict to an eigenspace of $D_{\mathcal{F}}$ corresponding to an eigenvalue $\alpha \neq 0$ (since we are only considering antibasic sections now), and we see that

such an eigenvalue, normalized antibasic eigensection pair λ_f, ψ_f satisfies

$$\begin{aligned}
\langle L_f^* L_f \psi_f, \psi_f \rangle_f &= \left\langle \left(D_{\text{tr}}^2 + \frac{df}{f} \cdot_Q D_{\text{tr}} + \frac{1}{f^2} D_{\mathcal{F}}^2 \right) \psi_f, \psi_f \right\rangle_f \\
&= \left\langle \left(D_{\text{tr}}^2 + \frac{df}{f} \cdot_Q D_{\text{tr}} + \frac{1}{f^2} \alpha^2 \right) \psi_f, \psi_f \right\rangle_f \\
&= \left\langle \left(\frac{1}{f} D_{\text{tr}} (f D_{\text{tr}}) + \frac{1}{f^2} \alpha^2 \right) \psi_f, \psi_f \right\rangle_f \\
&= \langle D_{\text{tr}} (f D_{\text{tr}}) \psi_f, \psi_f \rangle_f + \left\langle \frac{1}{f^2} \alpha^2 \psi_f, \psi_f \right\rangle_f \\
&= \langle D_{\text{tr}} \psi_f, D_{\text{tr}} \psi_f \rangle_f + \alpha^2 \left\langle \frac{1}{f^2} \psi_f, \psi_f \right\rangle_f \\
&\geq \frac{\alpha^2}{\max(f^2)} \rightarrow \infty
\end{aligned}$$

as $f \rightarrow 0$ uniformly. Thus, the eigenvalues of $L_f^* L_f$ go to $+\infty$ as $f \rightarrow 0$ uniformly. Since the eigenvalues of $L_f^* L_f$ are precisely the squares of the eigenvalues of L_f , we also get that the eigenvalues of L_f approach $\pm\infty$ as $f \rightarrow 0$ uniformly. Next, observe that

$$\|D_{M,f} - L_f\|_{O_p} = \left\| \frac{1}{2} \left(\frac{df}{f} \right) \cdot_Q - \frac{f}{2} \xi \cdot_M \Omega \cdot_M \right\|_{O_p} \leq \frac{1}{2} \max |f| \max \|\Omega\| + \frac{1}{2} \max \left| \frac{df}{f} \right|,$$

and the right hand side remains bounded as $f \rightarrow 0$ uniformly with $\frac{|df|}{f}$ bounded. Thus, since the spectrum is continuous as a function of the operator norm (see Lemma 5.1), the eigenvalues of $D_{M,f}$ go to $\pm\infty$ as $f \rightarrow 0$ uniformly with $\frac{|df|}{f}$ bounded. The n odd case is similar.

(2) Now, suppose that κ is an exact form, so that $\kappa = dh$ for some function h (which must be basic; otherwise κ would have a ξ^* component). Then we may multiply the leafwise metric by \tilde{f}^2 where $\tilde{f} = \exp(h)$, and then in the new metric $\tilde{\kappa} = 0$. Then, given any positive function f , $g_f = f^2 \xi^* \otimes \xi^* + g_{\xi^\perp} = (f \tilde{f}^{-1})^2 \tilde{f}^2 \xi^* \otimes \xi^* + g_{\xi^\perp}$. Suppose that $f \rightarrow 0$ uniformly with $\frac{df}{f}$ uniformly bounded; then $f \tilde{f}^{-1} \rightarrow 0$ uniformly and $\frac{d(f \tilde{f}^{-1})}{f \tilde{f}^{-1}} = \frac{d(f)}{f} - \frac{d(\tilde{f})}{\tilde{f}}$ is also uniformly bounded. By the result in (2') above, the nonzero eigenvalues in $\{\mu_k(f)\}$ approach $\pm\infty$. \square

Remark 3.3. *Example 4.3 shows that in the case that \mathcal{F} is not taut, the methods of the proof for part (2) do not work. In this example, the only eigenvalue of $D_{\mathcal{F}}$ is 0, corresponding to the basic sections, and yet the spectrum of $D_{\mathcal{F}}$ is \mathbb{R} . So the conclusion of Corollary 2.9 does not hold even though κ is basic. We conjecture that the conclusion (2) is false for general Riemannian foliations.*

4. EXAMPLES

Example 4.1. *Consider $M = T^2 = \mathbb{R}^2 / (2\pi\mathbb{Z})^2$, the Euclidean two-dimensional torus, with a constant linear flow $\xi = a\partial_x + b\partial_y$, where $a^2 + b^2 = 1$. The spinor bundle ΣM is $\mathbb{C}^2 \times M$, and we consider the Clifford multiplication $(c\partial_x + d\partial_y) = \begin{pmatrix} 0 & -c + di \\ c + di & 0 \end{pmatrix}$. The bundle $Q = \xi^\perp = \text{span}\{-b\partial_x + a\partial_y\}$, and $\Sigma Q = \mathbb{C} \times M$. Covariant derivatives are the same as directional derivatives. The standard metric is $g = dx^2 + dy^2$, and we consider the perturbed metric*

$$g_f = g_t = dx^2 + dy^2 + (t^2 - 1)(\xi^*)^2 = dx^2 + dy^2 + (t^2 - 1)(a dx + b dy)^2$$

with $f(t) = t$. Since the foliation for this and the original metric is totally geodesic, $\nabla^{\Sigma M} = \nabla^{\Sigma Q \oplus \Sigma Q}$. Then

$$\begin{aligned} D_{M,t} &= \sum e_j \cdot_M \nabla_{e_j}^{\Sigma M} + \xi_t \cdot_t \nabla_{\xi_t}^{\Sigma M} \\ &= \sum e_j \cdot_M \nabla_{e_j}^{\Sigma M} + \frac{1}{t} \xi \cdot_M \nabla_{\xi}^{\Sigma M} \\ &= D_M + \left(\frac{1}{t} - 1 \right) \xi \cdot_M \nabla_{\xi}^M. \end{aligned}$$

We now compute the eigenvalues of $D_{M,t}$. Observe that

$$\xi \cdot_M \nabla_{\xi}^M = \begin{pmatrix} 0 & -a + bi \\ a + bi & 0 \end{pmatrix} (a\partial_x + b\partial_y).$$

Consider the space $V_{m,n} = \left\{ \begin{pmatrix} c \\ d \end{pmatrix} \exp(i(mx + ny)) : c, d \in \mathbb{C} \right\}$, so that the Hilbert sum $\bigoplus_{m,n \in \mathbb{Z}} V_{m,n} = L^2(\Sigma M)$. We see that

$$\begin{aligned} &D_{M,t} \left(\begin{pmatrix} c \\ d \end{pmatrix} \exp(i(mx + ny)) \right) \\ &= \left(\begin{pmatrix} 0 & -\partial_x + i\partial_y \\ \partial_x + i\partial_y & 0 \end{pmatrix} + \left(\frac{1}{t} - 1 \right) \begin{pmatrix} 0 & -a + bi \\ a + bi & 0 \end{pmatrix} (a\partial_x + b\partial_y) \right) \left(\begin{pmatrix} c \\ d \end{pmatrix} \exp(i(mx + ny)) \right) \\ &= \left(\begin{pmatrix} 0 & -im - n \\ im - n & 0 \end{pmatrix} + \left(\frac{1}{t} - 1 \right) \begin{pmatrix} 0 & -a + bi \\ a + bi & 0 \end{pmatrix} (iam + ibn) \right) \left(\begin{pmatrix} c \\ d \end{pmatrix} \exp(i(mx + ny)) \right). \end{aligned}$$

The matrix is

$$\begin{pmatrix} 0 & -im - n + \frac{1}{t}(-t+1)(-a+ib)(iam+ibn) \\ im - n + \frac{1}{t}(-t+1)(a+ib)(iam+ibn) & 0 \end{pmatrix}.$$

The eigenvalues are $\pm\sqrt{q}$, where

$$q = m^2 + n^2 - (am + bn)^2 + \frac{1}{t^2} (am + bn)^2.$$

So, in the case where $\frac{b}{a}$ is rational, the set of basic sections of ΣM is

$$\left\{ \begin{pmatrix} c \\ d \end{pmatrix} \exp(i(mx + ny)) : c, d \in \mathbb{C}, m, n \in \mathbb{Z}, am + bn = 0 \right\}.$$

Also, $\Sigma Q = \mathbb{C}$, and the basic Dirac operator is $D_b = i\theta$, where $\theta \perp \xi$. It has eigenvalues

$$\left\{ m\sqrt{1 + \frac{b^2}{a^2}} : m \in \mathbb{Z} \right\}$$

with eigensections of the form $\{\exp(i(mx + ny)) : m, n \in \mathbb{Z}, am + bn = 0\}$. Actually, M/\mathcal{F} is a circle of radius $\frac{2\pi}{\sqrt{1 + \frac{b^2}{a^2}}}$. As can be seen above, the eigenvalues $D_{M,t}$ are

$$\pm\sqrt{m^2 + n^2 - (am + bn)^2 + \frac{1}{t^2} (am + bn)^2}$$

with $m, n \in \mathbb{Z}$. The eigenvalues with $am + bn = 0$ are independent of t and trivially converge to the eigenvalues of $D_b \oplus -D_b$. All other eigenvalues go to $\pm\infty$ as $t \rightarrow 0$.

On the other hand, if $\frac{b}{a}$ is irrational, the basic sections of ΣM are $\left\{ \begin{pmatrix} c \\ d \end{pmatrix} : c, d \in \mathbb{C} \right\}$, since each leaf is dense. The basic Dirac operator is the zero operator and only has the eigenvalue 0. Also, since $am + bn \neq 0$ for all $(m, n) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$, the expression above implies that every eigenvalue besides 0 goes to $\pm\infty$ as $t \rightarrow 0$.

These results are consistent with our theorem. We also find the spectrum of the operator

$$\xi \cdot_M \nabla_{\xi}^M = \begin{pmatrix} 0 & -a + bi \\ a + bi & 0 \end{pmatrix} (a\partial_x + b\partial_y).$$

Applied to an element of $\left\{ \begin{pmatrix} c \\ d \end{pmatrix} \exp(i(mx + ny)) : c, d \in \mathbb{C}, m, n \in \mathbb{Z}, am + bn = 0 \right\}$, we get

$$(\xi \cdot_M \nabla_\xi^M) \begin{pmatrix} c \\ d \end{pmatrix} \exp(i(mx + ny)) = (iam + ibn) \begin{pmatrix} 0 & -a + bi \\ a + bi & 0 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} \exp(i(mx + ny)),$$

and the matrix restricted to this subspace is

$$(iam + ibn) \begin{pmatrix} 0 & -a + bi \\ a + bi & 0 \end{pmatrix} = \begin{pmatrix} 0 & (-a + ib)(iam + ibn) \\ (a + ib)(iam + ibn) & 0 \end{pmatrix}.$$

The eigenvalues are obviously $\pm(am + bn)$, so that in the irrational slope case 0 is a limit point of the eigenvalues of $D_{\mathcal{F}}$. In fact, the eigenvalues are dense in \mathbb{R} . Note that the whole spectrum is \mathbb{R} because it is closed, even though there exists an orthonormal basis of $L^2(\Sigma M)$ consisting of eigensections. The problem is that $(D_{\mathcal{F}} - \lambda I)^{-1}$ for any λ not in the spectrum, but this operator is not a bounded operator.

Example 4.2. Consider $M = T^3 = \mathbb{R}^3 / (2\pi\mathbb{Z})^3$, the Euclidean 3-torus, with a constant linear flow $\xi = a\partial_x + b\partial_y + c\partial_z$, where $a^2 + b^2 + c^2 = 1$. The spinor bundle ΣM is $\mathbb{C}^2 \times M$, and we consider the Clifford multiplication $(c\partial_x + d\partial_y + e\partial_z) = \begin{pmatrix} ie & -c + di \\ c + di & -ie \end{pmatrix}$. The bundle $Q = \xi^\perp = \text{span}\{-b\partial_x + a\partial_y, -c\partial_x + a\partial_z\}$, and $\Sigma Q = \mathbb{C}^2 \times M$. Covariant derivatives are the same as directional derivatives. The standard metric is $g = dx^2 + dy^2 + dz^2$, and we consider the perturbed metric

$$g_f = g_t = g + (t^2 - 1)(\xi^*)^2 = dx^2 + dy^2 + (t^2 - 1)(a dx + b dy + cdz)^2$$

with $f(t) = t$. Since the foliation for this and the original metric is totally geodesic, $\nabla^{\Sigma M} = \nabla^{\Sigma Q}$. Then

$$\begin{aligned} D_{M,t} &= \sum e_j \cdot_M \nabla_{e_j}^{\Sigma M} + \xi_t \cdot_t \nabla_{\xi_t}^{\Sigma M} \\ &= \sum e_j \cdot_M \nabla_{e_j}^{\Sigma M} + \frac{1}{t} \xi \cdot_M \nabla_\xi^{\Sigma M} \\ &= D_M + \left(\frac{1}{t} - 1\right) \xi \cdot_M \nabla_\xi^M. \end{aligned}$$

We now compute the eigenvalues of $D_{M,t}$. Observe that

$$\xi \cdot_M \nabla_\xi^M = \begin{pmatrix} ic & -a + bi \\ a + bi & -ic \end{pmatrix} (a\partial_x + b\partial_y + c\partial_z).$$

Consider the space $V_{m,n,k} = \left\{ \begin{pmatrix} r \\ s \end{pmatrix} \exp(i(mx + ny + kz)) : r, s \in \mathbb{C} \right\}$, so that the Hilbert sum $\bigoplus_{m,n,k \in \mathbb{Z}} V_{m,n,k} = L^2(\Sigma M)$.

We see that for $\varphi = \begin{pmatrix} r \\ s \end{pmatrix} \exp(i(mx + ny + kz))$,

$$\begin{aligned} &D_{M,t}\varphi \\ &= \left(\begin{pmatrix} i\partial_z & -\partial_x + i\partial_y \\ \partial_x + i\partial_y & -i\partial_z \end{pmatrix} + \left(\frac{1}{t} - 1\right) \begin{pmatrix} ic & -a + bi \\ a + bi & -ic \end{pmatrix} (a\partial_x + b\partial_y + c\partial_z) \right) \varphi \\ &= \left(\begin{pmatrix} -k & -im - n \\ im - n & k \end{pmatrix} + \left(\frac{1}{t} - 1\right) \begin{pmatrix} ic & -a + bi \\ a + bi & -ic \end{pmatrix} (iam + ibn + ick) \right) \varphi. \end{aligned}$$

One can check that the eigenvalues of $D_{M,t}$ restricted to such sections are

$$\pm \sqrt{k^2 + n^2 + m^2 + \frac{(1-t^2)}{t^2} (am + bn + ck)^2}.$$

As $t \rightarrow 0^+$, then $\lambda \approx \pm \frac{1}{t} |am + bn + ck|$ if $am + bn + ck \neq 0$, and $\lambda = \pm \sqrt{k^2 + n^2 + m^2}$ otherwise. So as $t \rightarrow 0^+$, if $am + bn + ck = 0$ (i.e. basic eigensections of D_M), then the eigenvalues are $\pm \sqrt{k^2 + n^2 + m^2}$ and do not change with t . Otherwise, if $am + bn + ck \neq 0$, then all the eigenvalues go to $\pm\infty$. This is consistent with our theorem.

Example 4.3. Consider the Carrière example from [8] in the 3-dimensional case. This foliation is not taut, and we will show that the spectrum of $D_{\mathcal{F}}$ is all of \mathbb{R} in this case, and its only eigenvalue is 0, corresponding to the basic sections. Choose $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ to be a symmetric matrix in $\mathrm{SL}_2(\mathbb{Z})$, and let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

Note that the eigenvalues of A are $\lambda = \frac{3+\sqrt{5}}{2}$, $\frac{1}{\lambda} = \frac{3-\sqrt{5}}{2}$ corresponding to normalized eigenvectors

$$\begin{aligned} V_1 &= \begin{pmatrix} \frac{\frac{1}{2}\sqrt{5}+\frac{1}{2}}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}} \\ \frac{1}{\sqrt{\frac{1}{2}\sqrt{5}+\frac{5}{2}}} \end{pmatrix} = \begin{pmatrix} 0.85065 \\ 0.52573 \end{pmatrix} =: G\partial_x + K\partial_y, \\ V_2 &= \begin{pmatrix} \frac{-\frac{1}{2}\sqrt{5}+\frac{1}{2}}{\sqrt{-\frac{1}{2}\sqrt{5}+\frac{5}{2}}} \\ \frac{1}{\sqrt{-\frac{1}{2}\sqrt{5}+\frac{5}{2}}} \end{pmatrix} = \begin{pmatrix} -0.52573 \\ 0.85065 \end{pmatrix} =: -K\partial_x + G\partial_y, \end{aligned}$$

respectively. Let the hyperbolic torus $M = \mathbb{T}_A^3$ be the quotient of $\mathbb{T}^2 \times \mathbb{R}$ by the equivalence relation which identifies (m, t) to $(A(m), t+1)$. We may also think of it as $\mathbb{T}^2 \times [0, 1]$ with $(m, 0)$ identified with $(Am, 1)$.

We choose the bundle-like metric so that the vectors V_1, V_2, ∂_t form an orthonormal basis at $t=0$ and in general $\lambda^t V_1, \lambda^{-t} V_2, \partial_t$ form an orthonormal basis for $t \in [0, 1]$. Note that at $t=0$, this is the standard flat metric on the torus. If we use $*$ to denote the adjoint/dual with respect to the $t=0$ metric, the metric is

$$g = dt^2 + \lambda^{-2t} (V_1^*)^2 + \lambda^{2t} (V_2^*)^2.$$

We have that the mean curvature of the flow is $\kappa = \kappa_b = -\log(\lambda) dt$, since $\chi_{\mathcal{F}} = \lambda^t V_2^*$ is the characteristic form and $d\chi_{\mathcal{F}} = \log(\lambda) \lambda^t dt \wedge V_2^* = -\kappa \wedge \chi_{\mathcal{F}}$. We also have that $\varphi_0 = 0$ for this flow.

We choose the trivial spin structure, so that the spin bundle is $M \times \mathbb{C}^2$ with spinor connection, with Clifford multiplication

$$c(\lambda^{-t} V_2) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, c(\lambda^t V_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, c(\partial_t) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

We need to calculate the covariant derivatives of spinors. We calculate for $\xi = e_0 = \lambda^{-t} V_2$, $e_1 = \lambda^t V_1$, $e_2 = \partial_t$.

$$\begin{aligned} [\lambda^{-t} V_2, \lambda^t V_1] &= [e_0, e_1] = 0, \\ [\lambda^{-t} V_2, \partial_t] &= [e_0, e_2] = +(\log \lambda) \lambda^{-t} V_2 = (\log \lambda) e_0, \\ [\lambda^t V_1, \partial_t] &= [e_1, e_2] = -(\log \lambda) \lambda^t V_1 = -(\log \lambda) e_1. \end{aligned}$$

Then by the Koszul formula, the Christoffel symbols are

$$\begin{aligned} \Gamma_{00}^2 &= \langle \nabla_{e_0} e_0, e_2 \rangle = \frac{1}{2} (-\langle [e_0, e_2], e_0 \rangle - \langle [e_0, e_2], e_0 \rangle) = -\log \lambda, \\ \Gamma_{12}^1 &= -\Gamma_{11}^2 = -\Gamma_{02}^1 = -\log \lambda \end{aligned}$$

similarly. Now we use the formula

$$\nabla_X^{\Sigma M} \psi = X(\psi) + \frac{1}{2} \sum_{i < j} \langle \nabla_X^M e_i, e_j \rangle e_i \cdot_M e_j \cdot_M \psi.$$

Then

$$\begin{aligned} \nabla_{\lambda^{-t} V_2}^{\Sigma M} \varphi &= \nabla_{e_0}^{\Sigma M} \varphi = s^{-1} \lambda^{-t} V_2(\varphi) + \frac{1}{2} \sum_{i < j} \langle \nabla_{e_0}^M e_i, e_j \rangle e_i \cdot_M e_j \cdot_M \varphi \\ &= s^{-1} \lambda^{-t} V_2(\varphi) + \frac{\log \lambda}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \varphi = s^{-1} \lambda^{-t} V_2(\varphi) - \frac{\log \lambda}{2} e_1 \cdot_M \varphi, \\ \nabla_{\lambda^t V_1}^{\Sigma M} \varphi &= \nabla_{e_1}^{\Sigma M} \varphi = \lambda^t V_1(\varphi) + \frac{1}{2} \sum_{i < j} \langle \nabla_{e_1}^M e_i, e_j \rangle e_i \cdot_M e_j \cdot_M \varphi \\ &= \lambda^t V_1(\varphi) - \frac{\log \lambda}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \varphi = \lambda^t V_1(\varphi) - \frac{\log \lambda}{2} e_0 \cdot_M \varphi, \\ \nabla_{\partial_t}^{\Sigma M} \varphi &= \partial_t \varphi. \end{aligned}$$

With $\xi = \lambda^{-t}V_2$, the connection satisfies

$$\begin{aligned}\nabla_{\xi}^{\Sigma M}\varphi &= \lambda^{-t}V_2(\varphi) + \frac{\log \lambda}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \varphi \\ &= \nabla_{\xi}^{\Sigma Q}\varphi + \frac{1}{2}\Omega \cdot_M \varphi + \frac{1}{2}\xi \cdot_M \kappa \cdot_M \varphi \\ &= \nabla_{\xi}^{\Sigma Q}\varphi - \frac{\log \lambda}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \varphi \\ &= \nabla_{\xi}^{\Sigma Q}\varphi + \frac{\log \lambda}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \varphi,\end{aligned}$$

so

$$\nabla_{\xi}^{\Sigma Q}\varphi = s^{-1}\lambda^{-t}V_2\varphi.$$

Now we compute

$$\begin{aligned}D_{\mathcal{F}}\varphi &= \xi \cdot_M \nabla_{\xi}^{\Sigma Q}\varphi \\ &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \lambda^{-t}V_2\varphi \\ &= \begin{pmatrix} i\lambda^{-t}V_2 & 0 \\ 0 & -i\lambda^{-t}V_2 \end{pmatrix} \varphi.\end{aligned}$$

So to determine the spectrum of $D_{\mathcal{F}}$, we consider $D_{\mathcal{F}} - \mu I$ and determine when it has a bounded inverse. We apply this to a section of the form

$$\varphi = \begin{pmatrix} a_{bc} \\ f_{bc} \end{pmatrix} e^{2\pi i(bx+cy)}.$$

Then

$$\begin{aligned}(D_{\mathcal{F}} - \mu I)\varphi &= \begin{pmatrix} i\lambda^{-t}V_2 - \mu & 0 \\ 0 & -i\lambda^{-t}V_2 - \mu \end{pmatrix} \begin{pmatrix} a_{bc} \\ f_{bc} \end{pmatrix} e^{2\pi i(bx+cy)} \\ &= \begin{pmatrix} i\lambda^{-t}(2\pi i(-Kb + Gc)) - \mu & 0 \\ 0 & -i\lambda^{-t}(2\pi i(-Kb + Gc)) - \mu \end{pmatrix} \varphi \\ &= \begin{pmatrix} -\lambda^{-t}2\pi(-Kb + Gc) - \mu & 0 \\ 0 & \lambda^{-t}2\pi(-Kb + Gc) - \mu \end{pmatrix} \varphi.\end{aligned}$$

Suppose that μ is actually an eigenvalue of $D_{\mathcal{F}}$. Then φ must satisfy the condition $\varphi(t+1, 2x+y, x+y) = \varphi(t, x, y)$, μ must be constant, and $\mu = \pm\lambda^{-t}2\pi(-Kb + Gc)$. So only $b = c = 0$ is possible, corresponding to the double eigenvalue 0. The eigensections are exactly the sections that depend on t alone, the basic sections. What is in the other part of the spectrum of $D_{\mathcal{F}}$? We have

$$(D_{\mathcal{F}} - \mu I)^{-1}\varphi = \begin{pmatrix} -p - \mu & 0 \\ 0 & p - \mu \end{pmatrix}^{-1} \varphi = \begin{pmatrix} -\frac{1}{p+\mu} & 0 \\ 0 & \frac{1}{p-\mu} \end{pmatrix} \varphi$$

acting on sections of the form φ , which exists as long as $p-\mu \neq 0$ and $p+\mu \neq 0$, where $p = \lambda^{-t}2\pi(-Kb + Gc) = \lambda^{-t}2\pi(-0.52573b + 0.85065c)$ takes on every number in the range

$$\lambda^{-1}2\pi(-0.52573b + 0.85065c) \leq p \leq 2\pi(-0.52573b + 0.85065c);$$

that is,

$$-1.2617b + 2.0415c \leq p \leq -3.3033b + 5.3448c.$$

So μ is in the spectrum if and only if $\pm\mu$ is in the set where $-1.2617b + 2.0415c \leq \mu \leq -3.3033b + 5.3448c$ for any integers b, c . Thus, every $\mu \in \mathbb{R}$ is in the spectrum.

5. APPENDIX

We include the following well-known result for completeness, although it certainly is contained in more general perturbation theory of linear operators in the literature.

Lemma 5.1. *Let A and B be two unbounded, essentially self-adjoint operators with discrete spectrum and the same domain on a Hilbert space such that the eigenspaces according to each eigenvalue are finite-dimensional and the eigenvalues approach ∞ in absolute value. If $\|A - B\|_{op} \leq \varepsilon$ for some $\varepsilon > 0$ and*

$$\dots \leq \lambda_{j-1} \leq \lambda_j \leq \lambda_{j+1} \leq \dots$$

with $j \in \mathbb{Z}$ are the eigenvalues of A , counted with multiplicities. Then there is a numbering of the eigenvalues

$$\dots \leq \mu_{j-1} \leq \mu_j \leq \mu_{j+1} \leq \dots$$

of B such that

$$|\lambda_j - \mu_j| \leq \varepsilon$$

for all j .

Proof. First, we prove the result for the case of nonnegative operators. Let A and B be nonnegative, satisfy $\|A - B\|_{op} \leq \varepsilon$, and have domain \mathcal{D} . For any subspace S of \mathcal{D} ,

$$\sup_{\substack{\alpha \in S \\ \|a\|=1}} \|A\alpha\| \leq \sup_{\substack{\alpha \in S \\ \|a\|=1}} \|(A - B)\alpha\| + \|B\alpha\| \leq \varepsilon + \sup_{\substack{\alpha \in S \\ \|a\|=1}} \|B\alpha\|,$$

so in particular

$$\lambda_k = \inf_{\substack{S \subset \mathcal{D} \\ \dim S = k}} \left(\sup_{\substack{\alpha \in S \\ \|a\|=1}} \|A\alpha\| \right) \leq \varepsilon + \inf_{\substack{S \subset \mathcal{D} \\ \dim S = k}} \left(\sup_{\substack{\alpha \in S \\ \|a\|=1}} \|B\alpha\| \right) = \varepsilon + \mu_k.$$

Reversing the roles of A and B , we do obtain $|\lambda_k - \mu_k| \leq \varepsilon$ for the nonnegative case.

Next, for arbitrary operators A and B that satisfy the hypothesis, consider the nonnegative operators $A' = |A| + A$, $B' = |B| + B$, so that $\|A' - B'\|_{op} \leq 2\varepsilon$. The eigenvalues of A' and B' are $|\lambda_k| + \lambda_k$ and $|\mu_k| + \mu_k$, respectively, and the previous argument shows that $|\lambda_k - \mu_k| \leq \varepsilon$ for all nonnegative eigenvalues λ_k and μ_k of A and B . Similarly, we apply the previous argument to $A'' = |A| - A$ and $B'' = |B| - B$ to show that $|\lambda_k - \mu_k| \leq \varepsilon$ for all negative eigenvalues λ_k and μ_k of A and B . \square

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