NEW COHOMOLOGICAL INVARIANTS OF FOLIATIONS

GEORGES HABIB AND KEN RICHARDSON

Abstract. Given a smooth foliation on a closed manifold, basic forms are differential forms that can be expressed locally in terms of the transverse variables. The space of basic forms yields a differential complex, because the exterior derivative fixes this set. The basic cohomology is the cohomology of this complex, and this has been studied extensively. Given a Riemannian metric, the adjoint of the exterior derivative maps the orthogonal complement of the basic forms to itself, and we call the resulting cohomology the "antibasic cohomology". Although these groups are defined using the metric, the dimensions of the antibasic cohomology groups are invariant under diffeomorphism and metric changes. If the underlying foliation is Riemannian, the groups are foliated homotopy invariants that are independent of basic cohomology and ordinary cohomology of the manifold. For this class of foliations we use the codifferential on antibasic forms to obtain the corresponding Laplace operator, develop its analytic properties, and prove a Hodge theorem. We then find some topological and geometric properties that impose restrictions on the antibasic Betti numbers.

1. Introduction

The ordinary Hodge decomposition theorem on a closed Riemannian manifold (M, g) of dimension n gives an L^2-orthogonal decomposition of differential forms:

$$\Omega^k(M) = \operatorname{im}(d_{k-1}) \oplus \mathcal{H}^k \oplus \operatorname{im}(\delta_{k+1}),\ 0 \le k \le n,$$

where $d_k : \Omega^k(M) \to \Omega^{k+1}(M)$ is the exterior derivative with L^2-adjoint $\delta_{k+1} : \Omega^{k+1}(M) \to \Omega^k(M)$, and where $\mathcal{H}^k = \ker(\Delta_k)$ is the space of harmonic k-forms. Note also that

$$\ker(d_k) = \operatorname{im}(d_{k-1}) \oplus \mathcal{H}^k \text{ and } \ker(\delta_k) = \mathcal{H}^k \oplus \operatorname{im}(\delta_{k+1}).$$

From this we get that the de Rham cohomology groups satisfy $H^k(M) \cong \mathcal{H}^k$. Now, an alternative way of looking at this is to define a "new" de Rham homology $H^k_\delta(M)$ using δ instead of d: $\delta^2 = 0$, so

$$H^k_\delta(M) = \frac{\ker \delta_k}{\operatorname{im} \delta_{k+1}},\ 0 \le k \le n,$$

is well-defined. By the equations above for ker δ_k and $\ker d_k$, $H^k_\delta(M) \cong H^k_d(M)$. So no one ever defines $H^k_\delta(M)$ separately, because it does not provide anything new, and it seems to require a metric.

We consider however the situation where M is endowed with a smooth foliation $\mathcal{F}$ of codimension q. Many researchers have studied the properties of basic forms on foliations (see [13] for the original work and the expositions [15], [11], [20] and the references therein). Specifically, the basic forms are differential forms on M that locally depend only on the transverse variables. Because the exterior derivative preserves the set $\Omega^*_b(M)$ of basic forms, one can define the basic cohomology groups as

$$H^k_b(M, \mathcal{F}) := \frac{\ker\left(d : \Omega^k_b(M) \to \Omega^{k+1}_b(M)\right)}{\operatorname{im}\left(d : \Omega^{k-1}_b(M) \to \Omega^k_b(M)\right)},\ 0 \le k \le q.$$

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These cohomology groups are invariants of the foliation and can in general be infinite dimensional even when M is compact. The isomorphism classes of these groups are invariant under any homotopy equivalence between foliations that preserve the leaves. For certain classes of foliations, such as Riemannian foliations, these cohomology groups are finite dimensional.

Let $L^2\left(\Omega_b^*(M)\right)$ denote the completion of the space of smooth basic forms with respect to the L^2 inner product on differential forms on M. This is a subspace of the Hilbert space of differential forms with respect to this same inner product. Since the latter Hilbert space is complete, the subspace is the same as the closure of the space of smooth basic forms with respect to the L^2 norm. Since d preserves the smooth basic forms as mentioned previously, the formal adjoint δ of d with respect to the L^2 inner product preserves the smooth forms inside the L^2 orthogonal complement $\left(\Omega_b^*(M)\right)^{\perp}$, and we denote the set of smooth forms in this subspace by $\Omega_a^*(M,g)$, the set of "antibasic forms". Because $\delta^2=0$ on this space, we may define the "antibasic cohomology groups" as

$$H_a^k(M,\mathcal{F},g) := \frac{\ker\left(\delta : \Omega_a^k(M,g) \to \Omega_a^{k-1}(M,g)\right)}{\operatorname{im}\left(\delta : \Omega_a^{k+1}(M,g) \to \Omega_a^k(M,g)\right)},\ 0 \le k \le n.$$

We see that $H_a^k(M,\mathcal{F},g)$ depends on the choice of g, but we show that the isomorphism classes of these groups are independent of this choice (Theorem 2.1) and are in fact invariants of the smooth foliation structure (Corollary 2.2). For that reason, we henceforth remove the background metric g from the notation. Unlike the case of the de Rham cohomology of ordinary manifolds defined using δ above, these cohomology groups provide new invariants of the foliation, which are not necessarily isomorphic to either the basic or ordinary de Rham cohomology groups.

We are interested in whether these new foliation invariants give obstructions to certain types of geometric structures on the manifolds and foliations. In Theorem 2.5 we show that if the foliation is codimension one on a connected manifold, and if the mean curvature form of the normal bundle is everywhere nonzero, then $H_a^0(M)=\{0\}$, and $H_a^j(M)=H^j(M)$ for $j\ge 1$.

Starting with Section 3, we consider the case of Riemannian foliations, where the normal bundle carries a holonomy invariant metric; c.f. [14], [11], [20]. As is customary, we choose a bundle-like metric, one such that the leaves of the foliation are locally equidistant. In this particular case, the geometry forces many consequences for the antibasic cohomology. One crucial property of Riemannian foliations that allows us to proceed with analysis is that the L^2 orthogonal projection P_b from all forms to basic forms preserves smoothness. This was shown in [12] and [1], and it is false in general for non-Riemannian foliations (see Example 9.4). As a consequence, it is also true that the L^2 orthogonal projection P_a from all forms to antibasic forms preserves smoothness. In Proposition 3.1, we derive explicit formulas for the commutators $[d,P_a]$ and $[\delta,P_a]$, which are zeroth order operators that are in general not pseudodifferential. These formulas allow us to express the antibasic Laplacian $\Delta_a=\left(P_a(d+\delta)P_a\right)^2$ in terms of elliptic operators on all forms in Theorem 4.1. That is, Δ_a can be written in terms of the ordinary Laplacian Δ on M by the formula

$$\Delta_a = \left(\Delta + \delta P_b \varepsilon^* + P_b \varepsilon^* \delta\right) P_a,$$

where ε^* is a zeroth order differential operator determined by the geometry of the foliation and defined explicitly in Proposition 3.1.

Because Δ_a and $D_a=P_a(d+\delta)P_a$ are similar to elliptic differential operators but are in general not pseudodifferential, we do not necessarily expect them to satisfy the usual properties of Laplace and Dirac operators. However, in Section 5, we are able to show many of the functional analysis results with a few modifications. Specifically, we prove a version of Gårding's Inequality (Lemma 5.2), the elliptic estimates (Lemma 5.5), and the essential self-adjointness of both D_a and Δ_a (Corollary 5.12). Also, we show that elliptic regularity holds (Proposition 5.13), and finally we prove the spectral theorem (Theorem 5.17) for $\Delta_a=D_a^2$ and D_a, showing that there exists a complete orthonormal basis of $L^2\left(\Omega_a^*(M)\right)$ consisting of smooth eigenforms of D_a, and the eigenvalues of Δ_a

have finite multiplicity and accumulate only at $+\infty$. In all of these cases, the proofs are a bit more complicated than usual because of the antibasic projection and the issue of operators not being pseudodifferential.

In Section 6, we are able to prove the Hodge theorem and decomposition (Theorem 6.1 and Corollary 6.5) for the antibasic forms, again only for the Riemannian foliation case. For these foliations, there is an alternate way of expressing the antibasic cohomology, using $d_a = P_a d P_a$ as a differential. Then it turns out that if $f : (M, \mathcal{F}) \to (M', \mathcal{F}')$ is a foliated map, which takes leaves into leaves, then $P_a f^* P_a'$ induces a linear map on d_a-cohomology. We show that for Riemannian foliations, the isomorphism classes of antibasic cohomology groups are foliated homotopy invariants; see Theorem 6.8 and Corollary 6.9. We do know in general that the antibasic Betti numbers are foliated diffeomorphism invariants, but it is an open question whether they are foliated homotopy invariants; see Problem 1 and the preceding discussion.

In Section 7, we prove identities for antibasic cohomology in special cases. If the foliation is Riemannian, then $H_a^0(M, \mathcal{F}) \cong \{0\}$ and

$$\dim H^1(M) \leq \dim H_b^1(M, \mathcal{F}) + \dim H_a^1(M, \mathcal{F});$$

see Proposition 7.5 and Proposition 7.6. If in addition the normal bundle is involutive, then for all k,

$$H^k(M) \cong H_b^k(M, \mathcal{F}) \oplus H_a^k(M, \mathcal{F}),$$

by Proposition 7.1. In the special case where the Riemannian foliation is the set of orbits of a connected, compact Lie group of isometries, we show that antibasic cohomology can be computed using only the subspace of invariant differential forms; see Proposition 7.8.

The case of Riemannian flows is investigated in Section 8. In this setting, we are able to characterize $H_a^1(M, \mathcal{F})$. We prove in Proposition 8.1 that when the flow is taut, meaning that there exists a metric for which the leaves are minimal,

$$\dim\left(H_a^{r+1}(M, \mathcal{F})\right) \geq \dim\left(H_b^r(M, \mathcal{F})\right)$$

for $r \geq 0$. In the particular case where $H^1(M) = \{0\}$ and M is connected, we get $H_a^1(M) \cong \mathbb{R}$ always (Theorem 8.4). On the other hand, if M is connected and the flow is nontaut, we have that $H_a^1(M) \cong \{0\}$.

In Section 9, we compute the antibasic cohomology of specific foliations in low dimensions. These examples include Riemannian and non-Riemannian foliations and illustrate the results we have proved.

2. Basic and antibasic cohomology of foliations

Let M be a smooth, closed manifold, and let $\mathcal{F}$ be a smooth foliation on M of codimension q and dimension p (i.e. the dimension of M is $n = p + q$). The subspace $\Omega_b(M) \subseteq \Omega(M)$ of basic differential forms is defined as

$$\Omega_b(M) = \{\beta \in \Omega(M) : X \lrcorner \beta = 0, X \lrcorner d\beta = 0 \text{ for all } X \in \Gamma(T\mathcal{F})\}$$

where $X \lrcorner$ denotes interior product with X. Since d maps basic forms to themselves, we may compute the basic cohomology

$$H_b^k(M, \mathcal{F}) := \frac{\ker\left(d : \Omega_b^k(M) \to \Omega_b^{k+1}(M)\right)}{\operatorname{im}\left(d : \Omega_b^{k-1}(M) \to \Omega_b^k(M)\right)}$$

for $0 \leq k \leq q$. Because d commutes with pullbacks, these vector spaces are smooth invariants of the foliation, meaning that foliated diffeomorphisms (diffeomorphisms that map leaves onto leaves) preserve the basic cohomology groups. In [8], it was shown for complete Riemannian foliations that the basic cohomology algebra is a topological invariant. In general, it is not true that the basic cohomology is topologically invariant, because there exist smooth foliations that are foliated homeomorphic (but not foliated diffeomorphic) with nonisomorphic basic cohomology groups (see the introduction of [8] for an example). However, in [2, Théorème 1, Corollaire 1], the authors show for arbitrary smooth foliations that smooth foliated homotopy equivalences induce isomorphisms on basic cohomology.

In general, $H_b^k(M,\mathcal{F})$ need not be finite dimensional, unless there are topological restrictions (such as the existence of a bundle-like metric), and even when such restrictions apply, Poincaré duality is not satisfied except in special cases, such as when the foliation is taut and Riemannian. Much work on these cohomology groups has been done (c.f. [1], [11], [20], [2], [5], [6], and the associated references).

Suppose next that M is endowed with a Riemannian metric g. For simplicity, we will assume that M is oriented and $\mathcal{F}$ is transversally oriented in what follows, to make the Hodge star operators well-defined. However, many of the results carry over to the general case with minor changes and simple adjustments to the proofs.

A metric on the bundle of differential forms is induced from g, and in fact for any $\alpha,\beta\in\Omega^r(M)$,

$$\langle\alpha,\beta\rangle=\int_M \alpha\wedge *\beta,$$

where $*$ is the Hodge star operator. The formal adjoint δ of d with respect to this metric satisfies

$$\delta=(-1)^{nr+n+1}*d*=(-1)^r*^{-1}d*$$

on $\Omega^r(M)$. We let the smooth part of the L^2-orthogonal complement of $\Omega_b^r(M)$ be

$$\Omega_a^r(M,g):=\Omega_b^r(M)^{\perp}=\{\alpha\in\Omega^r(M):\langle\alpha,\beta\rangle=0\text{ for all }\beta\in\Omega_b^r(M)\}$$

the space of **antibasic r-forms**. Observe that for all $\beta\in\Omega_b^{r-1}(M)$, $\alpha\in\Omega_a^r(M,g)$,

$$0=\langle d\beta,\alpha\rangle=\langle\beta,\delta\alpha\rangle,$$

so that δ preserves the antibasic forms, and again $\delta^2=0$. We now define the **antibasic cohomology groups** $H_a^r(M,\mathcal{F},g)$ for $0\le r\le n$ by

$$H_a^r(M,\mathcal{F},g):=\frac{\ker\left(\delta:\Omega_a^r(M,g)\to\Omega_a^{r-1}(M,g)\right)}{\operatorname{im}\left(\delta:\Omega_a^{r+1}(M,g)\to\Omega_a^r(M,g)\right)}.$$

Theorem 2.1. *Let $\mathcal{F}$ be a smooth foliation on a smooth, closed, oriented manifold M that is endowed with a metric g. The isomorphism classes of the groups $H_a^r(M,\mathcal{F},g)$ do not depend on the choice of g and are thus invariants of $(M,\mathcal{F})$.*

Proof. Consider a general change of metric from g to g'. Let $*$ denote the Hodge star operator for metric g, and let $*'$ denote the Hodge star operator for g'. Similarly we define δ and δ'. We define the invertible bundle maps A^r and B^r on $\Omega^r(M)$ by

$$\begin{aligned} A^r &:= *^{-1}*':\Omega^r(M)\to\Omega^r(M),\\ B^r &:= *'*^{-1}:\Omega^r(M)\to\Omega^r(M). \end{aligned}$$

Then observe that

$$A^rB^r=*^{-1}*'*'*^{-1}=\text{identity},$$

and also B^rA^r is the identity. Thus we also have that

$$\begin{aligned} A^r &= *^{-1}*'=*\left(*'\right)^{-1},\\ B^r &= *'*^{-1}=\left(*'\right)^{-1}*. \end{aligned}$$

With these definitions,

$$*'=*A^r=B^{n-r}*.$$

Then on $\Omega^r(M)$, the formal adjoint of d in the g' metric is

$$\begin{aligned}\delta' &= \pm * A^{n-r+1} d * A^r \\ &= B^{r-1}\delta A^r.\end{aligned}$$

Then we check

$$\begin{aligned}0 &= \left(\delta'\right)^2 = B^{r-2}\delta A^{r-1}B^{r-1}\delta A^r \\ &= B^{r-2}\delta^2 A^r.\end{aligned}$$

Consider the map on differential r-forms given by $\psi \mapsto \psi' = (A^r)^{-1}\psi = B^r\psi$, which is an isomorphism. Then we see that

$$\begin{aligned}\delta'\psi' &= B^{r-1}\delta A^r (A^r)^{-1}\psi \\ &= B^{r-1}(\delta\psi) = (\delta\psi)'.\end{aligned}$$

Restricting now to the foliation case and the antibasic forms, we must determine if B^r maps the g-antibasic r-forms to the g'-antibasic r forms. We check this by taking any g-antibasic r-form ψ and any basic form β:

$$\begin{aligned}0 &= \langle\beta,\psi\rangle = \int_M \beta\wedge *\psi \\ &= \int_M \beta\wedge *'\left(*'\right)^{-1} * \psi = \int_M \beta\wedge *' B^r\psi = \langle\beta, B^r\psi\rangle'.\end{aligned}$$

Hence B^r maps the g-basic forms to the g'-antibasic forms. By the above, $\delta'(B^r\psi) = (\delta\psi)'$ for antibasic r-forms ψ, so that $B^r(\ker\delta) = \ker\delta'$ and $B^r(\operatorname{im}\delta) = \operatorname{im}\delta'$, so that the antibasic cohomology groups corresponding to g and g' are isomorphic through the map $[\psi]\mapsto[B^r\psi]'$. □

Corollary 2.2. *Suppose that $(M,\mathcal{F})$ is a smooth foliation of a smooth, closed, oriented manifold M. Suppose that $F: M\to M'$ is a diffeomorphism, and let $\mathcal{F}'$ be the foliation induced on M'. Then for any two metrics g, g' on M and M', respectively, $H_a^r(M,\mathcal{F},g)\cong H_a^r(M',\mathcal{F}',g')$. Thus, the isomorphism class of $H_a^r(M,\mathcal{F},g)$ is a smooth foliation invariant.*

Proof. Given the setting as above, observe that F^*g' is another metric on M. By construction and the theorem above, $H_a^r(M',\mathcal{F}',g')\cong H_a^r(M,\mathcal{F},F^*g')\cong H_a^r(M,\mathcal{F},g)$. □

Notation 2.3. *Henceforth we will denote $\Omega_a^r(M) = \Omega_a^r(M,g)$ and $H_a^r(M,\mathcal{F})\cong H_a^r(M,\mathcal{F},g)$, with the particular background metric g understood.*

Lemma 2.4. *Let $(M,\mathcal{F})$ be a smooth foliation of codimension q on a closed, oriented manifold M with any Riemannian metric. Then $H_a^k(M,\mathcal{F}) = H^k(M)$ for $k>q$, and $H_a^q(M,\mathcal{F})$ is isomorphic to a subspace of $H^q(M)$.*

Proof. Since $\Omega_a^k(M) = \Omega^k(M)$ for $k>q$, $H_a^k(M,\mathcal{F}) = H^k(M)$ for $k>q$. We also have

$$H_a^q(M,\mathcal{F}) = \frac{\ker\left(\delta|_{\Omega_a^q(M)}\right)}{\operatorname{im}\left(\delta|_{\Omega_a^{q+1}(M)}\right)} = \frac{\ker\left(\delta|_{\Omega_a^q(M)}\right)}{\operatorname{im}\left(\delta|_{\Omega^{q+1}(M)}\right)} \subseteq \frac{\ker\left(\delta|_{\Omega^q(M)}\right)}{\operatorname{im}\left(\delta|_{\Omega^{q+1}(M)}\right)} = H^q(M).$$

□

In the case of codimension 1 foliations, we can say more.

Proposition 2.5. *Let M be a closed, connected, oriented Riemannian manifold with codimension 1 foliation $\mathcal{F}$. Assume that the mean curvature form of the normal bundle is everywhere nonzero. Then the only basic functions on M are constants, $H_a^0(M) = \{0\}$, $H_b^0(M,\mathcal{F}) = \mathbb{R}$, and $H_a^j(M,\mathcal{F}) = H^j(M)$, $H_b^j(M,\mathcal{F}) = \{0\}$ for $j\geq 1$.*

Proof. Since the normal bundle $(T\mathcal{F})^{\perp}$ has rank 1, it is involutive. Let ν be the transverse volume form of $\mathcal{F}$. Note that $T\mathcal{F} = \ker \nu$. By Rummler's formula [18],

$$d\nu = -\kappa_N \wedge \nu,$$

where κ_N is the mean curvature 1-form of $(T\mathcal{F})^{\perp}$. By assumption, κ_N is nonzero everywhere. Observe that any one-form may be written $\beta = a\nu + \gamma$, where a is a function and γ is orthogonal to ν. Note that a one-form β is basic if and only if $X \lrcorner \beta = 0$ and $X \lrcorner d\beta = 0$ for all $X \in \ker \nu$. The first condition implies $\beta = a\nu$, and the second condition implies

$$\begin{aligned} 0 &= X \lrcorner (da \wedge \nu + a d\nu) = X \lrcorner (da \wedge \nu - a\kappa_N \wedge \nu) \\ &= X \lrcorner [(da - a\kappa_N) \wedge \nu], \end{aligned}$$

which implies

$$X \lrcorner (da - a\kappa_N) = 0$$

for all $X \in \Gamma(T\mathcal{F})$, or

$$da = a\kappa_N + b\nu$$

for some function b. Since $\kappa_N \neq 0$ and is orthogonal to ν, the maximum and minimum of the function a on M must occur when $a = 0$, so $a \equiv 0$. Thus, there are no nonzero basic one-forms, so that $\Omega_a^1(M) = \Omega^1(M)$. Every function f on M can be written as $f = c + \delta\alpha$ for some one-form α and constant c by the Hodge theorem. Since α and $\delta\alpha$ are necessarily antibasic, we have the natural decomposition of f into its basic component c and antibasic component $\delta\alpha$. Therefore, every antibasic function is δ-exact, and every basic function is constant, so we have $H_a^0(M, \mathcal{F}) = \{0\}$, $H_b^0(M, \mathcal{F}) = \mathbb{R}$, and $H_a^j(M, \mathcal{F}) = H^j(M)$, $H_b^j(M, \mathcal{F}) = \{0\}$ for $j \geq 1$, because $\Omega_a^j(M) = \Omega^j(M)$ for $j \geq 1$. □

Remark 2.6. *In the next section, we consider the case of Riemannian foliations. Codimension one Riemannian foliations always have $\kappa_N = 0$, so the proof of the previous proposition does not apply. Indeed, it is not true that there are no basic one-forms, since the transverse volume form ν is always a basic one-form. Also, it is quite possible that there are nonconstant basic functions. The cohomological facts in this case are only different in degree 1: $H_a^0(M, \mathcal{F}) = \{0\}$, $H_b^0(M, \mathcal{F}) = \mathbb{R}$, $H_b^1(M, \mathcal{F}) = \mathbb{R}$, $H^1(M) \cong H_a^1(M, \mathcal{F}) \oplus H_b^1(M, \mathcal{F})$, and $H_a^j(M, \mathcal{F}) = H^j(M)$, $H_b^j(M, \mathcal{F}) = \{0\}$ for $j \geq 2$.*

Given two smooth foliations $(M, \mathcal{F})$ and $(M', \mathcal{F}')$, a map $f : (M, \mathcal{F}) \to (M', \mathcal{F}')$ is called **foliated** if f maps the leaves of $\mathcal{F}$ to the leaves of $\mathcal{F}'$, which implies $f_*(T\mathcal{F}) \subset T\mathcal{F}'$. It follows that the basic forms on $(M', \mathcal{F}')$ pull back to basic forms on $(M, \mathcal{F})$. Two foliated maps $f, g : (M, \mathcal{F}) \to (M', \mathcal{F}')$ are **foliated homotopic** if there exists a continuous map $H : [0, 1] \times M \to M'$ such that $H(0, x) = f(x)$ and $H(1, x) = g(x)$ and for all $t \in [0, 1]$ the map $H(t, \cdot)$ is foliated and smooth as a map from $(M, \mathcal{F})$ to $(M', \mathcal{F}')$. A foliated map $f : (M, \mathcal{F}) \to (M', \mathcal{F}')$ is a **foliated homotopy equivalence** if there exists a foliated map $h : (M', \mathcal{F}') \to (M, \mathcal{F})$ such that $f \circ h$ and $h \circ f$ are foliated homotopic to the identity on the two foliations.

It is proved in [2] (also in [8] for the case of foliated homeomorphisms) that foliated homotopic maps induce the same map on basic cohomology and that basic cohomology is a foliated homotopy invariant. We now examine whether or not antibasic cohomology satisfies the same property.

Note that since in general the codifferential δ does not commute with pullback f^* by a smooth map $f : (M, \mathcal{F}) \to (M', \mathcal{F}')$, we do not expect that pullback induces a linear map on antibasic cohomology. However, since it is true that on differential forms $d \circ f^* = f^* \circ d$, we also have that

$$(f^*)^{\dagger} \circ \delta = \delta \circ (f^*)^{\dagger},$$

where † denotes the formal L^2-adjoint. Note that f^* is not necessarily bounded on L^2. If we restrict to the case of closed manifolds, f^* does map smooth forms to smooth forms in L^2, so it is a densely defined operator on L^2. Here $(f^*)^\dagger$ is the formal adjoint defined on its domain. From unbounded operator theory, the domain of $(f^*)^\dagger$ is

$$\mathrm{Dom}\left((f^*)^\dagger\right) = \{\alpha \in L^2(\Omega(M)) : \exists \gamma \in L^2\left(\Omega\left(M'\right)\right) \text{ such that } \langle f^*\beta, \alpha\rangle_M = \langle \beta, \gamma\rangle_{M'}\ \forall \beta \in \Omega\left(M'\right)\}.$$

But it is known that if α is smooth, and if the linear map $\Phi_f(\beta) := \langle f^*\beta, \alpha\rangle_M$ is bounded, then $\Phi_f(\beta) = \langle \beta, \gamma\rangle_{M'}$ for some γ by the Riesz representation theorem. However, it turns out that $\Phi_f(\cdot)$ is unbounded for almost all choices of f (the rank of its differential must be constant, for instance). In the cases where Φ_f is bounded, $(f^*)^\dagger$ induces a linear map on antibasic cohomology. The usual proof applies in this case to show that maps $(f^*)^\dagger$ are invariant over the homotopy class of such f.

Another possible approach is to use the Hodge star operator $*$ and $*'$ on M and M', respectively, and to consider $*f^**'$ as a map that commutes with δ up to a sign. However, this would not apply in our case since $*f^**'$ does not necessarily preserve the antibasic forms.

Thus, we still have the following open problem:

Problem 1. *If the foliations $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ with Riemannian metrics are foliated homotopy equivalent, does that mean that their antibasic cohomology groups are isomorphic?*

Remark 2.7. *This problem is solved in the case of Riemannian foliations, as we see in Theorem 6.8 and Corollary 6.9. In this case, P_a preserves the smooth forms, so we show that the operator $P_a f^* P_a'$ induces a linear map on antibasic cohomology, which is an isomorphism when f is a foliated homotopy equivalence.*

3. Riemannian foliation setting

In the Riemannian foliation setting, we often restrict to basic forms. Let $(M, \mathcal{F})$ be a foliation of codimension q and dimension p, endowed with a bundle-like metric. Again, for simplicity of exposition, we assume the foliation and manifold are oriented.

From [12], the orthogonal projection $P_b : L^2(\Omega(M)) \to L^2(\Omega_b(M))$ maps smooth forms to smooth basic forms; this was also stated and used in [1]. Because of this, it is also true that

$$P_a = (I - P_b) : L^2(\Omega(M)) \to L^2\left(\Omega_b(M)^\perp\right)$$

maps smooth forms to smooth "antibasic forms". As described in [12], we have

$$d_b = P_b d P_b = d P_b,$$

(i.e. d restricts to the basic forms). Letting $\delta = \delta_k : \Omega^k(M) \to \Omega^{k-1}(M)$ be the L^2 adjoint of d_{k-1}, we then have

$$\delta_b = P_b \delta P_b = P_b \delta,$$

and note that the basic adjoint is $\delta_b = P_b\delta = P_b\delta P_b$. Note also that the formulas above imply that

$$\begin{aligned} (I - P_b)\, d\, (I - P_b) &= (I - P_b)\, d, \\ (I - P_b)\, \delta\, (I - P_b) &= \delta\, (I - P_b), \end{aligned}$$

or

$$\begin{aligned} d_a &= P_a d P_a = P_a d, \\ \delta_a &= P_a \delta P_a = \delta P_a. \end{aligned} \tag{3.1}$$

We see

$$\delta_a^2 = P_a \delta P_a P_a \delta P_a = \delta^2 P_a = 0.$$

The adjoint of δ_a restricted to antibasic forms is $d_a = P_a d P_a = P_a d$, and again

$$d_a^2 = P_a d P_a P_a d P_a = P_a d^2 = 0.$$

Also in [12], it is shown that

$$\begin{aligned} P_b\delta - \delta P_b &= \varepsilon \circ P_b = \left[-\left(P_a\kappa\right)\lrcorner + (-1)^p \left(\varphi_0\lrcorner\right)\left(\chi_{\mathcal{F}}\wedge\right)\right] \circ P_b, \\ dP_b - P_b d &= P_b \circ \varepsilon^* = P_b \circ \left[-\left(P_a\kappa\right)\wedge + (-1)^p \left(\chi_{\mathcal{F}}\lrcorner\right)\left(\varphi_0\wedge\right)\right] \end{aligned} \tag{3.2}$$

on $\Omega^*(M)$. Recall here that $\chi_{\mathcal{F}}$ is the characteristic volume form of $T\mathcal{F}$ and φ_0 is a $(p+1)$-form with the property that $v_1\lrcorner\cdots\lrcorner v_p\lrcorner\varphi_0 = 0$ for any set $\{v_j\}$ of p vectors in $T\mathcal{F}$. Note that φ_0 vanishes precisely when $(T\mathcal{F})^\perp$ is completely integrable. We observe that the only information about the foliation needed to obtain the formulas above in [12] is the fact that the orthogonal projection P_b maps smooth forms to smooth forms, that P_b commutes with $\bar{*}$, the transversal Hodge star-operator, and that $P_b(\alpha\wedge P_b\beta) = (P_b\alpha)\wedge(P_b\beta)$ for all smooth forms α,β. These facts are true for Riemannian foliations. From the formulas above and the notation $\kappa_a = P_a\kappa$, we obtain the following.

Proposition 3.1. *On an oriented Riemannian foliation $(M,\mathcal{F})$ on a closed, oriented manifold with bundle-like metric,*

$$\delta P_a - P_a\delta = \varepsilon \circ P_b = \left[-\kappa_a\lrcorner + (-1)^p\left(\varphi_0\lrcorner\right)\left(\chi_{\mathcal{F}}\wedge\right)\right]\circ P_b, \tag{3.3}$$

$$P_a d - dP_a = P_b \circ \varepsilon^* = P_b \circ \left[-\kappa_a\wedge + (-1)^p\left(\chi_{\mathcal{F}}\lrcorner\right)\left(\varphi_0\wedge\right)\right] \tag{3.4}$$

on $\Omega^*(M)$. The operation ε maps $\Omega_b(M)$ to $\Omega_b(M,\mathcal{F})^\perp$, and it follows that

$$\begin{aligned} P_b\varepsilon P_b &= P_b\varepsilon^* P_b = 0, \\ \varepsilon P_b &= P_a\varepsilon P_b,\ \varepsilon^* P_b = P_a\varepsilon^* P_b, \\ P_b\varepsilon P_a &= P_b\varepsilon,\ P_b\varepsilon^* P_a = P_b\varepsilon^*. \end{aligned} \tag{3.5}$$

4. The antibasic Laplacian

Again we assume that $(M,\mathcal{F})$ is a foliation of codimension q and dimension p, endowed with a bundle-like metric, with orientations on both the foliation and the manifold. Recall that the basic Laplacian is $\Delta_b = \delta_b d_b + d_b\delta_b =$ restriction of $P_b\delta d + dP_b\delta$ to $\Omega_b(M)$. We wish to do a similar restriction to antibasic forms. Let the subscript a denote the restriction to $\Omega_a(M)$, the antibasic forms. Then

$$\begin{aligned} \Delta_a &= \delta_a d_a + d_a\delta_a = \left(d_a + \delta_a\right)^2 \\ &= \text{restriction of } \delta P_a d + P_a d\delta \text{ to } \Omega_a(M). \end{aligned}$$

From the formulas (3.4) and (3.5),

$$\begin{aligned} \Delta_a &= \left.\left(\delta P_a d + P_a d\delta\right)\right|_{\Omega_a(M)} \\ &= \left.\left(\delta d P_a + \delta P_b\varepsilon^* P_a + dP_a\delta + P_b\varepsilon^* P_a\delta\right)\right|_{\Omega_a(M)} \\ &= \left.\left(\delta d + \delta P_b\varepsilon^* + d\delta + P_b\varepsilon^*\delta\right)\right|_{\Omega_a(M)} \\ &= \left.\left(\Delta + \delta P_b\varepsilon^* + P_b\varepsilon^*\delta\right)\right|_{\Omega_a(M)}. \end{aligned}$$

Thus Δ_a is the restriction of an elliptic operator on the space of all differential forms. Note that it is not clear whether this operator is differential or pseudodifferential or not, since P_b is not pseudodifferential in general, because it is not pseudolocal. Simple examples show that P_b can take a smooth function to a discontinuous function.

We summarize the results below.

Theorem 4.1. *The antibasic Laplacian Δ_a satisfies the following.*

$$\Delta_a P_a = \widetilde{\Delta} P_a = P_a\widetilde{\Delta}^* = P_a\overline{\Delta}P_a,$$

where $\widetilde{\Delta} = \Delta + \delta P_b\varepsilon^* + P_b\varepsilon^*\delta$, $\widetilde{\Delta}^* = \Delta + \varepsilon P_b d + d\varepsilon P_b$ is its adjoint, and $\overline{\Delta} = \Delta - \varepsilon P_b\varepsilon^*$.

Proof. The first equality was shown above. To prove that $\widetilde{\Delta}P_a = P_a\overline{\Delta}P_a$, we compute

$$\begin{aligned}\widetilde{\Delta}P_a &= P_a\widetilde{\Delta}P_a = P_a\left(\Delta + \delta P_b\varepsilon^* + P_b\varepsilon^*\delta\right)P_a \\ &= P_a\Delta P_a + P_a\delta P_b\varepsilon^* P_a \\ &= P_a\Delta P_a + P_a\left(P_a\delta\right)P_b\varepsilon^* P_a \\ &= P_a\Delta P_a + P_a\left(\delta P_a - \varepsilon P_b\right)P_b\varepsilon^* P_a \\ &= P_a\Delta P_a - P_a\varepsilon P_b\varepsilon^* P_a \\ &= P_a\left(\Delta - \varepsilon P_b\varepsilon^*\right)P_a.\end{aligned}$$

Here we have used the fact that $P_aP_b = 0$, $P_a^2 = P_a$, $P_b^2 = P_b$, and formula (3.3). Note that since $P_a\overline{\Delta}P_a$ is formally self-adjoint, $\widetilde{\Delta}P_a = \left(\widetilde{\Delta}P_a\right)^* = P_a\widetilde{\Delta}^*$. □

Corollary 4.2. *The antibasic Laplacian is the restriction of the ordinary Laplacian if the mean curvature is basic and the normal bundle of the foliation is involutive.*

Proof. If the mean curvature is basic and the normal bundle of the foliation is involutive, then $P_a\kappa = 0$ and $\varphi_0 = 0$, so that $\varepsilon = 0$. Then $\widetilde{\Delta} = \widetilde{\Delta}^* = \overline{\Delta}$ in this case, so by the theorem above $\Delta_aP_a = \Delta P_a$.

□

Also we show a few more facts about the projections and the operators d, δ, ε.

Proposition 4.3. *With notation as above,*

$$P_a\left(d + \varepsilon^*\right)P_a = \left(d + \varepsilon^*\right)P_a.$$

Proof. By (3.4), we have

$$\begin{aligned}P_a\left(d + \varepsilon^*\right)P_a &= P_adP_a + P_a\varepsilon^*P_a \\ &= \left(dP_a + P_b\varepsilon^*\right)P_a + P_a\varepsilon^*P_a \\ &= dP_a + \left(P_b + P_a\right)\varepsilon^*P_a \\ &= \left(d + \varepsilon^*\right)P_a.\end{aligned}$$

□

Now, we let the first order operator D^ε be defined as

$$D^\varepsilon = \delta + d + \varepsilon^*,$$

and the antibasic operator D_a^ε by

$$D_a^\varepsilon = P_a\left(\delta + d + \varepsilon^*\right)P_a.$$

Corollary 4.4. *We have*

$$D_a^\varepsilon = P_aD^\varepsilon P_a = D^\varepsilon P_a,$$

so that D_a^ε is the restriction of the elliptic operator D^ε.

Corollary 4.5. *Let $\Delta^\varepsilon := \Delta + \varepsilon^*\delta + \delta\varepsilon^*$. Then $P_a\Delta^\varepsilon P_a = \Delta^\varepsilon P_a$, so that the operator $P_a\Delta^\varepsilon P_a$ on antibasic forms is the restriction of an elliptic operator.*

Proof. By Proposition 4.3 and (3.1), we compute

$$\begin{aligned}P_a\Delta^\varepsilon P_a &= P_a\left(\Delta + \varepsilon^*\delta + \delta\varepsilon^*\right)P_a = P_a\left(\delta\left(d + \varepsilon^*\right) + \left(d + \varepsilon^*\right)\delta\right)P_a \\ &= P_a\delta P_a\left(d + \varepsilon^*\right)P_a + \left(d + \varepsilon^*\right)P_a\delta P_a \\ &= \delta P_a\left(d + \varepsilon^*\right)P_a + \left(d + \varepsilon^*\right)\delta P_a \\ &= \delta\left(d + \varepsilon^*\right)P_a + \left(d + \varepsilon^*\right)\delta P_a = \Delta^\varepsilon P_a.\end{aligned}$$

□

5. Functional analysis of the antibasic de Rham and Laplace operators

In this section, we show that the antibasic de Rham and Laplace operators on a Riemannian foliation have properties similar to the ordinary de Rham and Laplace operators on closed manifolds, namely that they have discrete spectrum consisting of eigenvalues corresponding to finite-dimensional eigenspaces. Note that we must work out the standard Sobolev and elliptic theory for these operators, because in fact they are not pseudodifferential and are not even restrictions of pseudodifferential operators to antibasic forms.

Throughout this section, we assume that $(M,\mathcal{F})$ is a foliation with bundle-like metric g, and as in the previous section, we denote the antibasic Hodge-de Rham and Laplace operators as

$$\begin{aligned} D_a &= P_a d P_a + P_a \delta P_a = d_a + \delta_a, \\ \Delta_a &= d_a \delta_a + \delta_a d_a = D_a^2. \end{aligned}$$

First, let H_a^k denote the Sobolev space $H^k(\Omega_a(M))$, defined as the completion of $\Omega_a(M)$ with respect to a choice of the k^{th} Sobolev norm $\|\bullet\|_k$; this is the same as the closure of $\Omega_a(M)$ inside the (complete) Sobolev space $H^k(\Omega(M))$. We notate the ordinary L^2 norm as the 0^{th} Sobolev norm $\|\bullet\|_0$, so that $H_a^0 = L^2\left(\Omega_a^k(M)\right)$. Note that Rellich's Theorem still holds on this subspace, i.e. the inclusion of $H_a^k \hookrightarrow H_a^\ell$ is compact for $k > \ell$. The proof follows easily from the standard case.

Also, note that the Sobolev embedding theorem holds for the antibasic forms, so that for any integer $m > \frac{\dim M}{2}$, the space $H_a^{k+m} \subseteq C^k(M)$. This follows from the fact that $H_a^{k+m} \subseteq H^{k+m}(\Omega(M))$.

Lemma 5.1. *There exists a constant $c > 0$ such that $\|D_a\psi\|_0 \le c\|\psi\|_1$ for all $\psi \in \Omega_a(M)$.*

Proof. By (3.4), for any $\psi \in \Omega_a(M)$,

$$\begin{aligned} D_a\psi &= (d_a + \delta_a)\psi = (P_a d + \delta)\psi = (dP_a + P_b\varepsilon^* + \delta)\psi \\ &= (d+\delta)\psi + P_b\varepsilon^*\psi. \end{aligned}$$

Then, since $d+\delta$ is a first order differential operator and ε^* is a bounded operator,

$$\begin{aligned} \|D_a\psi\|_0 &\le \|(d+\delta)\psi\|_0 + \|P_b\varepsilon^*\psi\|_0 \\ &\le c_1\|\psi\|_1 + \|\varepsilon^*\psi\|_0 \le c_1\|\psi\|_1 + c_2\|\psi\|_0 \end{aligned}$$

for some positive constants c_1 and c_2, so that there exists $c > 0$ independent of ψ such that $\|D_a\psi\|_0 \le c\|\psi\|_1$. □

Lemma 5.2. *(Gårding's Inequality) There exists a positive constant c such that $\|\psi\|_1 \le c(\|\psi\|_0 + \|D_a\psi\|_0)$ for all $\psi \in \Omega_a(M)$.*

Proof. By the ordinary Gårding's Inequality, since $d+\delta$ is an elliptic, first order operator on $\Omega(M)$, there exists a constant c_1 such that for all $\psi \in \Omega_a(M) \subseteq \Omega(M)$,

$$\|\psi\|_1 \le c_1(\|\psi\|_0 + \|(d+\delta)\psi\|_0).$$

Then, again by (3.4) and the proof of Lemma 5.1,

$$\begin{aligned} \|\psi\|_1 &\le c_1(\|\psi\|_0 + \|(d_a + \delta_a - P_b\varepsilon^*)\psi\|_0) \\ &\le c_1(\|\psi\|_0 + \|P_b\varepsilon^*\psi\|_0 + \|(d_a+\delta_a)\psi\|_0) \\ &\le c_1(\|\psi\|_0 + \|\varepsilon^*\psi\|_0 + \|(d_a+\delta_a)\psi\|_0), \end{aligned}$$

so since ε^* is bounded, the result follows. □

Lemma 5.3. *For all nonnegative integers k, there exists a positive constant c_k such that*

$$\|P_a\phi\|_k \le c_k\|\phi\|_k \text{ and } \|P_b\phi\|_k \le c_k\|\phi\|_k$$

for any differential form ϕ.

Proof. We use induction on k. Let ϕ be any differential form. Observe first that $\|P_a\phi\|_0 \leq \|\phi\|_0, \|P_b\phi\|_0 \leq \|\phi\|_0$. Next, suppose that the results have been shown for some nonnegative integer k. Since D^ε is elliptic on all forms, it satisfies the ordinary elliptic estimates: there exist constants b_1 and b_2 such that

$$\begin{aligned}
\|P_a\phi\|_{k+1} &\leq b_1 \|D^\varepsilon P_a\phi\|_k + b_2 \|P_a\phi\|_k \\
&= b_1 \|(d+\delta+\varepsilon^*) P_a\phi\|_k + b_2 \|P_a\phi\|_k \\
&= b_1 \|(P_a d - P_b\varepsilon^* + P_a\delta + \varepsilon P_b + \varepsilon^* P_a)\phi\|_k + b_2 \|P_a\phi\|_k \\
&= b_1 \|(P_a d - P_b\varepsilon^* P_a + P_a\delta + \varepsilon P_b + \varepsilon^* P_a)\phi\|_k + b_2 \|P_a\phi\|_k \\
&= b_1 \|(P_a d + P_a\delta + \varepsilon P_b + P_a\varepsilon^* P_a)\phi\|_k + b_2 \|P_a\phi\|_k \\
&\leq b_1 (\|P_a(d+\delta)\phi\|_k + \|\varepsilon P_b\phi\|_k + \|P_a\varepsilon^* P_a\phi\|_k) + b_2 \|P_a\phi\|_k .
\end{aligned}$$

Using the fact that ε is a zeroth order differential operator and the induction hypothesis,

$$\begin{aligned}
\|P_a\phi\|_{k+1} &\leq (\text{constant}) \|(d+\delta)\phi\|_k + (\text{constant}) \|\phi\|_k \\
&\leq (\text{constant}) \|\phi\|_{k+1} + (\text{constant}) \|\phi\|_k \leq (\text{constant}) \|\phi\|_{k+1} ,
\end{aligned}$$

since $d+\delta$ is a first order operator. Also,

$$\|P_b\phi\|_{k+1} = \|\phi - P_a\phi\|_{k+1} \leq \|\phi\|_{k+1} + \|P_a\phi\|_{k+1} \leq (\text{constant}) \|\phi\|_{k+1} .$$

By induction, the proof is complete. □

Lemma 5.4. *Let $D^\varepsilon = d+\delta+\varepsilon^*$ as an operator on all differential forms, and let $D_a = d_a + \delta_a$ be the antibasic de Rham operator. For all nonnegative integers k, there exists a positive constant c_k such that*

$$\|D^\varepsilon\psi\|_k - c_k \|\psi\|_k \leq \|D_a\psi\|_k \leq \|D^\varepsilon\psi\|_k + c_k \|\psi\|_k ,$$

for any antibasic form ψ.

Proof. For any antibasic ψ,

$$\|D_a\psi\|_k = \|D^\varepsilon\psi + (D^a - D^\varepsilon)\psi\|_k = \|D^\varepsilon\psi + P_a\varepsilon^*\psi\|_k .$$

It suffices to bound $\|P_a\varepsilon^*\psi\|_k$. This follows from a bound on $\|P_a\phi\|_k$ from Lemma 5.3, since ε^* is a zeroth order operator. □

Lemma 5.5. *(Elliptic Estimates for D_a) For every integer $k \geq 0$, there exists a positive constant C_k such that $\|\psi\|_{k+1} \leq C_k (\|\psi\|_k + \|D_a\psi\|_k)$ for all $\psi \in \Omega_a(M)$.*

Proof. Let k be a nonnegative integer. From the elliptic estimates for the operator D^ε on all forms, there exists a positive constant b_k such that for any $\psi \in \Omega_a(M)$,

$$\begin{aligned}
\|\psi\|_{k+1} &\leq b_k (\|\psi\|_k + \|D^\varepsilon\psi\|_k) \\
&\leq b_k (\|\psi\|_k + \|D_a\psi\|_k + c_k \|\psi\|_k)
\end{aligned}$$

for a positive constant c_k, by Lemma 5.4. The inequality follows by letting $C_k = \max(b_k, b_k(1+c_k))$. □

Remark 5.6. *The case $k=0$ is Gårding's Inequality, which we have shown independently in Lemma 5.2.*

Lemma 5.7. *The operator D_a on $\Omega_a(M)$ is formally self-adjoint.*

Proof. For any antibasic forms α and β,

$$\begin{aligned}\langle D_a\alpha, \beta\rangle &= \langle (P_a\,(d+\delta)\,P_a)\,\alpha, \beta\rangle \\ &= \langle \alpha, P_a\,(d+\delta)\,P_a\beta\rangle = \langle \alpha, D_a\beta\rangle\,.\end{aligned}$$

□

Lemma 5.8. *The domain of the closure of D_a is H_a^1.*

Proof. The graph of D_a is $G_a = \{(\omega, D_a\omega) : \omega \in \Omega_a(M)\} \subseteq H_a^0 \times H_a^0$. The closure of G_a is also a graph, by the following argument. We must show that for any $(0,\eta) \in \overline{G_a}$, $\eta = 0$. For any $(0,\eta) \in \overline{G_a}$, there is a sequence (ω_j) of smooth antibasic forms with $\omega_j \to 0$ and $D_a\omega_j \to \eta$ in $H_a^0 \subseteq L^2$. But then for any smooth antibasic form γ,

$$\langle D_a\omega_j, \gamma\rangle \to \langle \eta, \gamma\rangle\,, \text{ and } \langle \omega_j, D_a\gamma\rangle \to 0$$

as $j \to \infty$. But $\langle \omega_j, D_a\gamma\rangle = \langle D_a\omega_j, \gamma\rangle$ by Lemma 5.7, so $\langle \eta, \gamma\rangle = 0$ for all smooth γ, so $\eta = 0$. Thus $\overline{G_a} = \{(\omega, A\omega) : \omega \in \mathrm{dom}\,(A)\}$ for some operator A, which is defined to be the closure of D_a. Thus the domain is the set of all $\omega \in H_a^0$ such that there exists a sequence (ω_j) of smooth antibasic forms such that $\omega_j \to \omega$ in H_a^0 and $(D_a\omega_j)$ converges in H_a^0. By Gårding's Inequality (Lemma 5.2) and Lemma 5.1, $\mathrm{dom}\,(A) = H_a^1$. □

Lemma 5.9. *(Existence of Friedrichs' mollifiers) There exists a family of self-adjoint smoothing operators $\{F_\rho\}_{\rho\in(0,1)}$ on H_a^0 such that (F_ρ) is bounded in H_a^0, $F_\rho \to 1$ uniformly weakly in H_a^0 as $\rho \to 0$, and $[F_\rho, D_a]$ extends to a uniformly bounded family of operators on H_a^0.*

Proof. Let F_ρ^0 be defined as the usual Friedrichs' mollifiers; c.f. [16, Definition 5.21, Exercise 5.34]. Thus, these operators satisfy the properties above, except with H_a^0 replaced by $L^2(\Omega(M))$ and D_a replaced by any first order differential operator. Now let $F_\rho = P_aF_\rho^0$. Note that F_ρ is smoothing because F_ρ^0 is smoothing and since P_a maps smooth forms to smooth forms; its kernel is the kernel of F_ρ^0 followed by P_a. We now check the three properties. First, for any $\alpha \in H_a^0 \subseteq L^2(\Omega(M))$,

$$\|F_\rho\alpha\|_0 = \left\|P_aF_\rho^0\alpha\right\|_0 \le \left\|F_\rho^0\alpha\right\|_0 \le c\,\|\alpha\|_0$$

for some $c > 0$, by the first property of F_ρ^0. Next, for any $\alpha \in H_a^0$, for all smooth antibasic forms β,

$$\langle (F_\rho - 1)\,\alpha, \beta\rangle = \left\langle \left(P_aF_\rho^0 - 1\right)\alpha, \beta\right\rangle = \left\langle \left(F_\rho^0 - 1\right)\alpha, \beta\right\rangle,$$

which approaches 0 uniformly as $\rho \to 0$ by the corresponding property of F_ρ^0. Lastly, for any smooth antibasic forms ω, η,

$$\begin{aligned}\langle [F_\rho, D_a]\omega, \eta\rangle &= \left\langle P_aF_\rho^0D_a\omega, \eta\right\rangle - \left\langle D_aP_aF_\rho^0\omega, \eta\right\rangle \\ &= \left\langle F_\rho^0\,(\delta + d + P_b\varepsilon^*)\,\omega, \eta\right\rangle - \left\langle P_a\,(\delta + d + \varepsilon P_b)\,F_\rho^0\omega, \eta\right\rangle \\ &= \left\langle F_\rho^0\,(\delta + d)\,\omega, \eta\right\rangle + \left\langle F_\rho^0P_b\varepsilon^*\omega, \eta\right\rangle - \left\langle (\delta + d + \varepsilon P_b)\,F_\rho^0\omega, \eta\right\rangle \\ &= \left\langle \left[F_\rho^0, (\delta + d)\right]\omega, \eta\right\rangle + \left\langle F_\rho^0P_b\varepsilon^*\omega, \eta\right\rangle - \left\langle \varepsilon P_bF_\rho^0\omega, \eta\right\rangle.\end{aligned}$$

The operator $\left[F_\rho^0, (\delta + d)\right]$ is bounded by the corresponding property of F_ρ^0, and $F_\rho^0P_b\varepsilon^*$ and $\varepsilon P_bF_\rho^0$ are both zeroth order operators that are uniformly bounded in ρ on L^2, so we conclude that $[F_\rho, D_a]$ is uniformly bounded on H_a^0. □

Corollary 5.10. *Let $\{F_\rho\}$ be a family of Friedrichs' mollifiers. Then F_ρ and $[D_a, F_\rho]$ are uniformly bounded families of operators on H_a^k for any $k \ge 0$.*

Proof. We proceed by induction using the elliptic estimates in Lemma 5.5. □

Proposition 5.11. *Suppose that $\alpha, \beta \in H_a^0$ and $D_a\alpha = \beta$ weakly. Then $\alpha \in H_a^1 = \mathrm{dom}\,\overline{D_a}$, and $\overline{D_a}\alpha = \beta$.*

Proof. For any $\alpha, \beta \in H_a^0$, suppose $D_a\alpha = \beta$ weakly. Then for any smooth antibasic form γ and any $\rho \in (0,1)$,

$$\begin{aligned}\langle D_a F_\rho \alpha, \gamma\rangle &= \langle F_\rho \alpha, D_a \gamma\rangle \\ &= \langle \alpha, F_\rho D_a \gamma\rangle = \langle \alpha, D_a F_\rho \gamma\rangle + \langle \alpha, [F_\rho, D_a]\gamma\rangle \\ &= \langle \beta, F_\rho \gamma\rangle + \langle \alpha, [F_\rho, D_a]\gamma\rangle \\ &= \langle F_\rho \beta, \gamma\rangle + \langle \alpha, [F_\rho, D_a]\gamma\rangle\end{aligned}$$

by Lemma 5.9. Thus, for a constant $C > 0$ independent of ρ and γ,

$$|\langle D_a F_\rho \alpha, \gamma\rangle| \leq C \|\gamma\|_0 .$$

Then $\|D_a F_\rho \alpha\|_0 \leq C$. By Gårding's Inequality (Lemma 5.2) and the fact that F_ρ is a bounded operator in H_a^0, $\{F_\rho \alpha\}_{\rho\in(0,1)}$ is a bounded set in H_a^1. By the weak compactness of a ball in the Hilbert space H_a^1 (with equivalent metric $\langle \xi, \theta\rangle_1 = \langle D_a\xi, D_a\theta\rangle + \langle \xi, \theta\rangle$), there is a sequence $\rho_j \to 0$ and $\alpha' \in H_a^1$ such that $F_{\rho_j}\alpha \to \alpha'$ weakly in H_a^1. By Rellich's Theorem, the subsequence converges strongly in H_a^0, so $F_{\rho_j}\alpha \to \alpha'$ in H_a^0. But we know already that $F_{\rho_j}\alpha \to \alpha$ in H_a^0, so $\alpha = \alpha' \in H_a^1$. □

Corollary 5.12. *The antibasic de Rham and antibasic Laplacian are essentially self-adjoint operators.*

Proof. From the proposition above, the domain of the closure of the symmetric operator D_a is the domain of its H_a^0-adjoint, so that $\overline{D_a}$ is self-adjoint. Then $\Delta_a = D_a^2$ is also essentially self-adjoint. □

Proposition 5.13. *(Elliptic regularity) Suppose that $\omega \in \ker D_a \subseteq H_a^1$. Then ω is smooth.*

Proof. If $D_a\omega = 0$ for some $\omega \in H_a^1$. We will show by induction that $\omega \in H_a^k$ for all k, and then the Sobolev embedding theorem implies that ω is smooth. Suppose that we know $\omega \in H_a^{k-1}$ for some $k \geq 2$. Let $\{F_\rho\}$ be a family of Friedrich's mollifiers. Then from the elliptic estimates (Lemma 5.5), there is a constant $C_{k-1} > 0$ such that

$$\begin{aligned}\|F_\rho\omega\|_k &\leq C_{k-1}\left(\|F_\rho\omega\|_{k-1} + \|D_aF_\rho\omega\|_{k-1}\right) \\ &\leq C_{k-1}\left(\|F_\rho\omega\|_{k-1} + \|F_\rho D_a\omega\|_{k-1} + \|[D_a, F_\rho]\,\omega\|_{k-1}\right) \\ &= C_{k-1}\left(\|F_\rho\omega\|_{k-1} + \|[D_a, F_\rho]\,\omega\|_{k-1}\right).\end{aligned}$$

Thus $\|F_\rho\omega\|_k$ is bounded by Corollary 5.10. We now proceed as in the proof of Proposition 5.11 to say that there is a sequence $\rho_j \to 0$ such that $F_{\rho_j}\omega \to \omega'$ weakly in H_a^k and strongly to H_a^0. Thus, we get $\omega = \omega' \in H_a^k$. □

Corollary 5.14. *Eigenforms of D_a are smooth.*

Proof. The proof above also is easily modified if $D_a\omega = \lambda\omega$ to show that the eigenforms of D_a are smooth. □

We will now use a standard technique to derive the spectral theorem for D_a and Δ_a from these basic facts (c.f. [16, Chapter 5])

Lemma 5.15. *Let $\overline{G} = \{(\omega, D_a\omega) : \omega \in H_a^1\} \subseteq H_a^1 \times H_a^0$ denote the closure of the graph of D_a. Let $J : H_a^0 \times H_a^0 \to H_a^0 \times H_a^0$ be defined by $J(x, y) = (y, -x)$. Then there is an orthogonal direct sum decomposition*

$$H_a^0 \oplus H_a^0 = \overline{G} \oplus J\overline{G}.$$

Proof. Suppose $(x,y)\in\overline{G}^{\perp}$. Then, for all $\omega\in\Omega_a(M)$,

$$0=\langle(x,y),(\omega,D_a\omega)\rangle=\langle x,\omega\rangle+\langle y,D_a\omega\rangle,$$

so that $D_a y+x=0$ weakly. By Proposition 5.11, $y\in H_a^1$, so $(y,-x)\in\overline{G}$, so $(x,y)\in J\overline{G}$. □

Definition 5.16. *Let the operator $Q_a: H_a^0\to H_a^1$ be defined by the following equation: for any $\alpha\in H_a^0$, $(Q_a\alpha, D_aQ_a\alpha)$ is the orthogonal projection of $(\alpha,0)$ to $\overline{G}$ in $H_a^0\oplus H_a^0$.*

As $\|\alpha\|_0^2\ge\|Q_a\alpha\|_0^2+\|D_aQ_a\alpha\|_0^2\ge c\|Q_a\alpha\|_1^2$, then Q_a is bounded as an operator from H_a^0 to H_a^1. By Rellich's Theorem, Q_a is compact as an operator from H_a^0 to H_a^0. It is self-adjoint, positive, and injective, and has norm ≤ 1. By the spectral theorem for compact, self-adjoint operators, H_a^0 can be decomposed as a direct sum of finite-dimensional eigenspaces of Q_a, and the eigenvalues approach 0 as the only accumulation point. Given an eigenvector α of Q_a corresponding to the eigenvalue $\mu>0$, so that $0<\mu\le 1$, by Lemma 5.15 there exists η such that

$$\begin{aligned}(\alpha,0)&=(Q_a\alpha,D_aQ_a\alpha)+(-D_a\eta,\eta)\\&=\mu(\alpha,D_a\alpha)+(-D_a\eta,\eta),\end{aligned}$$

so that $(\mu-1)\alpha=D_a\eta$ and $\eta=-\mu D_a\alpha$. Letting $\lambda^2=\frac{1-\mu}{\mu}$ and $\beta=-\frac{1}{\mu\lambda}\eta$, we have

$$D_a\alpha=\lambda\beta,\ D_a\beta=\lambda\alpha.$$

Thus $\alpha\pm\beta$ are eigenforms of D_a with eigenvalues $\pm\lambda$. Thus, H_a^0 can be decomposed as an orthogonal direct sum of finite-dimensional eigenspaces of D_a. We now have the following.

Theorem 5.17. *(Spectral Theorem for the antibasic operators) The spectrum of the antibasic Laplacian and antibasic de Rham operators consists of real eigenvalues of finite multiplicity, with accumulation points at infinity. The smooth eigenforms of D_a are also the eigenforms of Δ_a and can be chosen to form a complete orthonormal basis of H_a^0.*

Proof. Besides the above computations, observe that $\Delta_a=D_a^2$. □

6. The antibasic Hodge decomposition and homotopy invariance

Let M be a closed manifold of dimension n endowed with a foliation of codimension q and a bundle-like metric. The basic Hodge decomposition theorem (proved in [12]) gives

$$\Omega_b^k(M)=\operatorname{im}(d_{b,k-1})\oplus\mathcal{H}_b^k\oplus\operatorname{im}(\delta_{b,k+1}),$$

where $d_{b,k}=d:\Omega_b^k(M)\to\Omega_b^{k+1}(M)$ is the exterior derivative restricted to basic forms with L^2-adjoint $\delta_{b,k+1}=P_b\delta:\Omega_b^{k+1}(M)\to\Omega_b^k(M)$, and where $\mathcal{H}_b^k=\ker(\Delta_{b,k})$ is the space of basic harmonic k-forms. Also

$$\ker(d_{b,k})=\operatorname{im}(d_{b,k-1})\oplus\mathcal{H}_b^k \text{ and } \ker(\delta_{b,k})=\mathcal{H}_b^k\oplus\operatorname{im}(\delta_{b,k+1}),$$

so the basic cohomology groups satisfy $H_b^k(M,\mathcal{F})\cong\mathcal{H}_b^k$. We now have the tools to prove the antibasic version.

First, note that there is an alternative de Rham complex that uses δ as a differential. Writing $\Omega^j=\Omega^j(M)$, the complex

$$\Omega^n\xrightarrow{\delta_n}\Omega^{n-1}\xrightarrow{\delta_{n-1}}\dots\xrightarrow{\delta_1}\Omega^0\longrightarrow 0$$

satisfies $\delta_{k-1}\delta_k=0$ for $0\le k\le n$, and the de Rham cohomology satisfies

$$H^k(M)=\frac{\ker\left(\delta_k:\Omega^k\to\Omega^{k-1}\right)}{\operatorname{im}\left(\delta_{k+1}:\Omega^{k-1}\to\Omega^k\right)}.$$

Abbreviating $\Omega_a^j=\Omega_a^j(M)$, the antibasic de Rham complex is a subcomplex

$$\Omega_a^n\xrightarrow{\delta_n}\Omega_a^{n-1}\xrightarrow{\delta_{n-1}}\dots\xrightarrow{\delta_1}\Omega_a^0\longrightarrow 0$$

We adopt a standard proof of Hodge decomposition to our case (c.f. [16, Chapter 6]) and utilize the analytic results proved in the previous section.

Theorem 6.1. *(**Antibasic Hodge Theorem**) Suppose that $(M, \mathcal{F})$ is a Riemannian foliation with bundle-like metric. Then for $0 \leq k \leq n$ the antibasic cohomology groups satisfy*

$$H_a^k(M, \mathcal{F}) \cong \mathcal{H}_a^k.$$

Proof. From Theorem 5.17, $\mathcal{H}_a^k = \ker\left(\Delta_a|_{\Omega_a^k}\right)$ is finite dimensional for all k. Consider the following subcomplex of the antibasic de Rham complex, with 0 being the codifferential, and the inclusion maps:

$$\begin{array}{ccccccccc} \ldots & \xrightarrow{0} & \mathcal{H}_a^{j+1} & \xrightarrow{0} & \mathcal{H}_a^{j} & \xrightarrow{0} & \mathcal{H}_a^{j-1} & \xrightarrow{0} & \ldots \\ & & \downarrow^{\imath} & & \downarrow^{\imath} & & \downarrow^{\imath} & & \\ \ldots & \xrightarrow{\delta_{j+2}} & \Omega_a^{j+1} & \xrightarrow{\delta_{j+1}} & \Omega_a^{j} & \xrightarrow{\delta_{j}} & \Omega_a^{j-1} & \xrightarrow{\delta_{j-1}} & \ldots \end{array}$$

We will show that the inclusion $\imath$ is a chain equivalence. We define the map $P : \Omega_a^j \to \mathcal{H}_a^j$ to be the restriction of the orthogonal projection $L^2\left(\Omega_a^j\right) \to \mathcal{H}_a^j$ to smooth antibasic forms. Then $P\imath = 1$ and $\imath P = 1 - f(D_a)$, where

$$f(\lambda) = \begin{cases} 1 & \text{if } \lambda \neq 0 \\ 0 & \text{if } \lambda = 0 \end{cases}$$

and where we have used Theorem 5.17 and the functional calculus. Let

$$g(\lambda) = \begin{cases} \frac{1}{\lambda^2} & \text{if } \lambda \neq 0 \\ 0 & \text{if } \lambda = 0 \end{cases}.$$

Then g is bounded on $\sigma(D_a)$, so the Green's operator $G_a = g(D_a)$ extends to a bounded operator on H_a^0. We see that $D_a^2 G_a = f(D_a) = 1 - \imath P$, and also

$$D_a^2 G_a = (\delta d_a + d_a \delta) G_a = \delta d_a G_a + G_a d_a \delta.$$

Since Δ_a commutes with d_a, we have $H_a = d_a G_a = G_a d_a$. Thus, H_a satisfies

$$1 - \imath P = \delta H_a + H_a \delta$$

and is thus a chain homotopy between $\imath P$ and 1, so $\imath$ is a chain equivalence.

□

Corollary 6.2. *On a Riemannian foliation on a closed manifold, the antibasic cohomology groups are finite dimensional.*

Remark 6.3. *After reading an early version of this paper, J.A. Álvarez-López observed that when $\mathcal{F}$ is Riemannian, the Hodge star operator maps the antibasic complex isomorphically to the p-basic complex investigated in* [19]*, where the author showed that every k-basic cohomology group is finite dimensional; the corollary also follows from that observation. If $p = 1$, M is oriented, and $\mathcal{F}$ is Riemannian, then $H_a^\bullet(M, \mathcal{F})$ is in fact isomorphic to the term $E_2^{1,\bullet}$ of the spectral sequence of $\mathcal{F}$; see* [19, Lemma 2.5]*. This agrees with computations made for the case $p = 1$.*

Remark 6.4. *For general foliations, the antibasic cohomology groups can be infinite-dimensional. See Example 9.3.*

The following corollary follows in the standard way.

Corollary 6.5. *We have the following L^2-orthogonal decomposition:*

$$\Omega_a^k = \mathcal{H}_a^k \oplus \operatorname{im}\left(\delta|_{\Omega_a^{k+1}}\right) \oplus \operatorname{im}\left(d_a|_{\Omega_a^{k-1}}\right).$$

Proof. We utilize the spectral theorem again, noting the eigenform decomposition $\Delta_a \geq 0$. For any smooth antibasic k-form α, $\Delta_a \alpha = (d_a + \delta_a)^2 \alpha = 0$ if and only if

$$\begin{aligned} 0 &= \langle (d_a + \delta_a)\alpha, (d_a + \delta_a)\alpha \rangle \\ &= \langle d_a\alpha, d_a\alpha \rangle + \langle \delta\alpha, \delta\alpha \rangle , \end{aligned}$$

if and only if $d_a\alpha = 0$ and $\delta\alpha = 0$. Because d_a and δ_a commute with Δ_a, the spaces of forms $\Omega_a^{k,+}$ with positive Δ_a eigenvalues are mapped isomorphically by $d_a + \delta$. Thus

$$\Omega_a^{k,+} \cong d_a \left(\Omega_a^{k,+} \right) \oplus \delta \left(\Omega_a^{k,+} \right) \subseteq \Omega_a^{k+1,+} \oplus \Omega_a^{k-1,+}.$$

Then

$$\Omega_a^{*} = \mathcal{H}_a^{*} + \operatorname{im}\left(\delta|_{\Omega_a^{*}} \right) + \operatorname{im}\left(d_a|_{\Omega_a^{*}} \right).$$

Also $\operatorname{im}\left(\delta|_{\Omega_a^{k+1}} \right)$ and $\operatorname{im}\left(d|_{\Omega_a^{k-1}} \right)$ are orthogonal:

$$\langle d_a\alpha, \delta\beta \rangle = \left\langle d_a^2\alpha, \beta \right\rangle = 0$$

for all α, β and likewise if $\gamma \in \mathcal{H}_a^k$, then for all antibasic forms η, θ we have

$$\begin{aligned} \langle d_a\eta, \gamma \rangle &= \langle \eta, \delta\gamma \rangle = 0, \\ \langle \delta\theta, \gamma \rangle &= \langle \theta, d_a\gamma \rangle = 0. \end{aligned}$$

Therefore, the result follows. □

Remark 6.6. *For the same reason that δ can be used in place of d in computing de Rham cohomology, the same reasoning shows from the Hodge theorem (in the Riemannian foliation case) that*

$$H_a^k(M, \mathcal{F}) \cong \frac{\ker\left(d_a : \Omega_a^k \to \Omega_a^{k+1} \right)}{\operatorname{im}\left(d_a : \Omega_a^{k-1} \to \Omega_a^k \right)}.$$

We now use the above formula for antibasic cohomology to prove the foliated homotopy invariance of antibasic cohomology in the case of Riemannian foliations.

Lemma 6.7. *Let $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ be Riemannian foliations of closed manifolds with bundle-like metrics, and let f be a foliated map from $(M, \mathcal{F}) \to (M', \mathcal{F}')$. Then $P_a f^* P_a'$ induces a linear map on antibasic cohomology.*

Proof. We consider antibasic cohomology through the isomorphism $H_a^k(M, \mathcal{F}) \cong \frac{\ker\left(d_a : \Omega_a^k \to \Omega_a^{k+1} \right)}{\operatorname{im}\left(d_a : \Omega_a^{k-1} \to \Omega_a^k \right)}$. Then we have the following equation on antibasic forms, using (3.4) and the fact that pullbacks by foliated maps preserve the basic forms:

$$\begin{aligned} P_a f^* P_a' d_a' &= P_a f^* P_a' d \\ &= P_a f^* \left(dP_a' + P_b' \varepsilon^{*\prime} \right) \\ &= P_a f^* dP_a' + P_a f^* P_b' \varepsilon^{*\prime} \\ &= P_a df^* P_a' + P_a P_b f^* P_b' \varepsilon^{*\prime} \\ &= P_a dP_a f^* P_a' = d_a P_a f^* P_a'. \end{aligned}$$

Therefore, $P_a f^* P_a'$ induces a linear map from d_a'-cohomology to d_a-cohomology. □

Theorem 6.8. *(Foliated Homotopy Axiom of Antibasic Cohomology) Let $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ be Riemannian foliations of closed manifolds with bundle-like metrics, and let f_1 and f_2 be two foliated maps from $(M, \mathcal{F}) \to (M', \mathcal{F}')$. If f_1 is foliated homotopic to f_2, then $P_a f_1^* P_a'$ and $P_a f_2^* P_a'$ induce the same map on antibasic cohomology.*

Proof. Again we view antibasic cohomology through the isomorphism $H_a^k(M,\mathcal{F}) \cong \frac{\ker\left(d_a:\Omega_a^k\to\Omega_a^{k+1}\right)}{\operatorname{im}\left(d_a:\Omega_a^{k-1}\to\Omega_a^k\right)}$. Let $H:[0,1]\times M\to M'$ be a foliated homotopy such that $H(0,x)=f_1(x)$ and $H(1,x)=f_2(x)$. Then, define $j_s:M\to[0,1]\times M$ by $j_s(x)=(s,x)$, and let $h:\Omega^k(M')\to\Omega^{k-1}(M)$ by

$$h(\sigma)=\int_0^1 j_s^*\left(\partial_t \lrcorner H^*\sigma\right)\,ds.$$

Note that h preserves the basic forms since H is a foliated homotopy. By a standard calculation, we have that

$$f_2^*-f_1^*=dh+hd$$

on $\Omega(M')$. Then we apply P_a on the left and P_a' on the right and use the equation $P_adP_a=P_ad$ to get

$$\begin{aligned}
P_af_2^*P_a'-P_af_1^*P_a' &= P_adhP_a'+P_ahdP_a' \\
&= P_adP_aP_ahP_a'+P_ah\left(P_a'd-P_b'\varepsilon^{*\prime}\right) \\
&= P_adP_aP_ahP_a'+P_ahP_a'P_a'd-P_ahP_b'\varepsilon^{*\prime} \\
&= d_a\left(P_ahP_a'\right)+\left(P_ahP_a'\right)d_a'-P_aP_bhP_b'\varepsilon^{*\prime} \\
&= d_a\left(P_ahP_a'\right)+\left(P_ahP_a'\right)d_a'.
\end{aligned}$$

Thus P_ahP_a' is a chain homotopy between $P_af_2^*P_a'$ and $P_af_1^*P_a'$. □

Corollary 6.9. *(Foliated Homotopy Invariance of Antibasic Cohomology) If $(M,\mathcal{F})$ and $(M',\mathcal{F}')$ are Riemannian foliations of closed manifolds with bundle-like metrics that foliated homotopy equivalent, then their antibasic cohomology groups are isomorphic.*

Proof. Suppose that $f_1:M\to M'$ is a foliated map such that there exists a foliated map $f_2:M'\to M$ such that $f_1\circ f_2$ and $f_2\circ f_1$ are each foliated homotopic to the identity. Then we have that

$$\begin{aligned}
P_a'\left(f_1\circ f_2\right)^*P_a' &= \mathrm{Id}':H_a^k\left(M'\right)\to H_a^k\left(M'\right), \\
P_a\left(f_2\circ f_1\right)^*P_a &= \mathrm{Id}:H_a^k(M)\to H_a^k(M).
\end{aligned}$$

Since the pullback by f_2 preserves the basic forms,

$$\begin{aligned}
\mathrm{Id}' &= P_a'f_2^*f_1^*P_a'=P_a'f_2^*\left(P_a+P_b\right)f_1^*P_a' \\
&= P_a'f_2^*P_aP_af_1^*P_a'+P_a'f_2^*P_bf_1^*P_a' \\
&= P_a'f_2^*P_aP_af_1^*P_a'+P_a'P_b'f_2^*P_bf_1^*P_a'=\left(P_a'f_2^*P_a\right)\left(P_af_1^*P_a'\right),
\end{aligned}$$

and similarly $(P_af_1^*P_a')(P_a'f_2^*P_a)=\mathrm{Id}$, so we must have that $P_af_1^*P_a'$ is an isomorphism from $H_a^k(M')$ to $H_a^k(M)$. □

7. Properties and applications

First we consider the simple case when the operators ε and ε^* are zero. In this case, the antibasic Betti numbers can be computed from the ordinary Betti numbers and basic Betti numbers.

Proposition 7.1. *Suppose that $(M,\mathcal{F})$ is a Riemannian foliation of a closed manifold of dimension n, such that the normal bundle $(T\mathcal{F})^\perp$ is involutive. Then for any bundle-like metric, the antibasic cohomology and basic cohomology add to the ordinary cohomology. That is, for $0\le k\le n$*

$$H^k(M)\cong H_b^k(M,\mathcal{F})\oplus H_a^k(M,\mathcal{F}).$$

Proof. First, we choose a bundle-like metric so that the mean curvature is basic; this can always be done [7]. The operator ε^* satisfies

$$\begin{aligned}
\varepsilon^* &= -\kappa_a\wedge+(-1)^k\left(\chi_{\mathcal{F}}\lrcorner\right)\left(\varphi_0\wedge\right) \\
&= 0
\end{aligned}$$

under the hypotheses, since $\varphi_0 = 0$ if and only if the normal bundle is involutive. Then, by Theorem 4.1, the antibasic Laplacian is precisely a restriction of the ordinary Laplacian. Similarly, by formula (3.2) and the results of [12], the basic Laplacian is a restriction of the ordinary Laplacian. Thus the ordinary Laplacian preserves the basic and antibasic forms, and it decomposes as a direct sum of the basic and antibasic Laplacians. The harmonic forms decompose into basic and antibasic parts, and the Hodge theorem implies the result. □

Corollary 7.2. *Suppose that $(M, \mathcal{F})$ is a Riemannian foliation of a closed manifold of dimension n, such that the normal bundle $(T\mathcal{F})^{\perp}$ is involutive. Then* $\dim H_b^k(M, \mathcal{F}) \leq \dim H^k(M)$, $\dim H_a^k(M, \mathcal{F}) \leq \dim H^k(M)$.

Remark 7.3. *It was essentially already known that* $\dim H_b^k(M, \mathcal{F}) \leq \dim H^k(M)$ *in this case, because using* [12] *and* [7] *we see that for a metric with basic mean curvature, $\delta_b = \delta$ when restricted to basic forms.*

Proposition 7.4. *Suppose that $(M, \mathcal{F}, g)$ is a Riemannian foliation of a closed manifold of dimension n with bundle-like metric, such that the mean curvature form is basic and the normal bundle $N\mathcal{F} = (T\mathcal{F})^{\perp}$ is involutive. Then the wedge product induces a bilinear product on basic and antibasic cohomology:*
$$\wedge : H_b^r(M, \mathcal{F}) \otimes H_a^s(M, \mathcal{F}) \to H_a^{r+s}(M, \mathcal{F}).$$

Proof. The operator $\varepsilon = 0$ under the assumptions, so that both d and δ restrict to both basic and antibasic forms. The wedge product of a basic and antibasic form is antibasic (since $P_b(P_b\alpha \wedge \beta) = P_b\alpha \wedge P_b\beta$ from [12]), so that the result follows from the standard result in de Rham cohomology. □

In more generality, if the mean curvature is basic but without the assumption on φ_0, the same result is true for $k = 0$.

Proposition 7.5. *Suppose that $(M, \mathcal{F}, g)$ is a Riemannian foliation of a closed manifold of dimension n. Then*
$$H^0(M) \cong H_b^0(M, \mathcal{F}) \oplus H_a^0(M, \mathcal{F}).$$
In particular, if M is connected, then $H_b^0(M, \mathcal{F}) \cong \mathbb{R}$ and $H_a^0(M, \mathcal{F}) \cong \{0\}$.

Proof. We first choose a bundle-like metric such that the mean curvature is basic. By Theorem 4.1, Δ_a is the restriction of $\Delta + \delta P_b \varepsilon^* + P_b \varepsilon^* \delta$ to $\Omega_a^*(M)$, and on functions this is $\Delta + \delta P_b \varepsilon^*$. But also $\varepsilon^* = -\kappa_a \wedge + (-1)^k (\chi_{\mathcal{F}} \lrcorner)(\varphi_0 \wedge) = 0$ on functions, so that Δ_a is the restriction of the ordinary Laplacian. Also, by the results of [12], Δ_b is the restriction of $\Delta + \varepsilon d + d\varepsilon$ to $\Omega_b^*(M)$, and on functions this is $\Delta + \varepsilon d$, but in our case $\varepsilon = (-1)^k (\varphi_0 \lrcorner)(\chi_{\mathcal{F}} \wedge)$ is zero on basic one-forms so that Δ_b is the restriction of Δ. Thus Δ is the orthogonal direct sum of the restrictions to basic and antibasic functions, and the result follows from the Hodge theorem. □

Proposition 7.6. *Suppose that $(M, \mathcal{F})$ is a Riemannian foliation on a closed, connected manifold. Then,*
$$\dim H^1(M) \leq \dim H_b^1(M, \mathcal{F}) + \dim H_a^1(M, \mathcal{F}).$$

Proof. First, we choose a bundle-like metric with basic mean curvature. Given a Δ-harmonic form β, consider $P_a\beta$. We see that
$$d_a(P_a\beta) = P_a d(P_a\beta) = P_a(P_a(d\beta) - P_b(\varepsilon^*\beta)) = 0.$$
Also,
$$\delta_a P_a \beta = \delta P_a \beta = P_a \delta \beta + \varepsilon(P_b\beta) = 0,$$
because $\varepsilon = 0$ on basic one-forms. Thus the map $\beta \mapsto P_a\beta$ maps harmonic one-forms to antibasic harmonic one-forms. The kernel of this map is the set of basic forms β such that $d\beta = 0$ and $0 = \delta\beta = (\delta_b - \varepsilon)\beta = \delta_b\beta$, since ε is zero on basic one-forms. Thus the kernel is the set of Δ_b-harmonic

forms. By the Hodge theorem, the result follows, since $H^1(M) \cong H_b^1(M,\mathcal{F}) \oplus P_a\left(\mathcal{H}^1(M)\right) \subseteq H_b^1(M,\mathcal{F}) \oplus \mathcal{H}_a^1(M,\mathcal{F})$. □

Remark 7.7. *Note that in general $H_b^1(M) \hookrightarrow H^1(M)$ is an injection for all foliations, so always $\dim H_b^1(M) \leq \dim H^1(M)$. Thus, if $H_a^1(M,\mathcal{F}) \cong \{0\}$, then $H_b^1(M) \cong H^1(M)$, so that every harmonic one-form is basic.*

Another simple class of examples of Riemannian foliations occurs when the orbits of a compact connected Lie group action all have the same dimension. In this case, we may choose a metric such that the Lie group acts by isometries. The Lie group acts on differential forms by pullback, and this action commutes with d and δ. Thus, if we decompose the differential forms according to the irreducible representations $\rho \in \widehat{G}$ of G, we have the L^2-orthogonal direct sum

$$\Omega^*(M) = \bigoplus_{\rho\in\widehat{G}} \Omega^{*,\rho}(M)$$

where $\Omega^{*,\rho}(M)$ is the space of differential forms of type $\rho : G \to U(V_\rho)$. That is,

$$\Omega^{*,\rho}(M) = \bigcup_{f\in \mathrm{Hom}_G(V_\rho,\Omega^*(M))} f(V_\rho).$$

Because of the metric invariance, both d and δ respect this decomposition. It is well-known that the harmonic forms are always invariant (i.e. belong to $\Omega^{*,\rho_0}(M)$, where ρ_0 is the trivial representation). Also, for the foliation $\mathcal{F}$ by G-orbits, we have $\Omega_b^*(M) \subseteq \Omega^{*,\rho_0}(M)$. We let d_j, δ_j refer to the restrictions of d, δ to Ω^j, and we let $d_{a,j}, \delta_{aj}, d_{b,j}, \delta_{bj}$ denote the corresponding restrictions to basic and antibasic forms. We use the superscript ρ to denote further restrictions to $\Omega^{*,\rho}(M)$. Then we have

$$\begin{aligned} d_j &= \bigoplus_{\rho\in\widehat{G}} d_j^\rho, \\ d_{bj} &= d_{bj}^{\rho_0}, \\ d_{aj} &= d_{aj}^{\rho_0} \oplus \bigoplus_{\substack{\rho\in\widehat{G}\\ \rho\neq\rho_0}} d_j^\rho, \\ \delta_{bj} &= \delta_{bj}^{\rho_0}, \\ \delta_{aj} &= \delta_{aj}^{\rho_0} \oplus \bigoplus_{\substack{\rho\in\widehat{G}\\ \rho\neq\rho_0}} \delta_j^\rho. \end{aligned}$$

Thus, in computing either the basic or antibasic cohomology, it is sufficient to restrict to invariant forms. The result below follows.

Proposition 7.8. *Let G be a connected, compact Lie group that acts on a connected closed manifold M by isometries. Let $(M,\mathcal{F},g)$ be the Riemannian foliation with bundle-like metric given by the G-orbits. Then*

$$H_b^j(M,\mathcal{F}) \cong \frac{\ker d_{bj}^{\rho_0}}{\mathrm{im}\, d_{b(j-1)}^{\rho_0}};\ H_a^j(M,\mathcal{F}) \cong \frac{\ker \delta_{aj}^{\rho_0}}{\mathrm{im}\, \delta_{a(j+1)}^{\rho_0}}.$$

In particular,

$$H_b^0(M,\mathcal{F}) \cong \mathbb{R};\ H_a^0(M,\mathcal{F}) \cong \{0\}.$$

Proof. The first part follows from the discussion above. Next, observe that all G-invariant functions are basic, so that $\Omega_b^0(M,\mathcal{F}) = \Omega^{0,\rho_0}(M)$, so that $\ker\left(d_{b0}^{\rho_0}\right) = \ker\left(d_0^{\rho_0}\right)$ consists of the constant functions, and $\ker \delta_{a0}^{\rho_0} = \{0\}$. The second part also follows from Proposition 7.5, since the mean curvature is always basic in this case. □

8. The case of Riemannian flows

In this section, we study tautness and cohomology for Riemannian flows. The following result shows a relationship between basic and antibasic cohomology when the flow is taut. We will see evidence of this behavior in Example 9.1 and Example 9.2. We will use standard techniques in the study of these flows, which can also be found for example in [3], [11], [4].

Proposition 8.1. *Suppose that $(M,\mathcal{F})$ is a Riemannian flow on a closed manifold with bundle-like metric g with basic mean curvature κ and characteristic form $\chi_{\mathcal{F}}$. If $\kappa=0$, then for all r, the map $\alpha\mapsto\chi_{\mathcal{F}}\wedge\alpha$ maps basic harmonic r-forms to antibasic harmonic $(r+1)$-forms injectively. Thus, $\dim\left(H_a^{r+1}(M,\mathcal{F})\right)\geq\dim\left(H_b^r(M,\mathcal{F})\right)$ whenever $[\kappa]=0\in H_b^1(M,\mathcal{F})$.*

Proof. Suppose that $[\kappa]=0$. We then choose a bundle-like metric such that $\kappa=0$. Note that φ_0 (from Rummler's formula) is basic for every Riemannian flow with bundle-like metric with basic mean curvature, and so it is certainly basic in this case. With this metric, for any basic harmonic r-form α,

$$\begin{aligned} d(\chi_{\mathcal{F}}\wedge\alpha) &= d\chi_{\mathcal{F}}\wedge\alpha-\chi_{\mathcal{F}}\wedge d\alpha \\ &= \varphi_0\wedge\alpha, \end{aligned}$$

which is basic, so that $d_a(\chi_{\mathcal{F}}\wedge\alpha)=0$. Let $\chi_{\mathcal{F}}^{\#}=\xi$, and choose the usual adapted orthonormal frame $\{b_i\}=\{e_i\}\cup\{\xi\}$ near a point, where the e_i are basic and ∇^Q-parallel at the point in question. Note that $\delta\chi_{\mathcal{F}}=0$ because the metric is bundle-like, and then

$$\begin{aligned} \delta(\chi_{\mathcal{F}}\wedge\alpha) &= -\sum_i b_i\lrcorner\nabla^M_{b_i}(\chi_{\mathcal{F}}\wedge\alpha) \\ &= -\sum_i b_i\lrcorner\left(\nabla^M_{b_i}\chi_{\mathcal{F}}\wedge\alpha+\chi_{\mathcal{F}}\wedge\nabla^M_{b_i}\alpha\right) \\ &= (\delta\chi_{\mathcal{F}})\,\alpha+\sum_i\nabla^M_{b_i}\chi_{\mathcal{F}}\wedge(b_i\lrcorner\alpha)-\nabla^M_\xi\alpha+\chi_{\mathcal{F}}\wedge\sum_i\left(b_i\lrcorner\nabla^M_{b_i}\alpha\right) \\ &= \sum_i\nabla^M_{b_i}\chi_{\mathcal{F}}\wedge(b_i\lrcorner\alpha)-\nabla^M_\xi\alpha-\chi_{\mathcal{F}}\wedge\delta\alpha. \end{aligned}$$

But since α is basic harmonic, $\delta\alpha=\delta_b\alpha-\varepsilon\alpha=0+\varphi_0\lrcorner(\chi_{\mathcal{F}}\wedge\alpha)=\chi_{\mathcal{F}}\wedge(\varphi_0\lrcorner\alpha)$. Thus, $\chi_{\mathcal{F}}\wedge\delta\alpha=0$, so that

$$\begin{aligned} \delta(\chi_{\mathcal{F}}\wedge\alpha) &= \sum_i\nabla^M_{b_i}\chi_{\mathcal{F}}\wedge(b_i\lrcorner\alpha)-\nabla^M_\xi\alpha \\ &= \sum_i\nabla^M_{e_i}\chi_{\mathcal{F}}\wedge(e_i\lrcorner\alpha)-\nabla^M_\xi\alpha \\ &= \sum_i(he_i)^\flat\wedge(e_i\lrcorner\alpha)-\nabla^M_\xi\alpha \\ &= \sum_{i,j}g(he_i,e_j)e^j\wedge(e_i\lrcorner\alpha)-\nabla^M_\xi\alpha \\ &= -\sum_j e^j\wedge((he_j)\lrcorner\alpha)-\nabla^M_\xi\alpha, \end{aligned}$$

where the skew-adjoint O'Neill tensor h satisfies $hX=\nabla^M_X\xi$ for $X\in\Gamma(N\mathcal{F})$. Now observe from one hand that $\left(\nabla^M_\xi\alpha\right)(\xi,e_{i_1},...,e_{i_{r-1}})=0$ since α is basic and $\kappa=0$. On the other hand, we use

$$\nabla^M_\xi Z=\nabla^Q_\xi Z+h(Z)-\kappa(Z)\,\xi=\nabla^Q_\xi Z+h(Z)$$

for $Z \in \Gamma(N\mathcal{F})$ to compute

$$\begin{aligned}\left(\nabla_\xi^M \alpha\right)(e_{i_1},...,e_{i_r}) &= \xi\left(\alpha\left(e_{i_1},...,e_{i_r}\right)\right) - \sum_k \alpha\left(e_{i_1},...,\nabla_\xi^M e_{i_k},...,e_{i_r}\right)\\ &= \xi\left(\alpha\left(e_{i_1},...,e_{i_r}\right)\right) - \sum_k \alpha\left(e_{i_1},...,he_{i_k},...,e_{i_r}\right)\\ &= \left(\nabla_\xi^Q \alpha\right)(e_{i_1},...,e_{i_r}) - \sum_k \alpha\left(e_{i_1},...,he_{i_k},...,e_{i_r}\right)\\ &= -\sum_k \alpha\left(e_{i_1},...,he_{i_k},...,e_{i_r}\right).\end{aligned}$$

Now we write

$$\begin{aligned}\nabla_\xi^M \alpha &= \frac{1}{r!}\sum_{(i_1,...,i_r)} \left(\nabla_\xi^M \alpha\right)(e_{i_1},...,e_{i_r})\, e^{i_1}\wedge ... \wedge e^{i_r}\\ &= -\frac{1}{r!}\sum \alpha\left(e_{i_1},...,he_{i_k},...,e_{i_r}\right) e^{i_1}\wedge ... \wedge e^{i_r}\\ &= -\frac{1}{r!}\sum (-1)^{k-1}\left(he_{i_k}\lrcorner\alpha\right)\left(e_{i_1},...,\widehat{e_{i_k}},...,e_{i_r}\right) e^{i_1}\wedge ... \wedge e^{i_r}\\ &= -\frac{r}{r!}\sum \left(he_\ell\lrcorner\alpha\right)\left(e_{i_1},...,e_{i_{r-1}}\right) e^{\ell}\wedge e^{i_1}\wedge ... \wedge e^{i_{r-1}}\\ &= -\sum_\ell e^\ell \wedge \left(he_\ell\lrcorner\alpha\right).\end{aligned}$$

Thus, substituting we have

$$\begin{aligned}\delta\left(\chi_{\mathcal{F}}\wedge\alpha\right) &= -\sum_j e^j\wedge\left((he_j)\lrcorner\alpha\right) - \nabla_\xi^M\alpha\\ &= -\sum_j e^j\wedge\left((he_j)\lrcorner\alpha\right) + \sum_\ell e^\ell\wedge (he_\ell)\lrcorner\alpha\\ &= 0,\end{aligned}$$

so that $\chi_{\mathcal{F}}\wedge\alpha \in \mathcal{H}_a^{r+1}(M,\mathcal{F})$. If α is nonzero, then $\chi_{\mathcal{F}}\wedge\alpha$ is nonzero, so the class $[\chi_{\mathcal{F}}\wedge\alpha]$ is nontrivial. □

Remark 8.2. *In particular, if* $[\kappa] = 0 \in H_b^1(M,\mathcal{F})$, *then* $\dim H_a^1(M,\mathcal{F}) \geq 1$.

Lemma 8.3. *Suppose that* $(M,\mathcal{F})$ *is a Riemannian flow on a closed, connected manifold, with a bundle-like metric chosen so that the mean curvature is basic. Then for any antibasic one-forms* α *and* β,

$$\langle\Delta_a\alpha,\beta\rangle = \langle\Delta\alpha,\beta\rangle - \int_M P_b\left(\chi_{\mathcal{F}},\alpha\right) P_b\left(\chi_{\mathcal{F}},\beta\right)|\varphi_0|^2\, dv_g.$$

Proof. From Theorem 4.1,

$$\Delta_a\alpha = \Delta\alpha + \delta P_b\varepsilon^*\alpha + P_b\varepsilon^*\delta\alpha.$$

Since $\varepsilon^*\delta\alpha = 0$ and

$$\varepsilon^*\alpha = -\chi_{\mathcal{F}}\lrcorner\left(\varphi_0\wedge\alpha\right) = -\left(\chi_{\mathcal{F}},\alpha\right)\varphi_0,$$

we have

$$\begin{aligned}\delta P_b\varepsilon^*\alpha &= -\delta P_b\left(\left(\chi_{\mathcal{F}},\alpha\right)\varphi_0\right) = -\delta\left(P_b\left(\chi_{\mathcal{F}},\alpha\right)\varphi_0\right)\\ &= df\lrcorner\varphi_0 - f\delta\varphi_0,\end{aligned}$$

where $f = P_b(\chi_{\mathcal{F}}, \alpha)$. Now, using $\delta\varphi_0 = \delta_b\varphi_0 - \varepsilon\varphi_0$ with $\varepsilon\varphi_0 = -\varphi_0 \lrcorner (\chi_{\mathcal{F}} \wedge \varphi_0)$, we write

$$\begin{aligned}
\langle \Delta_a \alpha, \beta \rangle &= \langle \Delta\alpha, \beta\rangle + \langle df \lrcorner \varphi_0 - f\delta\varphi_0, \beta\rangle \\
&= \langle \Delta\alpha, \beta\rangle + \langle \varphi_0, df \wedge \beta\rangle - \langle f\delta\varphi_0, \beta\rangle \\
&= \langle \Delta\alpha, \beta\rangle - \langle f(\delta_b - \varepsilon)\varphi_0, \beta\rangle \\
&= \langle \Delta\alpha, \beta\rangle + \langle f\varepsilon\varphi_0, \beta\rangle \\
&= \langle \Delta\alpha, \beta\rangle - \langle f(\varphi_0 \lrcorner \chi_{\mathcal{F}} \wedge \varphi_0), \beta\rangle \\
&= \langle \Delta\alpha, \beta\rangle - \left\langle f|\varphi_0|^2 \chi_{\mathcal{F}}, \beta\right\rangle \\
&= \langle \Delta\alpha, \beta\rangle - \int_M f|\varphi_0|^2 P_b(\chi_{\mathcal{F}}, \beta)\, dv_g \\
&= \langle \Delta\alpha, \beta\rangle - \int_M P_b(\chi_{\mathcal{F}}, \alpha) P_b(\chi_{\mathcal{F}}, \beta) |\varphi_0|^2\, dv_g.
\end{aligned}$$

This completes the proof. □

Theorem 8.4. *Suppose that $(M, \mathcal{F})$ is a Riemannian flow on a closed, connected manifold. If $H^1(M) = \{0\}$, then*

$$\dim H^1_a(M, \mathcal{F}) = 1.$$

If $[\kappa]$ is a nonzero class in $H^1_b(M, \mathcal{F}) \subseteq H^1(M)$, then

$$\dim H^1_a(M, \mathcal{F}) = 0.$$

Proof. We choose the bundle-like metric so that κ is basic-harmonic 1-form (as in [10]). We write any antibasic one-form α as

$$\alpha = f\chi_{\mathcal{F}} + \beta = (f_a\chi_{\mathcal{F}} + \beta) + (f_b\chi_{\mathcal{F}}) = \alpha_1 + \alpha_2,$$

where $f_a = P_a f$, $f_b = P_b f$, $\alpha_1 = f_a\chi_{\mathcal{F}} + \beta$, $\alpha_2 = f_b\chi_{\mathcal{F}}$ and β is an antibasic section of $N^*\mathcal{F}$. The L^2 inner product gives

$$\begin{aligned}
\langle \Delta_a\alpha, \alpha\rangle &= \langle \Delta_a\alpha_1, \alpha_1\rangle + \langle \Delta_a\alpha_2, \alpha_2\rangle + 2\langle \Delta_a\alpha_1, \alpha_2\rangle \\
&= \langle \Delta_a\alpha_1, \alpha_1\rangle + \langle \Delta_a\alpha_2, \alpha_2\rangle + 2\langle \Delta_a(f_a\chi_{\mathcal{F}}), f_b\chi_{\mathcal{F}}\rangle + 2\langle \Delta_a\beta, f_b\chi_{\mathcal{F}}\rangle \\
&= \langle \Delta\alpha_1, \alpha_1\rangle + \langle \Delta(f_b\chi_{\mathcal{F}}), f_b\chi_{\mathcal{F}}\rangle - \int_M f_b^2|\varphi_0|^2\, dv_g + 2\langle \Delta(f_b\chi_{\mathcal{F}}), f_a\chi_{\mathcal{F}}\rangle + 2\langle \Delta(f_b\chi_{\mathcal{F}}), \beta\rangle.
\end{aligned} \tag{8.1}$$

In the last equality, we use the formula in Lemma 8.3. In order to express each of the above inner product, we will compute $\Delta(f_b\chi_{\mathcal{F}})$, for any basic function f_b. To simplify the notation, we will omit the subscript "b" in f_b in the following computations. First we have (keep in mind that $f = f_b$ is basic)

$$\delta(f\chi_{\mathcal{F}}) = -df \lrcorner \chi_{\mathcal{F}} + f\delta\chi_{\mathcal{F}} = 0$$

since $\chi_{\mathcal{F}}$ is divergence free. Therefore, $d(\delta(f\chi_{\mathcal{F}})) = 0$. Next, using Rummler's formula $d\chi_{\mathcal{F}} = -\kappa \wedge \chi_{\mathcal{F}} + \varphi_0$, we write

$$\begin{aligned}
d(f\chi_{\mathcal{F}}) &= fd\chi_{\mathcal{F}} + df \wedge \chi_{\mathcal{F}} = -f\kappa \wedge \chi_{\mathcal{F}} + f\varphi_0 + df \wedge \chi_{\mathcal{F}}, \\
\delta d(f\chi_{\mathcal{F}}) &= \delta(-f\kappa \wedge \chi_{\mathcal{F}} + f\varphi_0 + df \wedge \chi_{\mathcal{F}}) \\
&= df \lrcorner (\kappa \wedge \chi_{\mathcal{F}} - \varphi_0) - f\delta(\kappa \wedge \chi_{\mathcal{F}} - \varphi_0) + \delta(df \wedge \chi_{\mathcal{F}}).
\end{aligned} \tag{8.2}$$

To express the divergence terms in the above equality, we consider an orthonormal frame $\{b_i\}$ of

TM and we compute for any basic 1-form θ,

$$\begin{aligned}\delta(\theta\wedge\chi_{\mathcal{F}}) &= -\sum_i b_i\lrcorner\nabla^M_{b_i}(\theta\wedge\chi_{\mathcal{F}})\\ &= -\sum_i b_i\lrcorner\left(\nabla^M_{b_i}\theta\wedge\chi_{\mathcal{F}}+\theta\wedge\nabla^M_{b_i}\chi_{\mathcal{F}}\right)\\ &= (\delta\theta)\,\chi_{\mathcal{F}}+\nabla^M_{\xi}\theta-\nabla^M_{\theta\#}\chi_{\mathcal{F}}-\theta\wedge\delta\chi_{\mathcal{F}}\\ &= (\delta\theta)\,\chi_{\mathcal{F}}+\nabla^M_{\xi}\theta-\nabla^M_{\theta\#}\chi_{\mathcal{F}}\\ &= (\delta_b\theta)\,\chi_{\mathcal{F}}+\left[\xi,\theta^{\#}\right]^{\flat}\\ &= (\delta_b\theta-(\kappa,\theta))\,\chi_{\mathcal{F}}.\end{aligned}$$

Therefore, we deduce for either $\theta=\kappa$ or $\theta=df$ that

$$\delta(\kappa\wedge\chi_{\mathcal{F}})=-|\kappa|^2\chi_{\mathcal{F}}\quad\text{and}\quad\delta(df\wedge\chi_{\mathcal{F}})=(\Delta_b f-(df,\kappa))\,\chi_{\mathcal{F}}$$

since κ is basic harmonic. Then we substitute into (8.2) to get

$$\begin{aligned}\delta d(f\chi_{\mathcal{F}}) &= df\lrcorner(\kappa\wedge\chi_{\mathcal{F}})-df\lrcorner\varphi_0-f\delta(\kappa\wedge\chi_{\mathcal{F}})+f\delta\varphi_0+\delta(df\wedge\chi_{\mathcal{F}})\\ &= (df,\kappa)\,\chi_{\mathcal{F}}-df\lrcorner\varphi_0+f|\kappa|^2\chi_{\mathcal{F}}+f\delta\varphi_0+(\Delta_b f-(df,\kappa))\,\chi_{\mathcal{F}}\\ &= -df\lrcorner\varphi_0+f|\kappa|^2\chi_{\mathcal{F}}+f\delta\varphi_0+(\Delta_b f)\,\chi_{\mathcal{F}}\\ &= -df\lrcorner\varphi_0+f|\kappa|^2\chi_{\mathcal{F}}+f(\delta_b-\varepsilon)\,\varphi_0+(\Delta_b f)\,\chi_{\mathcal{F}}\end{aligned}$$

since $\delta P_b=P_b\delta-\varepsilon P_b$ from (3.2). As $\varepsilon\varphi_0=-\varphi_0\lrcorner(\chi_{\mathcal{F}}\wedge\varphi_0)=-|\varphi_0|^2\chi_{\mathcal{F}}$, we arrive at (replace f by f_b)

$$\Delta(f_b\chi_{\mathcal{F}})=-df_b\lrcorner\varphi_0+f_b|\kappa|^2\chi_{\mathcal{F}}+f_b\delta_b\varphi_0+f_b|\varphi_0|^2\chi_{\mathcal{F}}+(\Delta_b f_b)\,\chi_{\mathcal{F}}.$$

In particular, one can easily get that

$$\langle\Delta(f_b\chi_{\mathcal{F}}),f_a\chi_{\mathcal{F}}\rangle=0\quad\text{and}\quad\langle\Delta(f_b\chi_{\mathcal{F}}),\beta\rangle=0,\tag{8.3}$$

since β is antibasic and orthogonal to ξ. Also, we have that

$$\langle\Delta(f_b\chi_{\mathcal{F}}),f_b\chi_{\mathcal{F}}\rangle=\int_M\left(f_b^2|\kappa|^2+f_b^2|\varphi_0|^2+|df_b|^2\right)dv_g.\tag{8.4}$$

Now substituting Equations (8.3) and (8.4) into Equation (8.1), we find that

$$\langle\Delta_a\alpha,\alpha\rangle=\langle\Delta\alpha_1,\alpha_1\rangle+\int_M\left(f_b^2|\kappa|^2+|df_b|^2\right)dv_g,$$

which is non-negative. Then $\langle\Delta_a\alpha,\alpha\rangle=0$ if and only if α_1 is harmonic (i.e. $\alpha_1\in H^1(M)$), f_b is constant and $f_b\kappa=0$. Recall that $\alpha=\alpha_1+\alpha_2$ with $\alpha_2=f_b\chi_{\mathcal{F}}$. In the case where $H^1(M)=\{0\}$ and α is Δ_a-harmonic 1-form, then $\alpha_1=0$ and $\alpha=f_b\chi_{\mathcal{F}}=(\text{constant})\,\chi_{\mathcal{F}}$. But this constant cannot be zero in view of Remark 8.2. Hence $\dim H^1_a(M,\mathcal{F})=1$. This proves the first part of the theorem. To prove the second part, we use the exact Gysin sequence for non-taut Riemannian flows established in [17]

$$0\to H^1_b(M)\to H^1(M)\to H^0_{\kappa,b}(M)\to\ldots$$

where $H^0_{\kappa,b}(M)\cong H^q_b(M)$, which is zero because the foliation is not taut. Thus, $H^1_b(M)\cong H^1(M)$. By the proof of Proposition 7.6, we get that $P_a\left(\mathcal{H}^1(M)\right)=0$. That means for every harmonic one-form ω, we have $P_a\omega=0$, and thus is basic. Hence $\langle\Delta_a\alpha,\alpha\rangle=0$ implies that α_1 is basic-harmonic and $f_b=0$. Thus both α_1 and α_2 are zero and then $H^1_a(M,\mathcal{F})=\{0\}$. □

Remark 8.5. *It might seem at first glance that Proposition 8.4 may contradict Proposition 7.1. But in fact, if $H^1(M) = \{0\}$ for some compact manifold M, then any Riemannian flow of M must have a normal bundle that is not involutive. The reason is as follows. First, the mean curvature can be chosen to be zero after a change in bundle-like metric, since the mean curvature must be exact. If the normal bundle is involutive, then $d\chi_{\mathcal{F}} = 0$ from Rummler's formula, and $\delta\chi_{\mathcal{F}} = 0$ (true for any Riemannian flow), so that $\chi_{\mathcal{F}}$ is a harmonic one-form and therefore represents a nontrivial class in $H^1(M)$, a contradiction. So Proposition 7.1 does not apply.*

9. Examples

We illustrate the antibasic cohomology and our theorems in some one-dimensional examples of foliations. These examples are certainly not meant to comprise a comprehensive list.

To simplify the exposition, we denote the Betti numbers for each example foliation $(M, \mathcal{F})$ as follows:

$$h^j = \dim H^j(M), \; h_b^j = \dim H_b^j(M, \mathcal{F}), h_a^j = \dim H_a^j(M, \mathcal{F}).$$

We start with the Hopf fibration, which is a taut Riemannian flow.

Example 9.1. *Using Theorem 8.4 above, we consider the Hopf fibration of $S^3 \subseteq \mathbb{C}^2 \to \mathbb{C}P^1$ via $(z_0, z_1) \to [z_0, z_1]$. The leaves of the foliation $\mathcal{F}$ are the orbits of the S^1 action $e^{it} \mapsto (e^{it}z_0, e^{it}z_1)$. This is a Riemannian flow, but the normal bundle is not involutive. The lengths of the circular leaves are constant, so the mean curvature is zero. By Theorem 8.4, $h_a^1 = 1$, and from Proposition 7.8, $h_a^0 \cong 0$. Also $H_a^2(S^3, \mathcal{F}) \subseteq H^2(S^3)$ because of Lemma 2.4, so that $h_a^2 = 0$, and $h_a^3 = h^3 = 1$. In summary, we have*

$$\begin{aligned}(h^0, h^1, h^2, h^3) &= (1, 0, 0, 1),\\ (h_b^0, h_b^1, h_b^2) &= (1, 0, 1),\\ (h_a^0, h_a^1, h_a^2, h_a^3) &= (0, 1, 0, 1).\end{aligned}$$

The following example is a Riemannian flow of a 3-manifold that is not taut.

Example 9.2. *We will compute the antibasic cohomology groups of the Carrière example from [3] in the 3-dimensional case. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. We denote respectively by V_1 and V_2 the eigenvectors associated with the eigenvalues λ and $\frac{1}{\lambda}$ of A with $\lambda > 1$ irrational. Let the hyperbolic torus $\mathbb{T}_A^3$ be the quotient of $\mathbb{T}^2 \times \mathbb{R}$ by the equivalence relation which identifies (m, t) to $(A(m), t + 1)$. The flow generated by the vector field V_2 is a transversally Lie foliation of the affine group. The Betti numbers of this closed manifold are $h^j = 1$, for $0 \leq j \leq 3$. We choose the bundle-like metric (letting (x, s, t) denote the local coordinates in the V_2 direction, V_1 direction, and $\mathbb{R}$ direction, respectively) as*

$$g = \lambda^{-2t}dx^2 + \lambda^{2t}ds^2 + dt^2.$$

The mean curvature of the flow is $\kappa = \kappa_b = \log(\lambda)\, dt$, since $\chi_{\mathcal{F}} = \lambda^{-t}dx$ is the characteristic form and $d\chi_{\mathcal{F}} = -\log(\lambda)\lambda^{-t}dt \wedge dx = -\kappa \wedge \chi_{\mathcal{F}}$. It is easily seen that the basic cohomology satisfies $h_b^j = 1$ for $j = 0, 1$ and $h_b^2 = 0$ (class of the mean curvature class being nonzero implies this; see [1]). The foliation has an involutive normal bundle, so that Proposition 7.1 applies, so that $h_a^2 = 1$, $h_a^3 = 1$ and $h_a^k = 0$ for $k = 0, 1$. In summary,

$$\begin{aligned}(h^0, h^1, h^2, h^3) &= (1, 1, 1, 1),\\ (h_b^0, h_b^1, h_b^2) &= (1, 1, 0),\\ (h_a^0, h_a^1, h_a^2, h_a^3) &= (0, 0, 1, 1).\end{aligned}$$

We now consider an example of a foliation that is not Riemannian (for any metric). This is a standard example of a foliation on a connected, compact manifold with infinite-dimensional basic cohomology; this example is from [9].

Example 9.3. *Let M be the closed 3-manifold defined as $\mathbb{R}\times T^2/\mathbb{Z}$, where $T^2=\mathbb{R}^2/\mathbb{Z}^2$ and $m\in\mathbb{Z}$ acts on $\mathbb{R}\times T^2$ by $m(t,x)=(t+m,A^m x)$, where A is the matrix $\begin{pmatrix}1&1\\0&1\end{pmatrix}$. We define the leaves of the foliation to be the t-parameter curves. Then observe that that leaf closures intersect each torus with a set of the form $S\times\{x_2\}$, where $x_2\in\mathbb{R}/\mathbb{Z}$ and S is a finite number of points for rational x_2 and is $\mathbb{R}/\mathbb{Z}$ for irrational x_2. Thus, the basic forms in the "coordinates" (t,x_1,x_2) have the form*

$$\begin{aligned}\Omega_b^0(M) &= \left\{f\in\Omega^0(M): f(t,x_1,x_2) \text{ is constant in } x_1,t\right\},\\ \Omega_b^1(M) &= \left\{f\,dx_2 : f\in\Omega_b^0(M)\right\},\\ \Omega_b^2(M) &= \left\{f\,dx_1\wedge dx_2 : f\in\Omega_b^0(M)\right\}.\end{aligned}$$

From this we can easily calculate that $h_b^0=1$, $h_b^1=1$, and $H_b^2(M,\mathcal{F})\cong\Omega_b^2(M)$, which is infinite dimensional. One may also check with a cell complex that the ordinary homology satisfies $H^j(M,\mathbb{Z})\cong\mathbb{Z}$ for $j=0,3$ and $H^j(M,\mathbb{Z})\cong\mathbb{Z}^2$ for $j=1,2$, so that the ordinary de Rham cohomology satisfies $h^j=1$ for $j=0,3$ and $h^j=2$ for $j=1,2$. We choose the metric in the $\partial_t,\partial_{x_1},\partial_{x_2}$ basis as

$$(g_{ij})=\begin{pmatrix}1&0&0\\0&1&-t\\0&-t&1+t^2\end{pmatrix}.$$

One can check the invariance with respect to the action of $m\in\mathbb{Z}$; it is chosen so that $\{e_1=\partial_t,e_2=\partial_{x_1},e_3=t\partial_{x_1}+\partial_{x_2}\}$ forms an orthonormal basis at each (t,x_1,x_2). Then the metric on covectors with basis $\{dt,dx_1,dx_2\}$ is

$$\left(g^{ij}\right)=\begin{pmatrix}1&0&0\\0&1+t^2&t\\0&t&1\end{pmatrix},$$

and the dual orthonormal basis is $\left\{e^1=dt,e^2=dx_1-t\,dx_2,e^3=dx_2\right\}$. Note also that $g=\det(g_{ij})=1$.

We now compute the antibasic forms with respect to this metric, which are the sets of smooth forms that are L^2-orthogonal to the sets of basic forms listed above. For any $x_2\in\mathbb{R}/\mathbb{Z}$, let C_{x_2} denote the torus $\{(t,x_1,x_2): t,x_1\in\mathbb{R}/\mathbb{Z}\}$. Then:

$$\begin{aligned}\Omega_a^0(M) &= \left\{f\in\Omega^0(M):\int_{C_{x_2}} f(t,x_1,x_2)\;dt\wedge dx_1=0 \text{ for all } x_2\in\mathbb{R}/\mathbb{Z}\right\},\\ \Omega_a^1(M) &= \left\{f\,dx_2+g\,(dx_1-t\,dx_2)+h\,dt : f\in\Omega_a^0(M), g,h\in\Omega^0(M)\right\},\\ \Omega_a^2(M) &= \left\{f\,dx_1\wedge dx_2+g\,dt\wedge(dx_1-t\,dx_2)+h\,dt\wedge dx_2 : f\in\Omega_a^0(M), g,h\in\Omega^0(M)\right\},\\ \Omega_a^3(M) &= \Omega^3(M).\end{aligned}$$

Immediately we have $h_a^3=1$. We compute the divergence on one-forms:

$$\begin{aligned}\langle df, a\,dt+c_1(dx_1-t\,dx_2)+c_2dx_2\rangle &= \int f_t a+f_{x_1}c_1+(t f_{x_1}+f_{x_2})\,c_2\\ &= \int f\left(-a_t-(c_1)_{x_1}-t\,(c_2)_{x_1}-(c_2)_{x_2}\right);\\ \delta\left(a\,dt+c_1(dx_1-t\,dx_2)+c_2dx_2\right) &= -a_t-\partial_1(c_1)-(t\partial_1+\partial_2)(c_2),\end{aligned}$$

which makes sense since all three vector fields are divergence-free.

We next compute the divergence of 2-forms, writing in terms of our frame and coframe. Note that in terms of an orthonormal frame $\{e_i\}$ with Christoffel symbols defined by $\nabla_{e_i}e_j=G_{ij}^k e_k$ or $d\left(e^k\right)=-G_{ij}^k e^i\wedge e^j$ (using Einstein summation convention here and subsequently) we have

$\delta\left(b_{ij}e^i\wedge e^j\right)=-e_i(b_{ij})e^j+e_j(b_{ij})e^i+b_{ij}\left(G^i_{\ell\ell}\delta^j_r-G^j_{\ell\ell}\delta^i_r-G^i_{jr}+G^j_{ir}\right)e^r$. *In what follows, we will assume that the two form is anti-symmetrized, so that* $b_{ji}=-b_{ij}$, *and then the formula above simplifies to* $\delta(b_{ij}e^i\wedge e^j)=-2e_i(b_{ij})e^j+2b_{ij}(G^i_{\ell\ell}\delta^j_r-G^i_{jr})e^r$. *In our case with* $e_1=\partial_t$, $e_2=\partial_{x_1}$, $e_3=t\partial_{x_1}+\partial_{x_2}$, *the covariant derivatives give* $\frac{1}{2}=G^2_{13}=G^1_{23}=G^1_{32}=-G^3_{12}=-G^3_{21}=-G^2_{31}$ *with all other* G^k_{ij} *zero. Also, all Lie brackets between these basis vector fields are zero except that* $[e_1,e_3]=-[e_1,e_3]=e_2$. *After a bit of calculation, we have (for antisymmetrized* $b_{ij}e^i\wedge e^j$*)*

$$\delta\left(b_{ij}e^i\wedge e^j\right)$$
$$=\left(-e_2\left(2b_{21}\right)-e_3\left(2b_{31}\right)\right)e^1+\left(-e_1\left(2b_{12}\right)-e_3\left(2b_{32}\right)-2b_{13}\right)e^2+\left(-e_1\left(2b_{13}\right)-e_2\left(2b_{23}\right)\right)e^3,$$

which implies also that (substituting $b_{ij}e^i\wedge e^j$ *below with* $\frac{1}{2}b_{ij}e^i\wedge e^j-\frac{1}{2}b_{ij}e^j\wedge e^i$ *above)*

$$\begin{aligned}&\delta\left(\sum_{i<j}b_{ij}e^i\wedge e^j\right)\\ =\;&\left(e_2\left(b_{12}\right)+e_3\left(b_{13}\right)\right)e^1+\left(-e_1\left(b_{12}\right)+e_3\left(b_{23}\right)-b_{13}\right)e^2+\left(-e_1\left(b_{13}\right)-e_2\left(b_{23}\right)\right)e^3.\end{aligned}\quad(9.1)$$

Finally we calculate divergence of 3*-forms:*

$$\begin{aligned}\delta\left(f\ e^1\wedge e^2\wedge e^3\right)&=-*d*\left(f\ e^1\wedge e^2\wedge e^3\right)=-*(df)\\&=-\frac{1}{2}\sum_{\sigma\in S_3}\operatorname{sgn}(\sigma)\,e_{\sigma_1}(f)\ e^{\sigma_2}\wedge e^{\sigma_3}\\&=-e_1(f)e^2\wedge e^3-e_2(f)e^3\wedge e^1-e_3(f)e^1\wedge e^2.\end{aligned}$$

From these formulas, we note that the divergence of a basic one-form (one of the type $c_2(x_2)\,dx_2$*) is always a basic function* ($-\partial_2c_2$)*, so that* δ *maps basic one-forms to basic functions and antibasic one-forms to antibasic functions. Then* $h^0_a=0$*, since in this case*

$$\begin{aligned}H^0(M)&=\frac{\Omega^0}{\operatorname{im}\delta|_{\Omega^1}}=\frac{\Omega^0_a\oplus\Omega^0_b}{\left(\operatorname{im}\delta|_{\Omega^1_a}\right)\oplus\left(\operatorname{im}\delta|_{\Omega^1_b}\right)}\\&=\frac{\Omega^0_a}{\left(\operatorname{im}\delta|_{\Omega^1_a}\right)}\oplus\frac{\Omega^0_b}{\left(\operatorname{im}\delta|_{\Omega^1_b}\right)}\\&=H^0_a(M,\mathcal{F})\oplus H^0_b(M,\mathcal{F})\cong H^0_a(M,\mathcal{F})\oplus H^0(M).\end{aligned}$$

(Note that the first and second step fail for foliations in general).

We now compute $H^2_a(M,\mathcal{F})$*. We have*

$$H^2_a(M,\mathcal{F})=\frac{\ker\delta|_{\Omega^2_a}}{\operatorname{im}\delta|_{\Omega^3}}\subseteq\frac{\ker\delta|_{\Omega^2}}{\operatorname{im}\delta|_{\Omega^3}}=H^2(M)\cong\mathbb{R}^2.$$

In the ordinary δ*-cohomology, the generators of* $H^2(M)$ *are* $\left[e^1\wedge e^2=dt\wedge(dx_1-tdx_2)\right]$ *and* $\left[e^2\wedge e^3=dx_1\wedge dx_2\right]$*. But* $b_{12}dt\wedge(dx_1-tdx_2)=b_{12}e^1\wedge e^2$ *is antibasic and* $b_{23}dx_1\wedge dx_2$ *is basic for any choice of constants* b_{12}, b_{23}*. Thus,* $h^2_a=1$.

We now consider 1*-forms. Observe that basic* 2*-forms have the form* $b_{23}(x_2)\,dx_1\wedge dx_2=b_{23}(x_2)\,e^2\wedge e^3$*, and from formula (9.1) above* $\delta\left(b_{23}(x_2)\,dx_1\wedge dx_2\right)=b'_{23}(x_2)e^2=b'_{23}(x_2)(dx^1-tdx^2)$ *for all basic functions* b_{23}*; note that the image is an antibasic one-form. It follows that the image of* δ_a *is a proper subset of the image of* δ *on* 2*-forms, which is contained in the space of* δ*-closed antibasic one-forms. Thus, we have*

$$H^1_a(M,\mathcal{F})=\frac{\ker\delta|_{\Omega^1_a}}{\operatorname{im}\delta|_{\Omega^2_a}}\twoheadrightarrow\frac{\ker\delta|_{\Omega^1_a}}{\operatorname{im}\delta|_{\Omega^2}}\subseteq\frac{\ker\delta|_{\Omega^1}}{\operatorname{im}\delta|_{\Omega^2}}=H^1(M)\cong\mathbb{R}^2.$$

In fact, we will show $H_a^1(M,\mathcal{F})$ is infinite-dimensional. Consider the subspace $\{F'(x_2)e^2 : F \in C^\infty(\mathbb{R}/\mathbb{Z})\}/\ker \delta|_{\Omega_a^2}$ of $H_a^1(M,\mathcal{F})$; The set $\{F'(x_2)e^2\}$ is clearly infinite-dimensional and is a subspace of $\ker \delta|_{\Omega_a^1}$. We wish to determine when two elements of this space are equivalent mod $\delta(\Omega_a^2)$. If $\delta(\beta) = F'(x_2)e^2$, then by Hodge theory $\beta = -F(x_2)e^2 \wedge e^3 +$ (harmonic 2-form) $+$ (element of $\operatorname{Im} \delta|_{\Omega^3}$). The first term is basic, and the second term is (constant)$e^2 \wedge e^3+$ (constant)$e^1 \wedge e^2$, so we may rewrite $\beta = -(F(x_2)+c_1)e^2 \wedge e^3+$ (antibasic 2-form). The form β can be antibasic if and only if $F(x_2) + c_1 = 0$, so we conclude that the space $\{F'(x_2)e^2\}/\delta(\Omega_a^2)$ is infinite-dimensional. Thus, $h_a^1 = \infty$.

In summary,

$$\begin{aligned}(h^0,h^1,h^2,h^3) &= (1,2,2,1),\\ (h_b^0,h_b^1,h_b^2) &= (1,1,\infty),\\ (h_a^0,h_a^1,h_a^2,h_a^3) &= (0,\infty,1,1).\end{aligned}$$

The following non-Riemannian flow is a simple example where the basic projection P_b and antibasic projection P_a do not preserve smoothness. In spite of that, the basic and antibasic cohomology can be calculated. From the calculations, we also see that the Hodge theorem is false for this foliation.

Example 9.4. *Let M be the flat torus $[0,2]\times[0,1]$ with opposite sides of the boundary identified. Let $\phi(x)$ be a smooth function on the circle $[0,2] \bmod 2$ such that ϕ is positive and ≤ 1 on $(0,1)$ and identically zero on $[1,2]$. Consider the foliation that whose tangent space at each point is spanned by the vector field $V(x,y) = \left(\phi(x), \sqrt{1-\phi(x)^2}\right)$. All the leaves in the region $0 < x < 1$ are noncompact and have $x = 0$ and $x = 1$ in their closure, and the leaves in the region $1 \leq x \leq 2$ are vertical circles. The set of basic functions is*

$$\Omega_b^0(M) = \left\{ \begin{array}{c} f : [0,2]\times[0,1] \to \mathbb{R} : f(x,y) = g(x) \text{ for a smooth function } g \\ \text{on } \mathbb{R}/2\mathbb{Z} \text{ such that } g(x) = \text{constant for } x \in [0,1] \bmod 2 \end{array} \right\}.$$

Since every basic normal vector field approaches 0 as $x \to 0^+$ and $x \to 1^-$, there are no bounded basic one-forms for $0 < x < 1$, so we have

$$\Omega_b^1(M) = \left\{ \begin{array}{c} \omega = h(x)\,dx : h \text{ is a smooth function on } \mathbb{R}/2\mathbb{Z} \\ \text{such that } h(x) = 0 \text{ for } x \in [0,1] \bmod 2 \end{array} \right\}.$$

Then, the set of antibasic forms are those smooth forms orthogonal to $\Omega_b^*(M)$ in L^2. We obtain

$$\begin{aligned}\Omega_a^0(M) &= \left\{ f : M \to \mathbb{R} : \int_0^1 f(x,y)\,dy = 0 \text{ for } x \in [1,2] \bmod 2 \text{ and } \int_0^1\int_0^1 f(x,y)\,dx\,dy = 0 \right\},\\ \Omega_a^1(M) &= \left\{ \alpha = a_1 dx + a_2 dy : \int_0^1 a_1(x,y)\,dy = 0 \text{ for } x \in [1,2] \bmod 2. \right\}.\end{aligned}$$

Note that in this example, the basic and antibasic projections are not smooth maps to differential forms. Observe that on functions,

$$P_b(f)(x,y) = \left\{ \begin{array}{ll} \text{average of } f \text{ over } [0,1]^2 & x \in (0,1) \\ \text{average of } f(x,\cdot) \text{ over } [0,1] & x \in (1,2) \end{array} \right. ,$$

so for instance

$$P_b(\sin(\pi x)) = \left\{ \begin{array}{ll} \frac{2}{\pi} & x \in (0,1) \\ \sin(\pi x) & x \in (1,2) \end{array} \right. ,$$

which is not a continuous function. Likewise, $P_a(\sin(\pi x)) = \sin(\pi x) - P_b(\sin(\pi x))$ *is not smooth. In any case, we may calculate the basic and antibasic cohomology groups.*

$$\begin{aligned}
H_b^0(M,\mathcal{F}) &= \ker\left(d:\Omega_b^0(M)\to\Omega_b^1(M)\right)\cong\mathbb{R},\\
H_b^1(M,\mathcal{F}) &= \frac{\Omega_b^1(M)}{\operatorname{im}\left(d:\Omega_b^0(M)\to\Omega_b^1(M)\right)}\\
&= \frac{\Omega_b^1(M)}{\left\{h(x)\,dx : \int_1^2 h(x)\,dx = 0 \text{ and } h(x)=0 \text{ for } x\in[0,1]\right\}}\\
&= \{[c\cdot \text{bump on } [0,1]]\}\cong\mathbb{R}.
\end{aligned}$$

Note that H_b^1 *is not represented by a basic harmonic form. Also,*

$$\begin{aligned}
H_a^0(M,\mathcal{F}) &= \frac{\Omega_a^0(M)}{\operatorname{im}\left(\delta:\Omega_a^1(M)\to\Omega_a^0(M)\right)}\\
&= \frac{\Omega_a^0(M)}{\left\{(a_1)_x+(a_2)_y : \int_0^1 a_1(x,y)\,dy = 0 \text{ for } x\in[1,2] \bmod 2\right\}} = \{0\},\\
H_a^1(M,\mathcal{F}) &= \frac{\ker\left(\delta:\Omega_a^1(M)\to\Omega_a^0(M)\right)}{\operatorname{im}\left(\delta:\Omega^2(M)\to\Omega_a^1(M)\right)}\\
&= \frac{\left\{a_1dx+a_2dy : \int_0^1 a_1(x,y)\,dy = 0 \text{ for } x\in[1,2] \bmod 2 \text{ and } (a_1)_x+(a_2)_y=0\right\}}{\{f_ydx-f_xdy\}}\\
&= \{[c\,dy]\}\cong\mathbb{R},\\
H_a^2(M,\mathcal{F}) &= H^2(M)\cong\mathbb{R}.
\end{aligned}$$

And we also have $h^0=h^2=1$, $h^1=2$. *In summary,*

$$\begin{aligned}
(h^0,h^1,h^2) &= (1,2,1),\\
(h_b^0,h_b^1) &= (1,1),\\
(h_a^0,h_a^1,h_a^2) &= (0,1,1).
\end{aligned}$$

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Lebanese University, Faculty of Sciences II, Department of Mathematics, P.O. Box 90656 Fanar-Matn, Lebanon and Université de Lorraine, CNRS, IECL, F-54506, Nancy, France

Email address, G. Habib: `ghabib@ul.edu.lb`

Department of Mathematics, Texas Christian University, Fort Worth, Texas 76129, USA

Email address, K. Richardson: `k.richardson@tcu.edu`