Modèles stochastiques individu-centrés avec des dynamiques allométriques : branchement, convergence, simulations numériques.

THÈSE

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Nous avons découvert moins d'une goutte d'eau dans un océan de concepts encore sans nom.

Quelques éléments de mathématiques, Jérémy Dousselin

- *Pour être bon, bosser c'est que la seconde condition.*

-*Ah oui ? Et c'est quoi, la première ?*

-*Aimer.*

Ulysse et Cyrano, Stéphane Servain, Xavier Dorison, Antoine Cristau

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Résumé

La première partie de ce travail s'attache à la conception et à l'étude d'un modèle individu-centré, structuré en un trait appelé *énergie*, décrivant une population d'individus consommant une ressource supposée constante au cours du temps. Afin de pouvoir comparer les comportements de différentes espèces vivantes, nous introduisons notamment l'énergie typique d'un individu à la naissance comme paramètre du modèle. Les mécanismes en jeu sont de deux types : des sauts aléatoires correspondant à des événements de naissance et mort ; et une évolution continue et déterministe des énergies individuelles entre ces instants de saut. Les taux de saut du processus dépendent de l'énergie des individus au cours du temps et, bien que notre modèle soit formulé de manière générale, nous nous intéressons en particulier au cas de taux *allométriques* (*i.e.* au cas de fonctions de type puissance). Les trajectoires individuelles sont indépendantes conditionnellement à l'état initial de la population, du fait de l'absence de compétition pour la ressource. Nous étudions donc un processus de branchement, et obtenons des conditions nécessaires sur les paramètres allométriques du modèle pour que ce processus soit surcritique, et que les trajectoires individuelles soient biologiquement acceptables (*i.e.* les énergies individuelles n'explosent pas, ne s'annulent pas, et les individus meurent en temps fini), et ce quelle que soit l'espèce considérée (c'est-à-dire quelle que soit l'énergie typique d'un individu à la naissance).
Dans une deuxième partie, nous modifions le modèle précédent en autorisant la ressource à varier au cours du temps, *via* un terme de renouvellement indépendant de l'état de la population, et un terme de consommation qui correspond à une compétition indirecte entre les individus. Après une étude préliminaire de la bonne définition des objets étudiés, nous considérons une suite de renormalisations du processus sous-jacent, et démontrons un résultat de tension pour les lois associées dans une asymptotique de grande population. Nous caractérisons les valeurs d'adhérence de cette suite comme des solutions d'un système d'équations intégro-différentiel, ce qui démontre au passage l'existence de solutions mesures à ce système. De plus, sous réserve que ce dernier admette une unique solution mesure, notre résultat de tension devient un résultat de convergence en loi vers cet unique processus. La nouveauté de nos résultats réside dans le caractère non-borné des taux de sauts, et nous étudions en particulier le cas allométrique.
Dans une dernière partie, nous montrons des résultats complémentaires sur le système d'équations intégro-différentiel décrit dans la deuxième partie. Nous présentons des simulations numériques illustrant notamment le résultat de convergence (sous réserve d'unicité de la solution mesure au système étudié) en grande population de la deuxième partie dans le cas allométrique, et discutons des nombreuses difficultés de mise en oeuvre d'un schéma numérique dans le cadre de notre modèle.

Mots-clés : allométries, modèle individu-centré, processus de branchement, problème de martingale, analyse asymptotique, Processus de Markov Déterministe par Morceaux, taux de saut non-bornés.

Abstract

The first part of this work focuses on the design and study of an individual-based model, structured in a trait called *energy*, describing a population of individuals consuming a resource assumed to be constant over time. In order to be able to compare the behaviour of various living species, we introduce the typical energy of an individual at birth as a parameter of the model. The mechanisms involved are of two types: random jumps corresponding to birth and death events; and a continuous and deterministic evolution of individual energies between jump times. The jump rates of the process depend on the energy of the individuals over time and, although our model is formulated in a general context, we especially focus on the case of *allometric* rates (*i.e.* they are assumed to be power functions). Individual trajectories are independent conditionally to the initial state of the population, because there is no competition for resources. We therefore study a branching process, and obtain necessary conditions on the allometric parameters of the model for this process to be supercritical, and for individual trajectories to be biologically relevant (individual energies do not explode, do not reach 0, and individuals die in finite time), for every living species we consider (*i.e.* for every typical energy of an individual at birth).
In the second part, we modify the previous model by allowing the resource to vary over time, adding a renewal term independent of the state of the population, and a consumption term corresponding to indirect competition between individuals. After a preliminary study to construct well-defined objects, we consider a sequence of renormalizations of the underlying process and show a tightness result for the associated laws in a large population asymptotic. We characterize the accumulation points of this sequence as solutions of an integro-differential system of equations, which proves the existence of measure solutions to this system. Furthermore, if such a measure solution is unique, then our tightness result becomes a convergence result towards this unique process. The originality of our results lies in the fact that we study unbounded jump rates, and in particular the allometric case.
In a final part, we show additional results for the integro-differential system of equations described in the second part. We present numerical simulations illustrating in particular the convergence (if the solution to the previous system is unique) result in a large population limit of the second part in the allometric case, and discuss the numerous difficulties encountered when we implement a numerical scheme within the framework of our model.

Keywords: allometry, individual-based model, branching process, martingale problem, asymptotic analysis, Piecewise Deterministic Markov Process, unbounded jump rates.

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1– Introduction en français

Dans cette section, nous présentons tout d'abord les questionnements de biologie évolutive qui sont à la genèse de ce travail de thèse. Ensuite, nous décrivons succinctement les résultats existants sur les modèles individu-centrés du même type que celui que nous développons. Nous décrivons ensuite les résultats principaux et les outils mathématiques utilisés dans chacun des trois chapitres de cette thèse. Pour conclure, nous donnons les perspectives de recherche liées à l'ensemble de ce travail.

1.1 Motivations biologiques

Depuis le travail de synthèse de Peters [Pet86], de nombreux articles dans le domaine de l'écologie évolutive [YI92, BMT93, OBB+13, DDA22] ont mis en évidence une relation très particulière entre la masse corporelle moyenne, ou la taille moyenne des individus, et le métabolisme moyen, ou taux métabolique moyen (qui peut être vu comme la vitesse moyenne de perte ou gain de masse des individus) d'une espèce à une autre. Cette relation est connue sous le nom de relation *allométrique*, ou simplement *allométrie*, présentée sous la forme $B \propto M^\alpha$, où B est une mesure du taux métabolique moyen au sein d'une espèce, M la masse moyenne des individus au sein de cette espèce, et α désigne le *coefficient allométrique*. Typiquement, les courbes obtenues justifiant ces relations allométriques sont similaires au graphe présenté en Figure 1.1. Il n'existe pas de consensus clair sur la valeur précise du coefficient allométrique α reliant masse moyenne et taux métabolique moyen : certains travaux démontrent qu'il existe un coefficient théorique unique basé sur une modélisation précise des systèmes vasculaires [SDF08], tandis que d'autres montrent, données à l'appui, que le coefficient peut dépendre de caractéristiques spécifiques au sein des espèces, et varier à travers la large gamme des organismes vivants [DOM+10].

Nourrie des travaux précurseurs d'Arrhénius et Kleiber [K+32], la Théorie Métabolique de l'Écologie a pour ambition de placer le métabolisme comme mécanisme central permettant de décrire les dynamiques écologiques de l'ensemble du vivant [BGA+04]. En s'appuyant sur la relation allométrique fondamentale reliant métabolisme moyen et masse moyenne au sein des espèces, cette théorie propose une large gamme de relations allométriques reliant des grandeurs physiologiques ou démographiques à la masse moyenne des individus au sein des espèces [PWS+12]. Ainsi, la mortalité des individus, le taux de croissance d'une population, la capacité de charge d'un milieu (quantité maximale d'individus qu'un environnement peut accueillir au vu des ressources disponibles), et jusqu'à la diversité des espèces au sein des écosystèmes [GAWB05], sont des grandeurs écologiques étroitement reliées au métabolisme, et vérifient également des relations allométriques dont l'expression des coefficients est théoriquement prédite [MM19]. Même si les allométries semblent être un ingrédient clé pour modéliser le comportement des espèces et leurs caractéristiques écologiques, elles sont encore principalement justifiées par des données brutes et des approches phénoménologiques, s'appuyant sur la mise

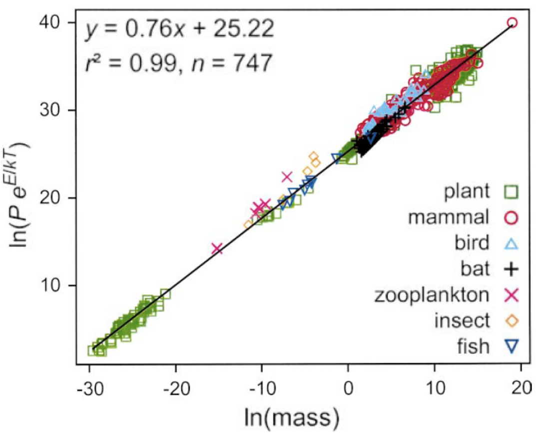


Figure 1.1: Taux moyen de production de biomasse (mesuré en grammes par individu et par an), corrigé en fonction de la température des organismes considérés, en fonction de la masse moyenne (mesurée en grammes) pour une grande variété d'organismes vivants, des eucaryotes unicellulaires aux plantes et aux mammifères (chaque point correspond donc à des grandeurs moyennes mesurées pour une espèce donnée). L'échelle est logarithmique, et l'équation de la droite de régression est présentée en haut à gauche du graphique, ainsi que le coefficient de régression r^2 et le nombre d'espèces étudiées n. La pente de la droite de régression s'identifie comme le coefficient allométrique reliant la production de biomasse moyenne à la masse corporelle moyenne, ici égal à 0.76. Ce graphique est extrait de [BGA+04].

en évidence statistique d'une relation log-linéaire entre métabolisme et masse des individus [NE01]. D'autres études sont basées sur des arguments physiques et physiologiques pour les individus, ajustés en fonction de la température des organismes [GCW+02]. La Théorie Métabolique de l'Écologie justifie *a posteriori* ces relations par des arguments d'optimisation énergétique et de dimensionnement [SDF08], négligeant tous les autres processus évolutifs et écologiques comme par exemple les phénomènes de compétition pour l'accès aux ressources ou la saisonnalité (c'est-à-dire la variabilité de l'environnement au cours du temps), qui devraient être pris en compte pour expliquer pleinement ce phénomène [JT92, Gla22, GM23]. De plus, ces allométries sont dites *interspécifiques*, dans le sens où elles font intervenir des grandeurs moyennes à l'échelle des espèces, et sont valides pour une très large gamme d'organismes vivants, même si la fluctuation de leurs masses se fait à des échelles différentes (voir Figure 1.1). Ces relations entre des grandeurs populationnelles sont alors souvent extrapolées à l'échelle individuelle [PWS+12]. De nombreux articles [KW97, HAA+07, DOM+10, MM19] montrent que ce passage d'une comparaison entre espèces à une comparaison entre individus au sein d'une même espèce n'est pas fondé. Ces travaux exhibent des relations allométriques dites *intraspécifiques*, c'est-à-dire valides au sein d'une espèce donnée, où les coefficients allométriques mesurés diffèrent fortement des coefficients allométriques prédits par la Théorie Métabolique de l'Écologie, comme présenté en Table 1.1.

Le questionnement fondamental qui sous-tend l'ensemble de ces travaux, que ce soit pour les adeptes ou les détracteurs de la Théorie Métabolique de l'Écologie, porte sur l'émergence des relations allométriques, observées de façon systématique au cours de la mise en oeuvre d'innombrables procédés expérimentaux [LL05]. Au centre de ce débat, ce sont des visions de l'écologie évolutive qui s'affrontent, mettant à l'honneur tantôt l'optimisation de facteurs internes et physiologiques comme le métabolisme [BGA+04],

Parameter (units)	General theory	Generation	Predicted scaling exponent	Observed scaling exponent
Max. intrinsic growth rate (r; 1/time)	$M^{\beta-1}$	200	0.17	0.18
		300	−0.11	0.13
		400	−0.07	0.38
Biomass (K_{bio}; biovolume)	$M^{1-\beta}$	200	−0.17	0.7
		300	0.11	−0.28
		400	0.07	0.35
Density (K_{pop}; cell density)	$M^{-\beta}$	200	−1.17	−0.47
		300	−0.89	−0.88
		400	−0.93	−0.59
Biomass product. (r_{bio}; biovol./time)	M^0	200	0	0.83
		300	0	−0.13
		400	0	0.41
Cell product. (r_{pop}; cell/time)	M^{-1}	200	−1	−0.17
		300	−1	−1.13
		400	−1	−0.59
Total energy use (E_{tot}; energy)	M^0	200	0	0.71
		300	0	0
		400	0	0.34

Table 1.1: Ce tableau est tiré de [MM19]. Les auteurs sélectionnent des individus au sein de la même espèce d'algue verte *Dunaliella tertiolecta* par centrifugation. Cette sélection artificielle permet de séparer les individus les plus gros des plus petits, pour introduire ces deux types d'individus dans de nouveaux milieux distincts, et les faire se reproduire indépendamment. Cette centrifugation suivie d'une sélection est répétée entre chaque génération, et cela a pour conséquence une évolution artificielle de la taille des individus au cours de l'expérience. Les auteurs mesurent différents paramètres démographiques aux générations 200, 300 et 400. La Théorie Métabolique de l'Écologie prédit alors un ensemble de relations allométriques vérifiées par ces paramètres, qui font intervenir la taille du corps M et le coefficient allométrique fondamental β reliant vitesse de perte de masse au cours du temps et taille du corps. Ces prédictions théoriques sont comparées aux coefficients allométriques observés au cours de l'expérience. Les coefficients allométriques observés pour lesquels les intervalles de confiance sur les mesures effectuées ne contiennent pas les prédictions théoriques sont présentés en rouge.

ou au contraire insistant sur les facteurs externes tels que la compétition ou la pression environnementale [Gla22]. Les relations allométriques sont vues comme un mécanisme structurant les écosystèmes, et une meilleure compréhension des phénomènes évolutifs aboutissant à de telles relations est encore à développer pour des réseaux trophiques complexes (*i.e.* ensemble d'espèces en interaction au sein d'un écosystème, incluant des dynamiques de production de ressources, de prédation, et de décomposition des organismes). Nous pouvons citer les travaux récents de Wickman et al. pour un système d'interaction proie-prédateur entre deux espèces, avec apport de ressources extérieures pour la population de proies [WLK24]. Dans cet article, les auteurs proposent de voir les allométries interspécifiques (décrivant des grandeurs moyennes sur les populations considérées) comme un phénomène émergent des dynamiques individuelles, structurées par des allométries intraspécifiques, et de mécanismes éco-évolutifs tels que la compétition (voir Figure 1.2).

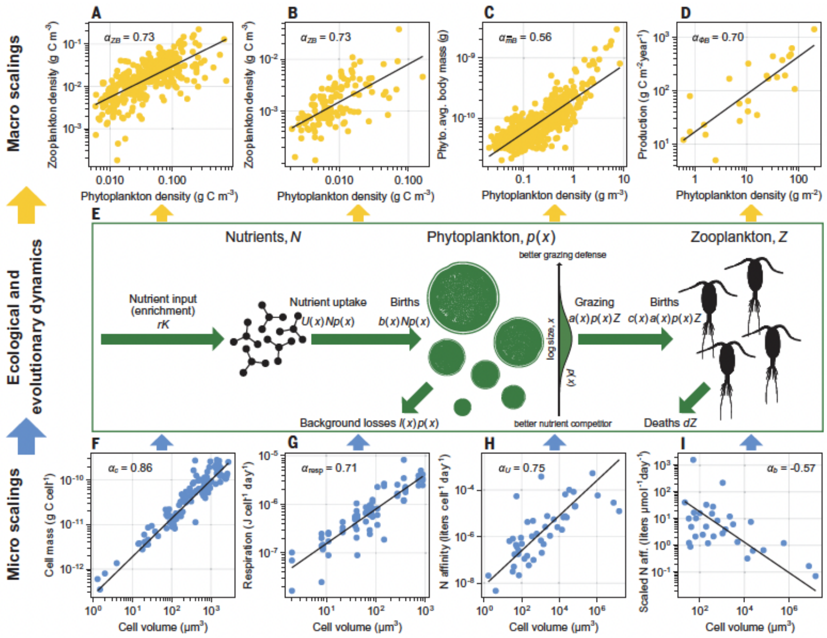


Figure 1.2: Cette figure est tirée de [WLK24]. L'objectif des auteurs est de concevoir un modèle d'interactions entre une population de phytoplanctons consommant une ressource dont l'apport est contrôlé, et une population de zooplanctons qui se nourrissent des phytoplanctons. Ils disposent de données expérimentales à l'échelle microscopique (ce sont les graphes en bleu au bas de la figure) et à l'échelle macroscopique (ce sont les graphes en jaune en haut de la figure). Plus précisément, les figures F à I présentent en échelle logarithmique les relations allométriques mesurées, dites microscopiques, reliant diverses propriétés individuelles des phytoplanctons à la taille de leur corps (chaque point représente un individu, contrairement à la Figure 1.1 où chaque point représentait une espèce), et les coefficients allométriques intraspécifiques α sont précisés. Les figures A à D présentent en échelle logarithmique les relations allométriques mesurées, dites macroscopiques, reliant diverses propriétés populationnelles moyennes des zooplanctons à la densité moyenne observée en phytoplanctons (cette fois-ci, chaque point représente une expérience indépendante et des grandeurs moyennes associées aux différentes populations de plancton étudiées), et les coefficients allométriques interspécifiques α sont précisés. Les auteurs proposent un modèle dont les mécanismes sont présentés en figure E, qui prend en entrée les données microscopiques, et propose en sortie des prédictions pour les valeurs des coefficients allométriques macroscopiques mesurés expérimentalement (ces coefficients théoriques ne sont pas présentés sur cette figure). Les coefficients obtenus à partir des données expérimentales en jaune sont compatibles avec les valeurs théoriques prédites, les auteurs expliquent donc l'apparition d'allométries interspécifiques comme phénomène émergent d'allométries intraspécifiques et de mécanismes éco-évolutifs.
(A et B) Densité de biomasse des zooplanctons par rapport à la densité de biomasse des phytoplanctons, respectivement pour la Manche et l'océan Atlantique. (C) Masse moyenne du corps cellulaire des phytoplanctons par rapport à la densité de biomasse des phytoplanctons. (D) Production de phytoplancton en fonction de la densité de biomasse des phytoplanctons. (E) Schéma du modèle éco-évolutif proposé. (F) Masse du corps cellulaire du phytoplancton en fonction du volume cellulaire. (G) Respiration du phytoplancton en fonction du volume cellulaire. (H et I) Affinité d'absorption d'azote du phytoplancton et absorption d'azote renormalisée en fonction du volume cellulaire.

Ainsi, en s'inspirant de travaux récents en écologie évolutive [WMA+19, WAB+22, WLK24], l'un des objectifs principaux de ce travail de thèse est de renverser la manière classique de justifier les relations allométriques précises proposées par la Théorie Métabolique de l'Écologie par un modèle théorique. Plutôt que de considérer les allométries comme un mécanisme impactant les dynamiques écologiques des écosystèmes (dans [TGW+23] par exemple, les auteurs utilisent une structure allométrique explicite pour prédire l'évolution de la structure de communautés marines dans un contexte de changement climatique), nous concevons un modèle où ce sont au contraire les mécanismes écologiques qui contraignent les coefficients allométriques à prendre des valeurs précises. Nous nous affranchissons donc de contraintes allométriques imposées *a priori* dans notre modèle, qui constituaient une des limites majeures des modèles existants [GAWB05, PWS+12, TGW+23], pour étudier de façon systématique les relations émergentes sur les allométries interspécifiques. Enfin, à notre connaissance, les modèles mathématiques proposés par les biologistes en écologie évolutive pour étudier les relations allométriques sont pour la plupart déterministes [YI92, DDA22, WLK24]. Nous présentons dans ce travail de thèse des modèles stochastiques qui rentrent dans la catégorie des modèles individu-centrés, et proposons donc un formalisme probabiliste pour aborder des questionnements similaires à ceux traités par les modèles déterministes antérieurs. De plus, nous obtenons dans le Chapitre II un résultat de tension pour notre modèle stochastique dans une asymptotique de grande population, et exhibons à la limite un modèle déterministe, régi par un système d'EDP proche des équations du type McKendrick-Von Foerster proposées par des modèles existants [DDA22].

1.2 Les modèles individu-centrés dans la littérature

Dans ce travail de thèse, nous proposons un modèle général pour décrire l'évolution d'un système d'interactions entre une population d'individus, caractérisés par un trait appelé *énergie*, et une ressource, d'abord supposée constante au cours du temps, puis variable avec un terme de consommation modélisant la compétition entre individus. Ce modèle s'inscrit dans la lignée des modèles individu-centrés structurés [FM04, CFM08, Tra08, CF15, Tch24], dont les ingrédients principaux résident dans:

- l'existence d'une structure, c'est-à-dire de traits caractérisant les individus et dont nous pouvons décrire l'évolution en tout temps,
- des mécanismes intervenant à l'échelle individuelle, dépendant du trait de l'individu considéré, et qui décrivent les dynamiques aléatoires de naissance et mort au sein de la population, ainsi que les dynamiques déterministes d'évolution du trait caractérisant les individus (celui-ci peut sauter brusquement lors d'un événement aléatoire, et évolue continument entre les sauts),
- l'étude d'un processus à valeurs mesures $(\mu_t)_{t\geq 0}$ de la forme

$$\mu_t := \sum_{u\in V_t} \delta_{\xi_t^u}, \tag{1.2.1}$$

  où V_t est un ensemble d'indices décrivant les individus vivants au temps t, et pour $u \in V_t$, $(\xi_t^u)_t$ est la trajectoire de l'individu u (c'est-à-dire l'évolution de son trait au cours du temps).

Ces modèles ont à l'origine été introduits par Fournier et Méléard [FM04], puis étoffés par Champagnat avec l'ajout d'une structure de traits variables pour étudier des phénomènes de mutation et de sélection [CFM08]. Ensuite, Tran propose l'ajout d'une structure d'âge [Tra08], et l'interaction avec une ressource dans le cas d'une structure en masse est étudiée par Campillo et Fritsch [CF15]. Tchouanti propose également dans des travaux récents [Tch24] de rajouter un terme de diffusion dans les dynamiques stochastiques individuelles du modèle.

Plus précisément, nous construisons le processus à valeurs-mesures $(\mu_t)_{t\geq 0}$ comme un Jumping Markov Process [JS96], c'est-à-dire que nous construisons une suite d'instants de saut (correspondant aux naissances et aux morts au sein de la population) entre lesquels le processus est déterministe et donné par un système d'équations différentielles ordinaires. La simulation numérique d'un tel processus est généralement coûteuse à la fois en temps de calcul et en espace de stockage, en particulier dans le cas de l'étude de grandes populations. L'un des résultats fondamentaux à la base de cette riche littérature sur les modèles individu-centrés est un théorème de convergence, dans une asymptotique de grande population, d'une suite de renormalisations du processus $(\mu_t)_{t\geq 0}$ vers la solution déterministe d'un système d'équations intégro-différentiel. En particulier, simuler un système d'EDP permet de réduire considérablement le temps de calcul nécessaire, et ouvre la voie à une étude numérique des dynamiques de population d'un point de vue évolutif [FCO17]. Dans ce travail de thèse, nous établissons notamment dans le Chapitre II un résultat de tension similaire au résultat de convergence du type précédent, mais dans un espace de mesure à poids.

1.3 Chapitre I – Un modèle stochastique individu-centré avec ressources constantes : contraintes vérifiées par les coefficients allométriques pour un processus de branchement surcritique

La première partie de notre travail s'attache à la conception et à l'étude d'un modèle individu-centré structuré en énergie décrivant une population d'individus consommant une ressource supposée constante au cours du temps. Nous utilisons délibérément le mot *énergie* au lieu de *masse* ou *taille*. En effet, les biologistes mesurent le transfert de biomasse plutôt que les masses elles-mêmes lorsqu'ils estiment les taux métaboliques des individus [YI92, DDA22]. C'est pourquoi nous préférons ce terme générique d'*énergie*, associé à un stock variable de ressources destinées à être investies par un individu pour sa subsistance ou sa reproduction. Nous introduisons comme paramètre du modèle une énergie typique $x_0 > 0$ caractérisant l'espèce considérée, qui représente l'énergie de chaque nouvel individu apparaissant dans la population lors d'une naissance. Ce mécanisme de reproduction spécifique rend notre modèle plus proche des modèles structurés par âge [Tra08, BCDF22] que des modèles de croissance-fragmentation [DHKR15, CCF16]. Nous supposons que la ressource consommée par les individus est maintenue à un niveau constant R, négligeant ainsi les interactions des individus à travers la consommation de la ressource. L'ensemble du système est régi par deux types de mécanismes :

- des sauts aléatoires correspondant aux naissances et morts dans la population, où les taux de naissance et mort dépendent de l'énergie d'un individu au cours du temps, et sont notés respectivement b et d,

- une évolution continue et déterministe de l'énergie des individus entre les sauts, où chaque énergie individuelle évolue à la vitesse $g(., R)$, qui dépend de l'état R de la ressource. Cette vitesse est décomposée en deux termes, $g(x,R) := f(x,R) - \ell(x)$, où $f(.,R) \geq 0$ représente un terme de croissance qui dépend à la fois de l'énergie individuelle et de l'état R de la ressource, et $\ell \geq 0$ est un terme modélisant la perte d'énergie des individus au cours du temps, c'est-à-dire leur métabolisme, et ne dépend que de l'énergie d'un individu. La fonction $(x,R) \mapsto g(x,R)$ peut donc prendre des valeurs positives ou négatives.

Nous définissons les trajectoires individuelles à valeurs dans $\mathbb{R}_+^* \times \{\partial\}$ où ∂ est un état cimetière représentant la mort de l'individu, de façon itérative. Nous construisons une suite d'instants de saut pour chaque trajectoire, représentant les événements de naissances et mort pour chaque individu. Nous rassemblons alors ces processus individuels en un processus populationnel, associé lui aussi à une suite d'instants de saut, qui est la réunion des instants de saut individuels. Les trajectoires individuelles sont indépendantes conditionnellement à la condition initiale pour la population, car nous supposons qu'il n'y a pas de compétition pour la ressource. Nous étudions donc un processus de branchement et comme les individus sont tous identiques à la naissance, nous pouvons nous focaliser plus particulièrement sur l'étude d'une trajectoire individuelle d'énergie initiale x_0. Nous étudions alors un Processus de Markov Déterministe par Morceaux [Dav84], dont le générateur étendu est donné pour toute fonction φ vérifiant de bonnes conditions de régularité par

$$\mathcal{L}\varphi : x \mapsto \begin{cases} g(x,R)\varphi'(x) + b(x)\left(\varphi(x-x_0) - \varphi(x)\right)\mathbb{1}_{\{x>x_0\}} + d(x)(\varphi(\partial) - \varphi(x)) & \text{if } x \neq \partial, \\ \varphi(\partial) & \text{if } x = \partial. \end{cases}$$

Le terme $g(x,R)\varphi'(x)$ représente l'évolution déterministe de l'énergie individuelle entre les instants de saut à vitesse g ; le terme $b(x)\left(\varphi(x-x_0) - \varphi(x)\right)\mathbb{1}_{\{x>x_0\}}$ signifie qu'à un taux $b(.)$ dépendant de l'énergie de l'individu, un événement de naissance survient et le parent perd une quantité d'énergie x_0 pour le transmettre à son enfant (pour empêcher les énergies individuelles de prendre des valeurs négatives, nous imposons que les naissances surviennent, si et seulement si l'énergie du parent est strictement supérieure à x_0) ; et le terme $d(x)(\varphi(\partial) - \varphi(x))$ signifie qu'à un taux $d(.)$ dépendant de l'énergie de l'individu, ce dernier meurt et son énergie rejoint l'état cimetière ∂. Par rapport aux modèles individu-centrés développés précédemment dans la littérature, les nouveaux ingrédients que nous proposons sont les suivants.

- Les modèles précédents étudient le cas classique de la mitose cellulaire [CF15, BDG19, RT24], où la masse d'un individu à la naissance est égale à celle du parent multipliée par un rapport $a \in (0,1)$, éventuellement aléatoire et tirée suivant un noyau de probabilité. Dans notre cas, nous souhaitons pouvoir distinguer les espèces étudiées *via* l'énergie typique d'un individu à la naissance x_0, introduite comme paramètre du modèle. Ainsi, nous remplaçons le mécanisme de division "multiplicatif" précédent, par un mécanisme "additif", où l'on retranche l'énergie caractéristique x_0 à l'énergie des individus concernés par un événement de naissance. Ce mécanisme de division spécifique sera conservé dans l'ensemble de la thèse et amène de nombreuses complications, à la fois théoriques pour l'étude du processus de branchement du Chapitre I ou pour prouver le résultat de tension du Chapitre II ; et numériques pour la conception de nos algorithmes dans le Chapitre III.

- Les modèles structurés en âge [Tra08] ou en masse [CF15] étudiés précédemment font l'hypothèse d'un trait croissant au cours du temps en dehors des événements de naissance. Nous autorisons le trait des individus – l'énergie dans notre cas – à être non-monotone (*i.e.* la fonction $(x,R) \mapsto g(x,R)$ peut changer de signe), ce qui permet par exemple de modéliser des populations d'individus subissant des famines [YKR18].

- Dans l'ensemble des modèles individu-centrés décrits auparavant, les taux de sauts individuels sont bornés [FM04, CFM08, Tra08, CF15]. L'ambition principale de notre travail est de s'affranchir de ces hypothèses, notamment pour pouvoir étudier le cas de taux de sauts allométriques.

Le fait d'autoriser les taux de naissance, mort ou croissance des individus à être non-bornés pose d'emblée un problème de bonne définition de notre processus. En effet, les trajectoires individuelles peuvent alors exploser ou toucher 0 en temps fini, ou la suite des instants de saut admettre un point d'accumulation fini, ce qui ne reflète pas les comportements biologiques attendus. Une première partie de notre travail consiste donc, en toute généralité, à spécifier les hypothèses qui garantissent la bonne définition de notre processus en tout temps dans le cas de taux non-bornés. Après ce travail préliminaire, nous nous intéressons plus particulièrement au cas où les paramètres fonctionnels du modèle sont allométriques. Précisément, nous supposons qu'il existe $(\beta, \delta) \in \mathbb{R}$ et $(C_\beta, C_\delta, C_\alpha, C_\gamma, \alpha, \gamma) \in \mathbb{R}_+^*$ tels que pour tout $x > 0$ et $R \geq 0$,

- $b(x) := C_\beta x^\beta$,
- $d(x) := C_\delta x^\delta$,
- $\ell(x) := C_\alpha x^\alpha$,
- $f(x, R) := \phi(R) C_\gamma x^\gamma$, et donc $g(x, R) := \phi(R) C_\gamma x^\gamma - C_\alpha x^\alpha$,

avec $0 \leq \phi \leq 1$ une fonction croissante et typiquement sigmoïdale, qui représente une réponse fonctionnelle, par exemple de la forme Holling de type II ou III [YI92]. L'allure typique d'une trajectoire individuelle dans le cadre allométrique est présentée en Figure 1.3.

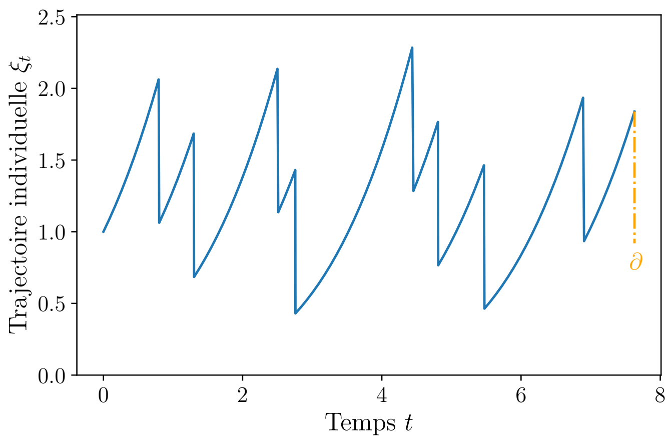


Figure 1.3: Allure typique de l'évolution d'une énergie individuelle ξ_t en fonction du temps, dans le cas allométrique avec les paramètres $x_0 = 1$, $\alpha = \gamma = \delta + 1 = 3/4$, $\beta = -0,2$, $C_\beta = 2$, $C_\delta = 0.5$, $C_\gamma = 2$, $C_\alpha = 1$ et $\phi(R) = 2/3$. Chaque discontinuité correspond à un instant de saut où l'individu donne naissance à un descendant et perd l'énergie x_0. La ligne orange verticale en pointillés représente la mort de l'individu.

Ainsi, les paramètres b, d, f, respectivement ℓ font intervenir des fonctions de type puissance, associées respectivement aux coefficients allométriques β pour le taux de naissance, δ pour le taux de mort, γ pour le taux de croissance, et α pour le taux métabolique. Dans ce chapitre, notre objectif principal est de mettre en évidence des relations entre ces coefficients allométriques, nécessaires pour vérifier les hypothèses suivantes:

Hypothèses I: *Pour toute espèce considérée (c'est-à-dire pour toute valeur de x_0),*

- *le modèle est biologiquement pertinent (i.e. presque sûrement, les énergies individuelles n'explosent pas et ne s'annulent pas, et les individus meurent en temps fini),*
- *la probabilité de survie de la population dans un milieu où la ressource est suffisante est strictement positive (i.e. avec une quantité suffisante de ressources, le processus de branchement que nous avons construit est surcritique).*

Nous proposons donc un modèle interspécifique général dans l'esprit de la Théorie Métabolique de l'Écologie, mais où les hypothèses allométriques sont faites à l'échelle de l'individu, et nous étudions la loi de reproduction d'un processus de branchement. Notre but n'est pas d'expliquer l'émergence des allométries et leurs aspects évolutifs, mais plutôt d'identifier les valeurs admissibles des coefficients allométriques β, δ et γ en fonction du coefficient fondamental α reliant métabolisme et énergie individuelle. Des relations précises entre ces coefficients allométriques sont justifiées par la Théorie Métabolique de l'Écologie [BGA+04, MM19] et présentées sous la forme

$$\alpha = \gamma = \delta + 1 = \beta + 1. \tag{1.3.2}$$

Nous introduisons également les relations allométriques suivantes, définies dans le cas où $C_\gamma > C_\alpha$:

$$\alpha = \gamma = \delta + 1 \quad \text{and} \quad \beta \geq \alpha - 1 + C_\delta/(C_\gamma - C_\alpha). \tag{1.3.3}$$

Le résultat principal de ce chapitre est le Théorème I.2.1, dont un énoncé informel est le suivant.

> **Théorème :** *Sous les Hypothèses I, les coefficients allométriques* $(\alpha, \gamma, \delta, \beta)$ *vérifient* (1.3.2) *ou* (1.3.3).
> *Réciproquement, si* (1.3.2)*, alors les Hypothèses I sont vérifiées. De plus, il existe des coefficients* $(\alpha, \gamma, \delta, \beta)$ *vérifiant* (1.3.3) *tels que les Hypothèses I sont vérifiées.*

Pour démontrer ce résultat, nous utilisons notamment des techniques de couplage pour comparer les trajectoires individuelles. Par ailleurs, nous avons été inspirés pour une partie de ces travaux par le concept de pseudotrajectoire asymptotique développé par Benaïm et Hirsch [Ben99], qui permet d'établir un lien entre le comportement asymptotique des trajectoires d'un processus stochastique et les courbes intégrales associées à un flot déterministe. Notre vision interspécifique s'appuie sur des raisonnements où $x_0 \to 0$ ou $+\infty$ pour déduire des contraintes mathématiques de nos hypothèses biologiques.

Bien que nous retrouvions les relations allométriques précises proposées par la Théorie Métabolique de l'Écologie, elles ne constituent pas le seul jeu de paramètres compatible avec les Hypothèses I. Nos arguments purement probabilistes et provenant de raisonnements mathématiques éclairent d'un jour nouveau les approches habituelles basées sur des résultats expérimentaux. Nous explorons les différences de comportement entre les jeux de paramètres (1.3.2) et (1.3.3) dans des simulations numériques. Notamment, au vu de ces simulations, nous conjecturons que pour tout $x_0 > 0$ et R tel que $g(x_0, R) > 0$, l'espérance du nombre de descendants directs d'un individu d'énergie initiale x_0 est finie dans le cas (1.3.2) et infinie dans le cas (1.3.3). Le travail présenté dans ce chapitre a fait l'objet d'un article, soumis à la revue *Annals of Applied Probability*, et en cours de relecture.

1.4 Chapitre II – Un modèle stochastique individu-centré avec ressources variables : un résultat de tension avec des taux de croissance, de naissance et de mort non-bornés

La deuxième partie de notre travail propose une modification du modèle du Chapitre I, en autorisant la ressource à varier au cours du temps, *via* un terme de renouvellement, et un terme de consommation représentant une compétition indirecte entre les individus. L'objectif est double : tout d'abord, il est naturel de souhaiter intégrer des dynamiques de compétition pour la ressource pour une étude plus aboutie des relations allométriques [JT92, Gla22, GM23] ; ensuite, ce travail s'inscrit plus généralement dans la lignée des modèles individu-centrés d'évolution darwinienne où compétition, mutation et sélection sont des ingrédients centraux [CFM08, Tra08, FCO17]. Contrairement à la partie précédente, nous ne bénéficions plus d'une propriété de branchement pour simplifier l'étude du processus populationnel. Les trajectoires individuelles dépendent maintenant de l'évolution de la ressource au cours du temps, qui dépend elle-même des énergies individuelles. Ainsi, nous étudions un processus $(\mu_t, R_t)_{t\geq 0}$, où μ est une mesure ponctuelle finie de la forme (1.2.1) représentant l'état de la population, et R la quantité de ressource. Les individus sont repérés par des indices $u \in \mathcal{U} := \bigcup_{n\in\mathbb{N}} (\mathbb{N}^*)^{n+1}$, et l'ensemble des individus vivants au temps t est donné par un ensemble aléatoire V_t. Nous construisons à nouveau le processus populationnel de manière itérative, en introduisant des mesures ponctuelles de Poisson $\mathcal{N}$ et $\mathcal{N}'$ sur $\mathcal{U} \times \mathbb{R}_+^* \times \mathbb{R}^+$ pour déterminer les instants de saut correspondant à des naissances, respectivement des morts, au sein de la population. Comme dans le Chapitre I, entre les instants de saut, chaque énergie individuelle ξ_t^u pour $u \in V_t$ vérifie

$$\frac{\mathrm{d}\xi_t^u}{\mathrm{d}t} = g(\xi_t^u, R_t) := f(\xi_t^u, R_t) - \ell(\xi_t^u), \tag{1.4.4}$$

où $f(x, R)$ représente la vitesse de consommation de la ressource R par un individu d'énergie x, et $\ell(x)$ le taux métabolique d'un individu d'énergie x. L'évolution de la ressource R_t est donnée par

$$\frac{\mathrm{d}R_t}{\mathrm{d}t} = \rho(R_t, \mu_t) := \varsigma(R_t) - \chi \int_{\mathbb{R}_+^*} f(x, R_t)\mu_t(\mathrm{d}x), \tag{1.4.5}$$

où ς est un terme de renouvellement choisi de telle sorte que la ressource reste dans un compact $[0, R_{\max}]$ au cours du temps, avec $R_{\max} > 0$. Le paramètre $\chi > 1$ est l'inverse du coefficient d'efficacité de conversion énergétique (*i.e.* $1/\chi$ représente la proportion de ressources consommées par un individu effectivement convertie en énergie individuelle), et μ_t est donnée par (1.2.1). Le terme intégral représente donc la vitesse de consommation totale de la ressource par la population, pondérée par le paramètre χ. La fonction ς est choisie de façon assez générale, notre modèle englobe donc les modèles de chemostat où la ressource est abiotique [CF15], mais peut aussi suivre l'évolution d'une ressource biotique [BLLD11, YKR18, WLK24]. Sur un ouvert $\mathfrak{V} \subseteq \mathcal{M}_P(\mathbb{R}_+^*) \times [0, R_{\max}] \times \mathbb{R}^+$, où $\mathcal{M}_P(\mathbb{R}_+^*)$ est l'ensemble des mesures ponctuelles finies, nous définissons un flot à valeurs dans $\mathcal{M}_P(\mathbb{R}_+^*) \times [0, R_{\max}]$ sous la forme

$$\begin{array}{rccl} X & : & \mathfrak{V} & \to & \mathcal{M}_P(\mathbb{R}_+^*) \times [0, R_{\max}] \\ & & (\mu, R, t) & \mapsto & X_t(\mu, R) := \left(X_t^\mu(\mu, R), X_t^{\mathfrak{R}}(\mu, R)\right). \end{array}$$

Ce flot déterministe représente l'évolution des énergies individuelles (cette information est regroupée dans la mesure $X_t^\mu(\mu, R)$ de la forme (1.2.1)) et de la ressource $X_t^{\mathfrak{R}}(\mu, R)$ à partir d'une condition initiale (μ, R), et suivant le système d'équations couplées (1.4.4) et (1.4.5). Une des principales difficultés qui survient dans notre cas comparé à d'autres modèles existants [Tra08, CF15] est que ce flot n'est pas défini globalement pour tout $(\mu, R, t) \in \mathcal{M}_P(\mathbb{R}_+^*) \times [0, R_{\max}] \times \mathbb{R}^+$, mais seulement localement sur l'ouvert $\mathfrak{V}$. De plus, pour un individu donné d'indice u, la fonction $t \mapsto g(\xi_t^u, R_t)$ peut changer de signe au cours du temps, donc l'évolution de ξ_t^u n'a donc aucune raison d'être monotone. De la même manière que dans le Chapitre I, à ces difficultés pour définir la partie déterministe de notre processus s'ajoute une potentielle accumulation de temps de saut en temps fini. Ces problèmes sont essentiellement dûs au choix de taux de saut non-bornés b, d et g, et un travail préliminaire est nécessaire pour assurer la bonne définition du processus en tout temps. Une fois ce travail effectué, nous montrons que le processus $(\mu_t, R_t)_{t\geq 0}$ peut s'écrire

$$\begin{aligned}(\mu_t, R_t) = X_t(\mu_0, R_0) \\ &+ \int_0^t \int_{\mathcal{U}\times\mathbb{R}_+^*} \mathbb{1}_{\{u\in V_{s-}\}} \mathbb{1}_{\{h\leq b(\xi_{s-}^u)\}} [X_{t-s}(\mu_{s-} + \delta_{x_0} + \delta_{\xi_{s-}^u - x_0} - \delta_{\xi_{s-}^u}, R_s) \\ &\qquad - X_{t-s}(\mu_{s-}, R_s)] \mathcal{N}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h) \\ &+ \int_0^t \int_{\mathcal{U}\times\mathbb{R}_+^*} \mathbb{1}_{\{u\in V_{s-}\}} \mathbb{1}_{\{h\leq d(\xi_{s-}^u)\}} [X_{t-s}(\mu_{s-} - \delta_{\xi_{s-}^u}, R_s) \\ &\qquad - X_{t-s}(\mu_{s-}, R_s)] \mathcal{N}'(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h),\end{aligned}$$

où l'on adopte une convention pour pouvoir écrire le flot $X_t^\mu(\mu, R)$ pour tout $(\mu, R, t) \in \mathcal{M}_P(\mathbb{R}_+^*)\times[0, R_{\max}]\times\mathbb{R}^+$. Cette écriture formelle est classique dans la littérature (Définition 2.4. dans [Tra08], Section 4.1 dans [CF15]), et doit être comprise comme une somme téléscopique. Tout d'abord, les énergies et les ressources individuelles évoluent de manière déterministe, en suivant le flot $X_t(\mu_0, R_0)$. Ensuite, à chaque événement de naissance ou mort, nous remplaçons le flot actuel par un nouveau flot, modifié en suivant les mécanismes de naissance et de mort de notre modèle. Nous construisons ainsi un Jumping Markov Process [JS96] dont le générateur étendu est donné pour toute fonction de la forme $F_\varphi : (\mu, r) \mapsto F_\varphi(\mu, r) := F(\langle \mu, \varphi \rangle, r)$, avec $F : (x, r) \mapsto F(x, r)$ et φ des fonctions vérifiant certaines conditions de régularité (et en notant $\partial_r F_\varphi(\mu, r) := \partial_r F(\langle \mu, \varphi \rangle, r)$ et $\partial_x F_\varphi(\mu, r) := \partial_x F(\langle \mu, \varphi \rangle, r)$), par

$$\begin{aligned}\mathfrak{L}F_\varphi(\mu, r) = \rho(\mu, r)\partial_r F_\varphi(\mu, r) + \langle \mu, g(., r)\varphi'(.)\rangle \partial_x F_\varphi(\mu, r) \\ &+ \int_{\mathbb{R}_+^*} b(x) \Big(F_\varphi(\mu + \delta_{x_0} + \delta_{x-x_0} - \delta_x, r) - F_\varphi(\mu, r) \Big) \mu(\mathrm{d}x) \\ &+ \int_{\mathbb{R}_+^*} d(x) \Big(F_\varphi(\mu - \delta_x, r) - F_\varphi(\mu, r) \Big) \mu(\mathrm{d}x).\end{aligned}$$

Le terme $\rho(\mu, r)\partial_r F_\varphi(\mu, r)$ est relatif à l'évolution de la ressource au cours du temps, suivant (1.4.5) ; le terme $\langle \mu, g(., r)\varphi'(.)\rangle \partial_x F_\varphi(\mu, r)$ est relatif à l'évolution déterministe des énergies individuelles entre les instants de saut à vitesse g, suivant (1.4.4) ; le premier terme intégral signifie que pour chaque individu d'énergie x et vivant au temps t, à un taux $b(x)$ dépendant de l'énergie de l'individu, un événement de naissance survient et le parent perd une quantité d'énergie x_0 pour le transmettre à son enfant, qui apparaît alors dans la population comme une masse de Dirac en x_0 ; et le second terme intégral signifie que pour chaque individu d'énergie x et vivant au temps t, à un taux $d(x)$ dépendant de l'énergie de l'individu, un événement de mort survient et on retire cet individu de la population.

Nous introduisons ensuite une suite de processus renormalisés $(\mu_t^K, R_t^K)_{t\geq 0}$ construits sur le même modèle que précédemment. Le paramètre K est un paramètre d'échelle représentant le nombre d'individus constituant la population à l'instant initial. Dans le cas de l'étude d'un chemostat, ce paramètre peut également être vu comme le volume total de la cuve où les bactéries interagissent avec le substrat [CF15]. Nous souhaitons regarder le comportement de notre modèle dans une asymptotique de grande population, lorsque le paramètre K tend vers $+\infty$. Contrairement aux travaux précédents sur des renormalisations similaires de modèles individu-centrés, nous devons ajouter une hypothèse supplémentaire importante, due au fait que nos taux sont non-bornés, de la forme suivante (la forme précise est donnée en Section II.1.4.2).

Hypothèse II.1: *Il existe une fonction régulière et croissante ω, vérifiant des conditions du type "fonction de Lyapunov" associée au générateur étendu $\mathfrak{L}$ du processus.*

Déterminer une fonction de Lyapunov associée au générateur étendu d'un processus de Feller général permet par exemple d'énoncer des résultats de retour du processus dans des ensembles compacts en des temps exponentiels, mais constitue un problème difficile [MT93, CV23]. Sous réserve d'existence d'une telle fonction ω, nous définissons une topologie à poids sur l'ensemble $\mathcal{M}_\omega(\mathbb{R}_+^*)$ des mesures intégrant la fonction ω, appelée topologie ω-étroite. C'est la topologie la plus fine pour laquelle les applications $\mu \mapsto \langle \mu, f\rangle$ sont continues, avec f toute fonction continue sur $\mathbb{R}_+^*$ telle que f/ω soit bornée, et $\langle \mu, f\rangle := \int_{\mathbb{R}_+^*} f(x)\mu(\mathrm{d}x)$. Nous construisons alors un espace polonais $(\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w})$ qui peut être muni d'une distance qui métrise la topologie ω-faible. Nous travaillons également sous une hypothèse classique d'existence de moments pour la condition initiale μ_0^K, contrôlés uniformément en K [CFM08, Tra08, CF15], adaptée au poids ω :

Hypothèse II.2: *Il existe $p > 1$ tel que*

$$\sup_{K\in\mathbb{N}^*} \mathbb{E}\left(\left(\langle \mu_0^K, 1 + \mathrm{Id} + \omega\rangle \right)^p \right) < +\infty.$$

Le résultat principal de ce chapitre est le Théorème II.3.1. Il indique que pour tout $T > 0$, sous les Hypothèses II.1 et II.2, la suite des lois des processus $\left((\mu_t^K, R_t^K)_{t\in[0,T]}\right)_{K\geq 1}$ est tendue dans l'espace de Skorokhod $\mathbb{D}([0, T], (\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w}) \times [0, R_{\max}])$. De plus, toute valeur d'adhérence $(\mu_t^*, R_t^*)_{t\in[0,T]}$ de cette suite est un processus continu à valeurs dans $(\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w}) \times [0, R_{\max}]$, déterministe conditionnellement à la condition initiale (μ_0^*, R_0^*), et caractérisé par le système d'équations intégro-différentiel au sens faible suivant. Pour tout $t \in [0, T]$,

$$R_t^* = R_0^* + \int_0^t \rho(R_s^*, \mu_s^*)\mathrm{d}s \tag{1.4.6}$$

et pour toute fonction $\varphi : (t,x) \in [0,T] \times \mathbb{R}_+^* \mapsto \varphi_t(x)$ suffisamment régulière et contrôlée par le poids ω,

$$\langle \mu_t^*, \varphi_t \rangle = \langle \mu_0^*, \varphi_0 \rangle + \int_0^t \int_{\mathbb{R}_+^*} \Big(\partial_s \varphi_s(x) + g(x, R_s^*) \partial_x \varphi_s(x) \\ + b(x)(\varphi_s(x_0) + \varphi_s(x - x_0) - \varphi_s(x)) - d(x)\varphi_s(x) \Big) \mu_s^*(\mathrm{d}x)\mathrm{d}s. \quad (1.4.7)$$

Le schéma de preuve de ce résultat de tension suit un chemin bien balisé, proposé initialement par Fournier et Méléard [FM04], mais à chaque étape de la preuve, nous rencontrons des difficultés supplémentaires du fait de l'interaction avec la ressource et des taux non-bornés. Nous devons notamment formuler des critères de tension dans des espaces de mesure à poids, et non plus simplement pour l'espace des mesures finies. Nous adaptons les résultats de Roelly [Roe86], et Méléard-Roelly [MR93] pour travailler avec la topologie ω-étroite. Sous réserve qu'il existe une unique solution au système d'équations (1.4.6) et (1.4.7), notre résultat de tension devient un résultat de convergence vers la limite $(\mu_t^*, R_t^*)_{t\in[0,T]}$ ainsi identifiée. À notre connaissance, il s'agit du premier résultat de tension de ce type pour un modèle individu-centré avec des taux non-bornés. Nous montrons en particulier que notre théorème s'applique dans le cas allométrique (*i.e.* nous démontrons l'existence d'une fonction poids ω adaptée au cas allométrique). En particulier, notre résultat de tension s'applique pour une gamme de coefficients allométriques qui comprend le cas (1.3.2) proposé par la Théorie Métabolique de l'Écologie. Nous proposons également dans le Théorème II.5.3 une extension du Théorème II.3.1 sous l'hypothèse supplémentaire suivante concernant les valeurs d'adhérences $(\mu_t^*, R_t^*)_{t\in[0,T]}$. En notant $\varpi := 1 + \mathrm{Id} + \omega$, et de la même manière que nous avons défini la topologie ω-étroite sur l'espace $\mathcal{M}_\omega(\mathbb{R}_+^*)$, nous pouvons définir une topologie ϖ-étroite sur l'espace $\mathcal{M}_\varpi(\mathbb{R}_+^*)$ des mesures intégrant la fonction ϖ, et nous notons $(\mathcal{M}_\varpi(\mathbb{R}_+^*), \mathrm{w})$ l'espace polonais associé.

> **Hypothèse II.3:** *Toute valeur d'adhérence $(\mu_t^*)_{t\in[0,T]}$ est un processus continu à valeurs dans $(\mathcal{M}_\varpi(\mathbb{R}_+^*), \mathrm{w})$, et pour tout $t \in [0,T]$, $\langle \mu_t^*, d\varpi \rangle < +\infty$.*

Sous l'Hypothèse II.3, nous étendons le résultat du Théorème II.3.1 à une tension dans l'espace de Skorokhod $\mathbb{D}([0,T], (\mathcal{M}_\varpi(\mathbb{R}_+^*), \mathrm{w}) \times [0, R_{\max}])$. Conditionnellement à μ_0^*, le processus limite $(\mu_t^*)_{t\in[0,T]}$ est déterministe et caractérisé par (1.4.6)-(1.4.7). Ainsi, nous transposons notre questionnement probabiliste en un questionnement déterministe. Il reste à prouver un résultat de régularité et de contrôle d'une solution mesure à un système d'EDP, ce qui constitue une piste potentielle de collaboration avec des spécialistes du domaine. Le travail présenté dans ce chapitre est l'objet d'un article de recherche qui sera soumis prochainement dans un journal à comité de lecture.

1.5 Chapitre III – Lien entre modèles stochastiques et déterministes, et illustration numérique avec des paramètres allométriques

Le Chapitre III se distingue des deux premiers chapitres de par sa nature exploratoire, et présente de nombreuses conjectures étayées par des simulations numériques de nos processus. La dernière partie de notre travail présente d'abord des résultats complémentaires sur les solutions au système (1.4.6)-(1.4.7) décrit dans la deuxième partie. Nous travaillons notamment sous l'hypothèse où une solution mesure μ_t^* admet une densité u_t par rapport à la mesure de Lebesgue sur $\mathbb{R}_+^*$ pour tout $t \in [0,T]$, et cela nous permet de reformuler le système d'équations (1.4.6)-(1.4.7). Cela donne, sous réserve d'existence des termes mis en jeu, pour tout $t \in [0,T]$ et $x \in \mathbb{R}_+^* \setminus \{x_0\}$,

$$\partial_t u_t(x) + \partial_x\Big(g(x,R_t^*)u_t(x)\Big) = b(x+x_0)u_t(x+x_0) - (b(x)+d(x))u_t(x),$$

avec

$$\frac{\mathrm{d}R_t^*}{\mathrm{d}t} = \varsigma(R_t^*) - \chi \int_{\mathbb{R}_+^*} f(x,R_t^*)u_t(x)\mathrm{d}x$$

et la condition de bord

$$\int_{\mathbb{R}_+^*} b(y)u_t(y)\mathrm{d}y = \Big(u_t(x_0+) - u_t(x_0-)\Big)g(x_0,R_t^*).$$

Nous proposons également une version simplifiée du système précédent, à ressources fixées $R \geq 0$, sous la forme

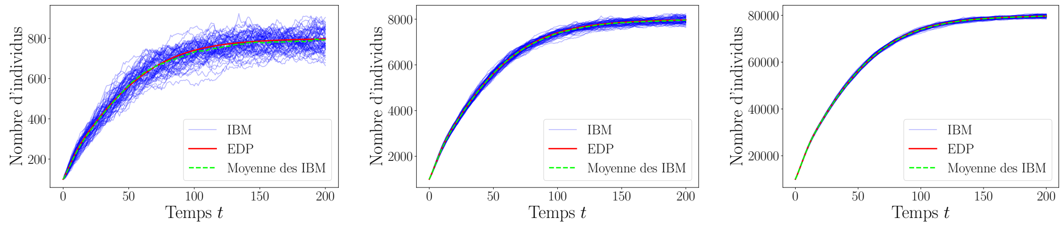
$$\begin{cases} \partial_t u_t(x) + \partial_x\Big(g(x,R)u_t(x)\Big) = b(x+x_0)u_t(x+x_0) - (b(x)+d(x))u_t(x), \\ \displaystyle\int_{\mathbb{R}_+^*} b(y)u_t(y)\mathrm{d}y = \Big(u_t(x_0+) - u_t(x_0-)\Big)g(x_0,R). \end{cases}$$

Nous cherchons alors des solutions particulières au système à ressources fixées R, dites *décorrélées*, sous la forme $u : (t,x) \mapsto u_R e^{\Lambda_R t}, R)$, avec $u_R > 0$, $\Lambda_R \in \mathbb{R}$. Ces solutions sont associées à un problème aux valeurs propres pour lequel nous conjecturons l'existence de solutions. En particulier, pour $\Lambda_R = 0$, cela correspond à des équilibres non-triviaux pour notre système d'individus en interaction avec une ressource. Dans le cas allométrique, nous montrons l'unicité de la valeur pour la ressource R_{eq} à un tel équilibre, sous réserve d'existence. Tout d'abord, cela nous permet de formuler un critère explicite de validation des schémas numériques utilisés pour simuler le système d'EDP (1.4.6)-(1.4.7).

> **Critère:** *Un schéma d'approximation numérique utilisé pour simuler l'évolution temporelle du couple $(u_t, R_t)_t$ est considéré comme incorrect s'il vérifie $R_t \nrightarrow R_{\text{eq}}$ quand $t \to +\infty$.*

Ensuite, dans le Corollaire III.1.11, nous montrons que parmi les deux jeux de coefficients allométriques (1.3.2) et (1.3.3) identifiés dans le Chapitre I, le seul qui est compatible avec la valeur de ressource à l'équilibre R_{eq} vérifie la première relation (1.3.2), qui est précisément celle proposée par la Théorie Métabolique de l'Écologie. Il resterait à prouver l'existence d'un tel équilibre dans le cas allométrique.

Dans la suite de ce chapitre, nous présentons des simulations numériques illustrant notamment le résultat de convergence (sous réserve d'unicité de la solution mesure au système (1.4.6)-(1.4.7)) en grande population du Chapitre II dans le cas allométrique, comme en Figure 1.4. De nombreuses difficultés surgissent dans la mise en oeuvre de nos algorithmes, dues au caractère non-borné des taux de saut et à notre mécanisme de division spécifique lors des événements de naissance. Nous débattons en détail des méthodes mises en place pour les contourner. Les calibrations de nos algorithmes sont très spécifiques au jeu de paramètres allométriques que nous choisissons, et nous ne proposons donc pas une façon systématique de traiter les difficultés évoquées ci-après, mais plutôt des pistes de réflexion pour valider nos schémas numériques au regard du critère que nous nous sommes donné. Par exemple, nous discutons l'utilisation d'algorithmes d'acceptation-rejet [Gil76] pour la simulation du modèle individu-centré en Section III.3. Aussi, en Section III.4.1, pour la construction des schémas d'approximation numérique de l'EDP vérifiée par la densité u_t, nous proposons une grille de discrétisation en énergie spécifique pour respecter la dynamique des naissances qui fait intervenir le paramètre x_0, en s'inspirant de grilles proposées dans la littérature [BDG19, RT24]. Enfin, il nous faut calibrer précisément le pas de temps associé à notre schéma d'approximation numérique en fonction de la grille de discrétisation en énergie choisie.



(a) K=100 (b) K=1000 (c) K=10000

Figure 1.4: Évolution temporelle de la taille de la population pour des tailles initiales K petites (a), moyennes (b) et grandes (c). La courbe rouge est obtenue par résolution numérique de l'EDP vérifiée par u_t. Les courbes bleues représentent les trajectoires de 100 simulations stochastiques indépendantes du modèle individu-centré (IBM). La courbe verte en pointillés représente la valeur moyenne de ces simulations stochastiques

1.6 Perspectives

L'apport principal de ce travail de thèse réside dans la conception d'un modèle individu-centré général avec des paramètres fonctionnels non-bornés. Les modèles individu-centrés s'appliquent à de nombreux questionnements biologiques, parmi lesquels nous pouvons citer l'estimation des taux de naissance et mort cellulaires dans le cas d'un modèle d'épidémiologie développé pour traiter des données cubaines sur l'épidémie du SIDA [CTDA08] ; la modélisation informatique d'un chemostat sans entrée ni sortie de substrat [LVP09] ; l'étude des dynamiques de mutation et sélection pour un modèle de chemostat multi-ressources [CJM14] ; le contrôle de la propagation spatiale d'un biofilm à la surface d'un liquide [AHDP15] ; et une mesure de l'efficacité d'un traitement d'immunothérapie des tumeurs malignes [BCM+16]. Nos travaux pourraient permettre de mener des études similaires en utilisant des taux de saut non-bornés, et en ce qui concerne la motivation principale de cette thèse, de s'intéresser à des questionnements d'écologie évolutive liés à l'émergence des relations allométriques dans les écosystèmes.

Les modèles développés dans les chapitres I et II, avec ressource respectivement constante ou variable, peuvent être étendus dans au moins deux directions différentes. Tout d'abord, dans l'esprit de la dynamique adaptative [MGM+95, D+04], nous pouvons introduire un mécanisme de mutation à la naissance sur le coefficient allométrique α par exemple, mais cela peut se faire sur l'ensemble des paramètres du modèle, et étudier les phénomènes de sélection qui en résultent. Un objectif possible serait d'identifier l'atteinte du jeu de paramètres allométriques (1.3.2) proposé par la Théorie Métabolique de l'Écologie comme une Stratégie Évolutivement Stable [Smi82] pour le maintien des écosystèmes tels que nous les observons aujourd'hui. En parallèle, nous pouvons rajouter un niveau de complexité trophique en introduisant une espèce prédatrice qui consomme une espèce proie, elle-même en interaction avec une ressource inerte comme dans le cadre de notre modèle. L'enjeu serait alors, à l'image des travaux de Wickman, Litchman et Klausmeier [WLK24], de proposer une structure allométrique pour la population de proies, et d'identifier les relations allométriques émergentes pour la population prédatrice, engendrées par les dynamiques éco-évolutives de compétition *via* la prédation. Plus généralement, l'ambition serait de comprendre les bonnes mises à l'échelle des tailles typiques des espèces entre différents niveaux d'un réseau trophique pour respecter une structure allométrique stable. Comme dans le Chapitre I, cette structure allométrique devrait garantir la bonne définition des trajectoires individuelles, ainsi que la survie des populations au sein du réseau trophique considéré. L'ensemble de ces questionnements biologiques sont l'objet d'un travail en cours avec Sylvain Billiard, qui donnera lieu à un article de recherche transdisciplinaire destiné à un public de biologistes.

Mathématiquement, de nombreux questionnements restent ouverts et nous formulons un certain nombre de conjectures dans chacun des trois chapitres. Dans le Chapitre I, nous pourrions étudier plus finement la loi de reproduction du processus de branchement sous-jacent pour montrer que le jeu de paramètres allométriques (1.3.3) autorise cette loi a être d'espérance infinie (voir Conjecture I.2.3). Cela permettrait de distinguer les comportements biologiques associés aux relations allométriques (1.3.3) des comportements associés aux relations (1.3.2) proposée par la Théorie Métabolique de l'Écologie, pour lequel l'espérance de la loi de reproduction du processus de branchement est toujours finie. Dans le Chapitre II, notre résultat de tension nous incite à étudier un processus limite $(\mu_t^*, R_t^*)_{t\in[0,T]}$ caractérisé par (1.4.6)-(1.4.7). Tout d'abord, la question de l'unicité des solutions mesures à ce problème reste ouverte. Ensuite, nos propositions d'extension du Théorème II.3.1 se ramènent à une question d'existence et de contrôle de solutions régulières à ce problème d'EDP (*i.e.* à montrer l'Hypothèse II.3). Dans le Chapitre III, il serait intéressant de démontrer l'existence de solutions au problème aux valeurs propres que nous associons au système d'EDP à ressources fixées (voir Conjecture III.1.1), notamment l'existence d'un équilibre non-trivial. Cela complèterait le résultat d'unicité de la valeur de ressource à l'équilibre R_{eq} dans le cas allométrique que nous donnons en Corollaire III.1.9. Enfin, les questions de stabilité et convergence des schémas numériques utilisés pour les simulations n'ont pas été abordées, notamment car les méthodes exploratoires utilisées dans nos algorithmes mériteraient d'être systématisées. Cela pourrait permettre une estimation plus fine de la quantité de ressource à l'équilibre pour des paramètres allométriques généraux. Campillo, Champagnat et Fritsch ont en effet montré dans le cadre d'un modèle similaire qu'une population mutante peut envahir un chemostat, si et seulement si la ressource à l'équilibre pour le mutant est inférieure à la ressource à l'équilibre pour le résident [CCF17]. Ceci donnerait lieu à un critère de fitness d'une population mutante dans un chemostat, exprimé en fonction des paramètres allométriques.

2– Introduction

In this section, we first present the evolutionary biology questioning that is at the root of this thesis. We also briefly describe existing results on individual-based models, originating in the work of Fournier and Méléard [FM04]. Then, we provide the main results and mathematical tools used in each of the three distinct chapters of this thesis. To conclude, we highlight the future lines of research inspired by our work.

2.1 Biological motivations

Since the synthesis work of Peters [Pet86], a lot of papers in the field of evolutionary ecology [YI92, BMT93, OBB+13, DDA22] have stood for a very particular relationship between average body mass, or average size of individuals, and average metabolism, or average metabolic rate (which can be seen as the average rate of mass loss or gain of individuals), across several orders of magnitude of species size. It is known as an *allometric* relationship, or simply *allometry*, presented in the form $B \propto M^\alpha$, where B is a measure of the average metabolic rate and M is the average mass of individuals within various species, and α is the so-called *allometric* coefficient. Typically, the curves obtained to justify these allometric relationships are similar to the graph shown in Figure 2.1. There is no clear consensus on a precise value for the allometric coefficient α linking average mass and average metabolic rate: some authors prove that there exists a single theoretical coefficient based on precise modelling of vascular systems [SDF08], while others argue, with empirical data, that the coefficient may depend on specific characteristics within species, and vary across the wide range of living organisms [DOM+10].

Inspired by the historical work of Arrhenius and Kleiber [K+32], the Metabolic Theory of Ecology aims to consider metabolism as the central mechanism for describing the ecological dynamics of all living organisms. Based on the fundamental allometric relationship linking average metabolism and average mass within species, this theory proposes a wide range of allometric relationships linking physiological or demographic variables to the average mass of individuals [PWS+12]. Thus, the mortality of individuals, the growth rate of a population, the carrying capacity of an environment (maximum quantity of individuals that an environment can accommodate given the available resources), and even the diversity of species within ecosystems [GAWB05], are ecological quantities closely linked to metabolism, and also verify allometric relationships whose allometric coefficients are theoretically predicted [MM19]. Although allometric relationships appear to be a key ingredient in modelling the behaviour of species and their ecological aspects, they are still mainly justified by raw data and phenomenological approaches, based on statistical evidence of a log-linear relationship between metabolism and mass [NE01]. Other studies are based on physical and physiological arguments for individuals, corrected according to the temperature of the considered organisms [GCW+02]. The Metabolic Theory of Ecology justifies *a posteriori* these relationships by arguments of energy optimisation and dimen-

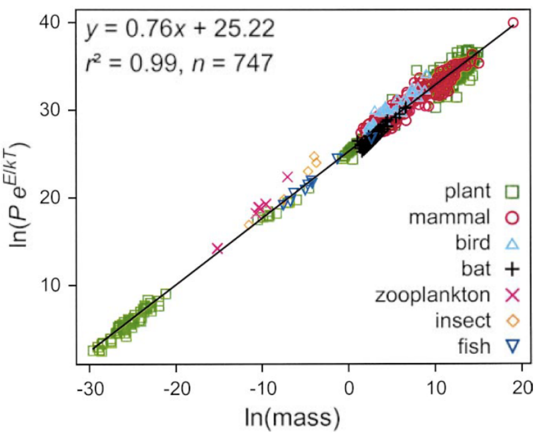


Figure 2.1: Temperature-corrected average rate of biomass production (measured in grams per individual per year), as a function of average mass (measured in grams) for a wide variety of living organisms, from unicellular eukaryotes to plants and mammals (thus, each dot on the graph corresponds to average values measured for a given species). The scale is logarithmic, and the equation of the regression line is presented in the top left corner of the graph, along with the regression coefficient r^2 and the number of studied species n. The slope of the regression line can be identified as the allometric coefficient relating average biomass production to average body mass, here equal to 0.76. This graph is taken from [BGA+04].

sioning [SDF08], neglecting all the other evolutionary and ecological processes such as, for example, competition for resources or seasonality (*i.e.* periodic variability of the environment over time), which should be taken into account to fully explain this phenomenon [JT92, Gla22, GM23]. In addition, these allometries are said to be *interspecific*, in the sense that they involve average quantities for species, and are valid for a very wide range of living organisms, even if the fluctuation of their masses occurs on different scales (see Figure 2.1). These relationships between populational average variables are then often extrapolated to the individual level [PWS+12]. Numerous articles [KW97, HAA+07, DOM+10, MM19] show that this shift from a comparison between species to a comparison between individuals within the same species is controversal. Thus gives rise to another kind of allometric relationships, said to be *intraspecific* (*i.e.* valid within a given species), where the measured allometric coefficients differ strongly from the allometric coefficients predicted by the Metabolic Theory of Ecology, as shown in Table 2.2.

The fundamental underlying questioning behind all these articles, for both supporters and detractors of the Metabolic Theory of Ecology, is about the evolutionary emergence of allometric relationships, systematically observed through numerous experimental procedures [LL05]. At the heart of this debate lie opposing visions of evolutionary ecology, where some authors emphasize the optimisation of internal and physiological factors such as metabolism [BGA+04], and others insist on external factors such as competition or environmental pressure [Gla22]. Allometric relationships are seen as a core mechanism structuring ecosystems, and a better understanding of the evolutionary phenomena leading to such relationships has yet to be developed for complex food webs (*i.e.* interacting species within an ecosystem, including dynamics of resource production, predation and decomposition of organisms). We can cite the recent work of Wickman et al. for a system of prey-predator interaction between two species, with external resources provided for the prey population [WLK24]. In this article, the authors propose to see interspecific allome-

Parameter (units)	General theory	Generation	Predicted scaling exponent	Observed scaling exponent
Max. intrinsic growth rate (r; 1/time)	$M^{\beta-1}$	200	0.17	0.18
		300	−0.11	0.13
		400	−0.07	0.38
Biomass (K_{bio}; biovolume)	$M^{1-\beta}$	200	−0.17	0.7
		300	0.11	−0.28
		400	0.07	0.35
Density (K_{pop}; cell density)	$M^{-\beta}$	200	−1.17	−0.47
		300	−0.89	−0.88
		400	−0.93	−0.59
Biomass product. (r_{bio}; biovol./time)	M^0	200	0	0.83
		300	0	−0.13
		400	0	0.41
Cell product. (r_{pop}; cell/time)	M^{-1}	200	−1	−0.17
		300	−1	−1.13
		400	−1	−0.59
Total energy use (E_{tot}; energy)	M^0	200	0	0.71
		300	0	0
		400	0	0.34

Figure 2.2: This table is taken from [MM19]. The authors select individuals within the same species of green algae *Dunaliella tertiolecta* by centrifugation. This process separates artificially the largest individuals from the smallest, then introduce these two types of individuals into new distinct environments, and make them reproduce independently. This centrifugation followed by selection is repeated between each generation, and this results in an artificial evolution of the size of individuals during the experiment. The authors measure different demographic parameters at generations 200, 300 and 400. The Metabolic Theory of Ecology then predicts a set of allometric relationships verified by these parameters, which involve body size M and the fundamental allometric coefficient β linking metabolism and body size. These theoretical predictions are compared to the allometric coefficients observed during the experiment. The observed allometric coefficients for which the confidence intervals on the measurements do not contain the theoretical predictions are shown in red.

tries (concerning average values for the species under consideration) as a phenomenon emerging from individual dynamics, structured by intraspecific allometries, and from eco-evolutionary mechanisms such as competition (see Figure 2.3).

Thus, drawing our inspiration from recent works in evolutionary ecology [WMA+19, WAB+22, WLK24], one of the main objectives of this thesis is to reverse the classical way of justifying the precise allometric relationships proposed by the Metabolic Theory of Ecology by a theoretical model. Rather than considering allometries as a mechanism impacting ecological dynamics of ecosystems (in [TGW+23] for example, the authors use an explicit allometric framework to predict the evolution of the structure of marine communities in a context of climate change), we design a model where it is on the contrary ecological mechanisms that constrain allometric coefficients to take precise values. We therefore free ourselves from allometric constraints imposed *a priori* by the model, which constituted one of the major limitations of existing models [GAWB05, PWS+12, TGW+23], for a systematic study of emerging interspecific allometric relationships. Finally, to the best of our knowledge, the mathematical models proposed by biologists in evolutionary ecology to study allometric relationships are mostly deterministic [YI92, DDA22, WLK24]. In this thesis, we present stochastic models, falling into the category of individual-based models, and therefore propose a probabilistic framework to tackle similar questionings as those treated by previous deterministic models. Furthermore, we obtain in Chapter II a tightness result for our stochastic model in a large-population asymptotic regime, and outline a limiting deterministic model, determined by a PDE system close to McKendrick-Von Foerster equations proposed by existing models [DDA22].

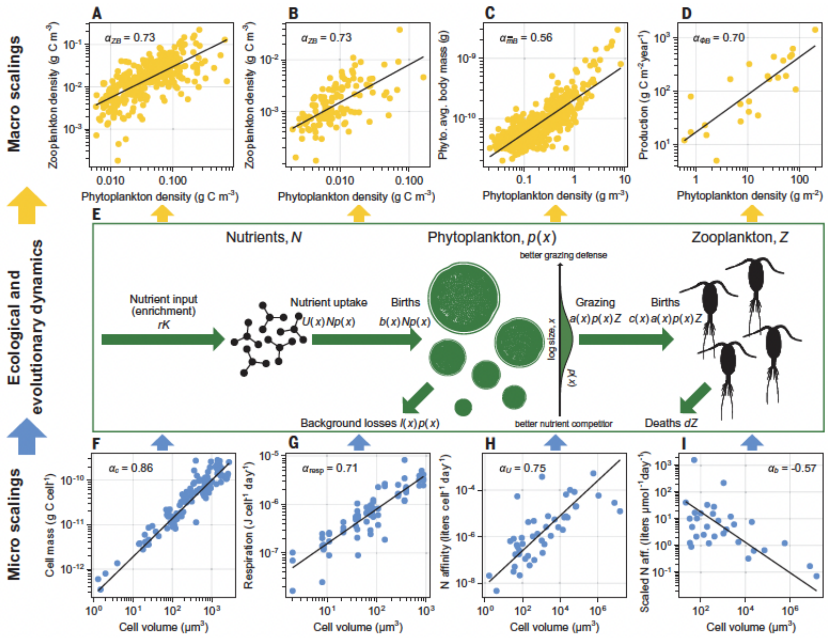


Figure 2.3: This figure is taken from [WLK24]. The goal of the authors is to design a model of interactions between a population of phytoplankton consuming a resource whose supply is controlled, and a population of zooplankton that feeds on phytoplankton. They have experimental data at the microscopic scale (these are the blue graphs at the bottom of the figure) and at the macroscopic scale (these are the yellow graphs at the top of the figure). More precisely, Figures F to I present in logarithmic scale the measured allometric relationships, referred to as microscopic relationships, linking various individual properties of phytoplankton to their body size (each dot represents an individual, unlike Figure 2.1 where each dot represented a species), and the intraspecific allometric coefficients α are specified. Figures A to D present in logarithmic scale the measured allometric relationships, known as macroscopic relationships, linking various average population properties of zooplankton to the average density observed in phytoplankton (this time, each dot represents an independent experiment and average quantities associated with the different populations of plankton), and the interspecific allometric coefficients α are specified. The authors propose a model whose mechanisms are presented on Figure E, which takes microscopic data as an input, and provides predictions for the values of the macroscopic allometric coefficients measured experimentally as an output (these theoretical coefficients are not presented on this figure). The coefficients obtained from the experimental data in yellow are compatible with the predicted theoretical values, therefore the authors identify interspecific allometries as an emerging phenomenon from intraspecific allometries and eco-evolutionary mechanisms.
(A and B) Zooplankton biomass density as a function of phytoplankton biomass density for the English Channel and Atlantic Ocean, respectively. (C) Average phytoplankton cell body mass as a function of phytoplankton biomass density. (D) Phytoplankton production as a function of phytoplankton biomass density. (E) Schematic diagram of the proposed eco-evolutionary model. (F) Phytoplankton cell body mass as a function of cell volume. (G) Phytoplankton respiration as a function of cell volume. (H and I) Phytoplankton nitrogen uptake affinity and renormalised nitrogen uptake as a function of cell volume.

2.2 Individual-based models in the literature

In this thesis, we propose a general model to describe the evolution of a population of individuals, characterised by a trait called *energy*, interacting with a resource, initially assumed to be constant over time, then fluctuating with a consumption term modelling competition between individuals. This model is in line with existing structured individual-based models [FM04, CFM08, Tra08, CF15, Tch24], whose main ingredients are:

- the existence of a structure, *i.e.* traits which characterise individuals and whose evolution can be described at any time,
- mechanisms operating at the individual level, depending on the trait of the individuals, and that describe the random dynamics of birth and death within the population, as well as the deterministic dynamics of evolution of the trait characterising the individuals (this trait can jump instantaneously during a random event, and evolves continuously between the jumps),
- the study of a measure-valued process $(\mu_t)_{t\geq 0}$ of the form

$$\mu_t := \sum_{u\in V_t} \delta_{\xi_t^u}, \tag{2.2.1}$$

  where V_t is a set of indices describing alive individuals at time t, and for $u \in V_t$, $(\xi_t^u)_t$ is the trajectory of individual u (*i.e.* the evolution of its trait over time).

This type of model was originally introduced by Fournier and Méléard [FM04], then extended by Champagnat with the addition of a variable trait structure to study mutation and selection phenomena [CFM08]. Tran then proposed the addition of an age structure [Tra08], and the interaction with a resource in the case of a mass structure was studied by Campillo and Fritsch [CF15]. Tchouanti also proposes in a recent work [Tch24] to add a diffusion term to the individual stochastic dynamics of the model.

More precisely, we construct the measured-valued process $(\mu_t)_{t\geq 0}$ as a Jumping Markov Process [JS96], *i.e.* we construct a sequence of jump times (corresponding to births and deaths in the population) between which the process is deterministic and given by a system of ordinary differential equations. Numerical simulations of such a process are generally costly both in terms of computing time and storage space, especially when studying large populations. One of the fundamental results underlying this rich literature on individual-based models is a theorem of convergence, in a large population asymptotic, of a sequence of renormalisations of the process $(\mu_t)_{t\geq 0}$ towards the deterministic solution of a system of integro-differential equations. In particular, simulating a system of PDEs considerably reduces the computing time required, and paves the way to a numerical study of population dynamics from an evolutionary point of view [FCO17]. In this thesis, we establish a tightness result similar to the previously mentioned convergence result, but in a measure space with a weighted topology.

2.3 Chapter I – An individual-based stochastic model with constant resources: constraints verified by allometric coefficients for a supercritical branching process

The first part of our work focuses on the design and study of an energy-structured individual-based model describing a population of individuals consuming a resource assumed to be constant over time. We deliberately use the word *energy* instead of *mass* or *size*. This is because biologists measure biomass transfer rather than the masses themselves when estimating the metabolic rates of individuals [YI92, DDA22]. This is why we prefer this generic term of *energy*, associated with a fluctuating stock of resources meant to be invested by an individual for its subsistence or reproduction. We introduce as a parameter of the model a typical energy $x_0 > 0$ characterising the species under consideration, which represents the energy of each new individual appearing in the population during a birth event. This specific reproduction mechanism makes our model closer to age-structured models [Tra08, BCDF22] than growth-fragmentation models [DHKR15, CCF16]. We assume that the resource consumed by individuals is maintained at a constant level R, thus neglect interactions of individuals through resource consumption. The whole system is ruled by two types of mechanism:

- random jumps corresponding to birth and death events in the population, where the birth and death rates depend on the energy of an individual over time, and are denoted by b and d respectively,

- a continuous and deterministic evolution of the energy of individuals between jumps, where each individual energy evolves at the speed $g(., R)$, which depends on the state R of the resource. We decompose this speed into two components, $g(x, R) := f(x, R) - \ell(x)$, where $f(., R) \geq 0$ is a growth term and depends on both the individual energy and the state R of the resource, and $\ell \geq 0$ is a term representing energy loss over time through metabolism, and depends only on individual energy. Thus, the function $(x, R) \mapsto g(x, R)$ can take positive or negative values.

We define individual trajectories iteratively, with values in $\mathbb{R}_+^* \times \{\partial\}$ where ∂ is a cemetery state representing the death of the individual. We construct a sequence of jump times for each trajectory, representing birth and death events for each individual. We then combine these individual processes into a population process, also associated with a sequence of jump times, which is the union of individual jump times. The individual trajectories are independent conditionally to the initial condition for the population, because we assume that there is no competition for resources. We therefore study a branching process and as individuals are all identical at birth, we focus more specifically on the study of an individual trajectory with initial energy x_0. We then study a Piecewise Deterministic Markov Process [Dav84], whose extended generator is given for any function φ satisfying appropriate regularity conditions by

$$\mathcal{L}\varphi : x \mapsto \begin{cases} g(x, R)\varphi'(x) + b(x)\left(\varphi(x - x_0) - \varphi(x)\right)\mathbb{1}_{\{x > x_0\}} + d(x)(\varphi(\partial) - \varphi(x)) & \text{if } x \neq \partial, \\ \varphi(\partial) & \text{if } x = \partial. \end{cases}$$

The term $g(x, R)\varphi'(x)$ represents the deterministic evolution of individual energy between random jumps at speed g; the term $b(x)\left(\varphi(x - x_0) - \varphi(x)\right)\mathbb{1}_{\{x > x_0\}}$ represents the fact that at rate $b(.)$ depending on the individual energy, a birth event occurs and the parent loses an amount of energy equal to x_0 to transmit it to its offspring (note that to prevent individual energies from taking non-positive values, we impose that this only happen if the energy of the parent is higher than x_0); and the term $d(x)(\varphi(\partial) - \varphi(x))$ represents the fact that at rate $d(.)$ depending on the individual energy, a death event occurs and individual energy goes to the cemetery state ∂. Compared to existing individual-based models in the literature, the new ingredients we propose are the following.

- Previous models study the classical case of cellular mitosis [CF15, BDG19, RT24], where the mass of an individual at birth is equal to that of the parent multiplied by a ratio $a \in (0,1)$, possibly random and drawn according to a probability kernel. In our case, we want to be able to distinguish between studied species using the typical energy of an individual at birth $x_0 > 0$, introduced as a parameter of the model. We therefore replace the previous 'multiplicative' division mechanism by an 'additive' mechanism, in which we subtract the characteristic energy x_0 from the energy of individuals concerned by a birth event. We keep this specific birth mechanism throughout the whole thesis, and this leads to several technical difficulties, both theoretical for studying the branching process of Chapter I or proving the tightness result of Chapter II; and numerical for designing our algorithms in Chapter III.

- The age-structured models [Tra08] or mass-structured models [CF15] previously studied assume that the trait of an individual is increasing over time apart from birth events. We allow the trait of individuals – energy in our case – to be non-monotonous (*i.e.* the function $(x, R) \mapsto g(x, R)$ can change sign), which allows for example to model populations of starving individuals [YKR18].

- In all the previously described individual-based models, individual jump rates are bounded [FM04, CFM08, Tra08, CF15]. The main ambition of our work is to free ourselves from these assumptions, in particular to be able to study the case of allometric jump rates.

Working with possibly unbounded birth, death and growth rates immediately raises a problem for the good definition of our process. This is because individual trajectories can then explode or reach 0 in finite time, or the sequence of jump times can admit a finite accumulation point, which does not reflect the expected biological behaviour. The first part of our work therefore consists, within a general framework, in specifying the assumptions that guarantee the good definition of our process at all times in the case of unbounded rates. After this preliminary work, we focus on the case where the functional parameters of the model are allometric. Precisely, we assume that there exist $(\beta, \delta) \in \mathbb{R}$ and $(C_\beta, C_\delta, C_\alpha, C_\gamma, \alpha, \gamma) \in \mathbb{R}_+^*$ such that for all $x > 0$ and $R \geq 0$,

- $b(x) := C_\beta x^\beta$,

- $d(x) := C_\delta x^\delta$,

- $\ell(x) := C_\alpha x^\alpha$,

- $f(x, R) := \phi(R) C_\gamma x^\gamma$, so that $g(x, R) := \phi(R) C_\gamma x^\gamma - C_\alpha x^\alpha$,

where $0 \leq \phi \leq 1$ is an increasing sigmoid-shaped function, which represents a functional response, for example with the shape of a Holling type II or III [YI92]. The typical shape of an individual trajectory in the allometric case is presented on Figure 2.4. Thus, we introduce the parameters b, d, f, respectively ℓ as power functions, associated respectively with the allometric coefficients β for the birth rate, δ for the death rate, γ for the growth rate, and α for the metabolic rate. In this chapter, our main objective is to highlight relationships between these allometric coefficients, that are necessary to verify the following assumptions.

Assumptions I: *For any species considered (i.e. for any value of* x_0*),*

- *the model is biologically relevant (i.e. almost surely, individual energies do not explode or reach 0, and individuals die in finite time),*

- *the probability of survival of the population in an environment where there are enough resources is positive (i.e. with a sufficient amount of resources, the branching process we construct is supercritical).*

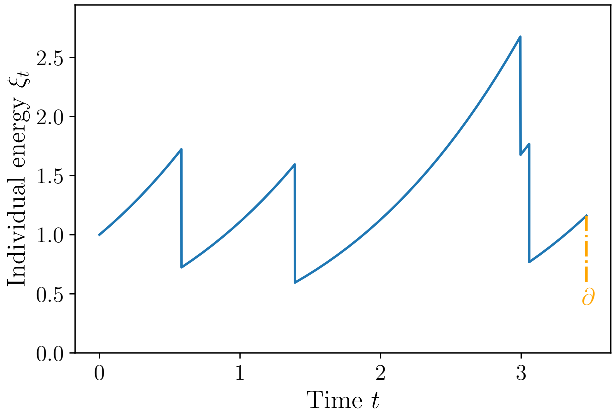


Figure 2.4: Typical shape of an individual trajectory ξ_t over time, in the allometric case with parameters $x_0 = 1$, $\alpha = \gamma = \delta + 1 = 3/4$, $\beta = -0.2$, $C_\beta = 2$, $C_\delta = 0.5$, $C_\gamma = 2$, $C_\alpha = 1$ and $\phi(R) = 2/3$. Each discontinuity corresponds to a jump time where the individual gives birth to offspring and loses the energy x_0. The vertical dotted orange line represents the death of the individual.

We therefore propose a general interspecific model in the spirit of the Metabolic Theory of Ecology, but where allometric assumptions are made at the individual level, and we study the offspring distribution of a branching process. Our goal is not to explain the emergence of allometries and their evolutionary aspects, but rather to identify the admissible values of allometric coefficients β, δ and γ to verify Assumptions I, depending on the fundamental coefficient α linking metabolism and invidual energy. Precise relationships between these allometric coefficients are justified by the Metabolic Theory of Ecology [BGA+04, MM19] and presented in the form

$$\alpha = \gamma = \delta + 1 = \beta + 1. \tag{2.3.2}$$

We also introduce the following allometric relationships, defined when $C_\gamma > C_\alpha$:

$$\alpha = \gamma = \delta + 1 \quad \text{and} \quad \beta \geq \alpha - 1 + C_\delta/(C_\gamma - C_\alpha). \tag{2.3.3}$$

The main result of this chapter is Theorem I.2.1, which can be informally stated as follows.

> **Theorem:** *Under Assumptions I, the allometric coefficients* $(\alpha, \gamma, \delta, \beta)$ *satisfy* (2.3.2) *or* (2.3.3).
> *Conversely, if* (2.3.2), *then Assumptions I are satisfied. Moreover, there exists coefficients* $(\alpha, \gamma, \delta, \beta)$ *satisfying* (2.3.3) *such that Assumptions I are satisfied.*

To prove this result, we mainly use coupling techniques to compare individual trajectories. Furthermore, we were inspired for a part of this work by the concept of asymptotic pseudotrajectory developed by Benaïm and Hirsch [Ben99], which allows us to establish a link between the asymptotic behavior of the trajectories of a stochastic process and the integral curves associated with a deterministic flow. Our interspecific vision relies on reasonings where $x_0 \to 0$ or $+\infty$ to deduce mathematical constraints from our biological assumptions.

Although we recover the precise allometric relationships proposed by the Metabolic Theory of Ecology, they do not constitute the only parameter set compatible with Assumptions I. Our purely probabilistic and mathematical reasonings shed new light on the usual approaches based on experimental results. We explore the differences on the population behavior between the two parameter sets (2.3.2) and (2.3.3) in numerical simulations. In particular, with these simulations, we conjecture that for all $x_0 > 0$ and R such that $g(x_0, R) > 0$, the expected number of direct offspring of an individual with initial energy x_0 is finite in the case (2.3.2) and infinite in the case (2.3.3). The work presented in this chapter is the subject of an article, submitted to the journal *Annals of Applied Probability*, and is currently under review.

2.4 Chapter II – An individual-based stochastic model with variable resources: a tightness result with unbounded growth, birth and death rates

The second part of our work proposes a modification of the model of Chapter I, by allowing the resource to vary over time, *via* a renewal term, and a consumption term representing indirect competition between individuals. We have two main goals: first, it is natural to integrate competition dynamics for the resource for a more exhaustive study of allometric relationships [JT92, Gla22, GM23]; then, this work is in line with individual-based models of darwinian evolution in which competition, mutation and selection are central ingredients [CFM08, Tra08, FCO17]. Contrary to the first chapter, we no longer benefit from a branching property to simplify the study of the population process. Individual trajectories now depend on the evolution of the resource over time, which in turn depends on individual energies. Thus, we study a process $(\mu_t, R_t)_{t\geq 0}$, where μ is a finite point measure of the form (2.2.1) representing the state of the population, and R the quantity of the resource. Individuals are identified by indices $u \in \mathcal{U} := \bigcup_{n\in\mathbb{N}}(\mathbb{N}^*)^{n+1}$, and the random set of alive individuals at time t is written V_t. We again construct the population process iteratively, introducing Poisson point measures $\mathcal{N}$ and $\mathcal{N}'$ on $\mathcal{U} \times \mathbb{R}_+^* \times \mathbb{R}^+$ to determine jump times corresponding to births and deaths in the population. As in Chapter I, between jump times, every individual trajectory ξ_t^u with $u \in V_t$ verifies

$$\frac{\mathrm{d}\xi_t^u}{\mathrm{d}t} = g(\xi_t^u, R_t) := f(\xi_t^u, R_t) - \ell(\xi_t^u), \tag{2.4.4}$$

where $f(x, R)$ represents the speed of consumption of the resource R by an individual with energy x, and $\ell(x)$ is the metabolic rate of an individual with energy x. The evolution of the resource is given by

$$\frac{\mathrm{d}R_t}{\mathrm{d}t} = \rho(R_t, \mu_t) := \varsigma(R_t) - \chi \int_{\mathbb{R}_+^*} f(x, R_t)\mu_t(\mathrm{d}x), \tag{2.4.5}$$

where ς is a renewal term chosen so that the resource remains in a compact $[0, R_{\max}]$ over time, with $R_{\max} > 0$. Also, $\chi > 1$ is the inverse of the conversion efficiency coefficient (*i.e.* $1/\chi$ represents the proportion of resources consumed by an individual actually converted into individual energy), and μ_t is given by (2.2.1). The integral term therefore represents the total consumption rate of the resource by the population, weighted by the parameter χ. The function ς is chosen quite generally, so we can model a chemostat where the resource is abiotic [CF15], but can also track the evolution of a biotic resource [BLLD11, YKR18, WLK24]. On an open set $\mathfrak{V} \subseteq \mathcal{M}_P(\mathbb{R}_+^*) \times [0, R_{\max}] \times \mathbb{R}^+$, where $\mathcal{M}_P(\mathbb{R}_+^*)$ is the set of finite point measures, we define a flow with values in $\mathcal{M}_P(\mathbb{R}_+^*) \times [0, R_{\max}]$ in the form

$$\begin{array}{rcl} X \quad : \qquad \mathfrak{V} & \to & \mathcal{M}_P(\mathbb{R}_+^*) \times [0, R_{\max}] \\ (\mu, R, t) & \mapsto & X_t(\mu, R) := \left(X_t^{\mu}(\mu, R), X_t^{\mathfrak{R}}(\mu, R)\right). \end{array}$$

This deterministic flow represents the evolution of individual energies (this information is summarized into the measure $X_t^{\mu}(\mu, R)$ of the form (2.2.1)) and of the resource $X_t^{\mathfrak{R}}(\mu, R)$ from an initial condition (μ, R), following the system of coupled equations (2.4.4) and (2.4.5). One of the main difficulties in our case compared to other existing models [Tra08, CF15] is that this flow is not defined globally for all $(\mu, R, t) \in \mathcal{M}_P(\mathbb{R}_+^*) \times [0, R_{\max}] \times \mathbb{R}^+$, but only locally on the open set $\mathfrak{V}$. Moreover, for a given individual of index u, the sign of the function $t \mapsto g(\xi_t^u, R_t)$ can change over time, hence the individual trajectory ξ_t^u has no reason to be monotonous between random jumps. As in Chapter I, besides these difficulties in defining the deterministic part of our process, there is a potential accumulation of random jump times. These problems are essentially due to the choice of unbounded jump rates b, d and g, and preliminary work is necessary to ensure that the process is well defined for every $t \geq 0$. Once this is done, we show that the process $(\mu_t, R_t)_{t\geq 0}$ can be written

$$\begin{aligned} (\mu_t, R_t) = X_t(\mu_0, R_0) & \\ + \int_0^t \int_{\mathcal{U}\times\mathbb{R}_+^*} & \mathbb{1}_{\{u\in V_{s-}\}}\mathbb{1}_{\{h\leq b(\xi_{s-}^u)\}}[X_{t-s}(\mu_{s-} + \delta_{x_0} + \delta_{\xi_{s-}^u - x_0} - \delta_{\xi_{s-}^u}, R_s) \\ & \qquad - X_{t-s}(\mu_{s-}, R_s)]\mathcal{N}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h) \\ + \int_0^t \int_{\mathcal{U}\times\mathbb{R}_+^*} & \mathbb{1}_{\{u\in V_{s-}\}}\mathbb{1}_{\{h\leq d(\xi_{s-}^u)\}}[X_{t-s}(\mu_{s-} - \delta_{\xi_{s-}^u}, R_s) \\ & \qquad - X_{t-s}(\mu_{s-}, R_s)]\mathcal{N}'(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h), \end{aligned}$$

where we adopt a convention to be able to write the flow $X_t^{\mu}(\mu, R)$ for all $(\mu, R, t) \in \mathcal{M}_P(\mathbb{R}_+^*) \times [0, R_{\max}] \times \mathbb{R}^+$. This formal writing is classic in the literature (Definition 2.4. in [Tra08], Section 4.1 in [CF15]), and should be understood as a telescopic sum. First, individual energies and resources evolve deterministically, following the flow $X_t(\mu_0, R_0)$. Then, at each birth or death event, we replace the current flow with a new one, modified according to our specific birth and death mechanisms. We thus construct a Jumping Markov Process [JS96] whose extended generator is given for any function of the form $F_\varphi : (\mu, r) \mapsto F_\varphi(\mu, r) := F(\langle \mu, \varphi\rangle, r)$, with $F : (x, r) \mapsto F(x, r)$ and φ functions verifying regularity conditions (and denoting $\partial_r F_\varphi(\mu, r) := \partial_r F(\langle \mu, \varphi\rangle, r)$ and $\partial_x F_\varphi(\mu, r) := \partial_x F(\langle \mu, \varphi\rangle, r)$), by

$$\begin{aligned} \mathfrak{L}F_\varphi(\mu, r) = \rho(\mu, r)\partial_r F_\varphi(\mu, r) &+ \langle \mu, g(., r)\varphi'(.)\rangle \partial_x F_\varphi(\mu, r) \\ &+ \int_{\mathbb{R}_+^*} b(x)\Big(F_\varphi(\mu + \delta_{x_0} + \delta_{x-x_0} - \delta_x, r) - F_\varphi(\mu, r)\Big)\mu(\mathrm{d}x) \\ &+ \int_{\mathbb{R}_+^*} d(x)\Big(F_\varphi(\mu - \delta_x, r) - F_\varphi(\mu, r)\Big)\mu(\mathrm{d}x). \end{aligned}$$

The term $\rho(\mu, r)\partial_r F_\varphi(\mu, r)$ is related to the evolution of the resource over time, according to (2.4.5); the term $\langle \mu, g(., r)\varphi'(.)\rangle \partial_x F_\varphi(\mu, r)$ is related to the deterministic evolution of individual energies between random jump times at speed g, according to (2.4.4); the first integral term means that for each individual of energy x and alive at time t, at a rate $b(x)$ depending on the individual energy, a birth event occurs and the parent loses an amount of energy x_0 to transmit it to its offspring, which then appears in the population as a Dirac mass at x_0; and the second integral term means that for each individual of energy x and alive at time t, at a rate $d(x)$ depending on the individual energy, a death event occurs and we remove this individual from the population.

Then, we introduce a sequence of renormalized processes $(\mu_t^K, R_t^K)_{t\geq 0}$ constructed with the same procedure as the previous process. The parameter K is a scaling parameter representing the number of individuals in the population at the initial time. In the case of studying a chemostat, this parameter can also be seen as the total volume of the vessel where bacteria interact with the substrate [CF15]. We want to understand the behavior of our model in a large population asymptotic, when the parameter K tends to $+\infty$. Importantly, unlike previous work on similar renormalizations of individual-based models, we have to add an additional assumption, due to the fact that our rates are unbounded, of the following form (the precise form is given in Section II.1.4.2):

> **Assumption II.1:** *There exists a regular and increasing function ω, satisfying conditions similar to the ones verified by "Lyapunov functions" associated to the extended generator $\mathfrak{L}$ of the process.*

Finding an appropriate Lyapunov function associated with the extended generator of a general Feller process entails, for example, a result of exponential returns in compact sets, but constitutes a difficult problem [MT93, CV23]. Provided that such a function ω exists, we define a weighted topology on the set $\mathcal{M}_\omega(\mathbb{R}_+^*)$ of measures integrating the function ω, called the ω-weak topology. It is the finest topology for which the applications $\mu \mapsto \langle \mu, f\rangle$ are continous, with f any continuous function on $\mathbb{R}_+^*$ such that f/ω is bounded, and $\langle \mu, f\rangle := \int_{\mathbb{R}_+^*} f(x)\mu(\mathrm{d}x)$. We then construct a Polish space $(\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w})$ that can be equipped with a distance that metrizes the ω-weak topology. We also work under a classical assumption of existence of moments [CFM08, Tra08, CF15], controlled uniformly in K, adapted to our context with the weight function ω:

> **Assumption II.2:** *There exists $p > 1$ such that*
> $$\sup_{K\in\mathbb{N}^*} \mathbb{E}\left(\left(\langle \mu_0^K, 1 + \mathrm{Id} + \omega\rangle\right)^p\right) < +\infty.$$

Then, we establish our main result in Theorem II.3.1. It states that for all $T > 0$, under Assumptions II.1 and II.2, the sequence of laws of the processes $\left((\mu_t^K, R_t^K)_{t\in[0,T]}\right)_{K\geq 1}$ is tight in the Skorokhod space $\mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w}) \times [0, R_{\max}])$. Furthermore, every accumulation point $(\mu_t^*, R_t^*)_{t\in[0,T]}$ is continuous with values in $.(\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w}) \times [0, R_{\max}]$, deterministic conditionally to the initial condition (μ_0^*, R_0^*), and characterized by the following weak integro-differential system of equations. For all $t \in [0,T]$,

$$R_t^* = R_0^* + \int_0^t \rho(R_s^*, \mu_s^*)\mathrm{d}s \tag{2.4.6}$$

and for every $\varphi : (t,x) \in [0,T] \times \mathbb{R}_+^* \mapsto \varphi_t(x)$ regular enough and dominated by ω,

$$\langle \mu_t^*, \varphi_t\rangle = \langle \mu_0^*, \varphi_0\rangle + \int_0^t \int_{\mathbb{R}_+^*} \Big(\partial_s\varphi_s(x) + g(x, R_s^*)\partial_x\varphi_s(x) \\ + b(x)(\varphi_s(x_0) + \varphi_s(x - x_0) - \varphi_s(x)) - d(x)\varphi_s(x)\Big)\mu_s^*(\mathrm{d}x)\mathrm{d}s. \tag{2.4.7}$$

The sketch of the proof for this tightness result follows the procedure initially proposed by Fournier and Méléard [FM04], but at each step of the proof, we encounter additional difficulties due to the interaction with the resource and unbounded rates. In particular, we must formulate tightness criteria in weighted measure spaces, and not only in the space of finite measures. We adapt the results of Roelly [Roe86], and Méléard-Roelly [MR93] to work with the ω-weak topology. If in addition, there exists a unique solution to the system (2.4.6) and (2.4.7), then our tightness result becomes a convergence result towards a fully indentified limit $(\mu_t^*, R_t^*)_{t\in[0,T]}$. To the best of our knowledge, this is the first tightness result of this kind for an individual-based model with unbounded rates. In particular, we show that our theorem applies in the allometric case (*i.e.* we construct an appropriate weight function ω for which the theorem holds true), and in line with the Metabolic Theory of Ecology, we verify that we can choose allometric coefficients verifying (2.3.2). We also propose in Theorem II.5.3 an extension of Theorem II.3.1 under the following additional assumption concerning the accumulation points $(\mu_t^*, R_t^*)_{t\in[0,T]}$. Writing $\varpi := 1 + \mathrm{Id} + \omega$, and in the same way as we defined the ω-weak topology on the space $\mathcal{M}_\omega(\mathbb{R}_+^*)$, we can define the ϖ-weak topology on the space $\mathcal{M}_\varpi(\mathbb{R}_+^*)$ of measures integrating the function ϖ, and we denote $(\mathcal{M}_\varpi(\mathbb{R}_+^*), \mathrm{w})$ the associated Polish space.

> **Assumption II.3:** *Every accumulation point $(\mu_t^*)_{t\in[0,T]}$ is a continuous process taking values in $(\mathcal{M}_\varpi(\mathbb{R}_+^*), \mathrm{w})$, and for all $t \in [0,T]$, $\langle \mu_t^*, d\varpi\rangle < +\infty$.*

Under Assumption II.3, we extend the result of Theorem II.3.1 to a tightness in the broader Skorokhod space $\mathbb{D}([0,T], (\mathcal{M}_\varpi(\mathbb{R}_+^*), \mathrm{w}) \times [0, R_{\max}])$. Conditionally to μ_0^*, the limiting process $(\mu_t^*)_{t\in[0,T]}$ is deterministic and characterized by (2.4.6)-(2.4.7). Thus, we transpose our probabilistic questioning into a deterministic problem. It remains to prove a result of regularity and control of a measure solution to a PDE system, which can lead to a potential collaboration with specialists in the field. The work presented in this chapter is the subject of an article that will soon be submitted to a peer-reviewed journal.

2.5 Chapter III – Link between stochastic and deterministic models, and a numerical illustration with allometric parameters

Chapter III differs from the first two chapters by its exploratory nature, and presents numerous conjectures supported by numerical simulations of our processes. The last part of our work first presents complementary results on the solutions to the system (2.4.6)-(2.4.7) described in the second chapter. In particular, we work under the assumption that a measure solution μ_t^* admits a density u_t with respect to Lebesgue measure on $\mathbb{R}_+^*$ for all $t \in [0,T]$, and this allows us to reformulate the system of equations (2.4.6)-(2.4.7). This gives, given the existence of the different terms in the following, for all $t \in [0,T]$ and $x \in \mathbb{R}_+^* \setminus \{x_0\}$,

$$\partial_t u_t(x) + \partial_x \Big(g(x, R_t^*) u_t(x) \Big) = b(x + x_0) u_t(x + x_0) - (b(x) + d(x)) u_t(x),$$

with

$$\frac{\mathrm{d}R_t^*}{\mathrm{d}t} = \varsigma(R_t^*) - \chi \int_{\mathbb{R}_+^*} f(x, R_t^*) u_t(x) \mathrm{d}x$$

and the boundary condition

$$\int_{\mathbb{R}_+^*} b(y)u_t(y)\mathrm{d}y = \Big(u_t(x_0+) - u_t(x_0-)\Big)g(x_0, R_t^*).$$

We also propose a simplified version of the previous system, with fixed resources $R \geq 0$, of the form

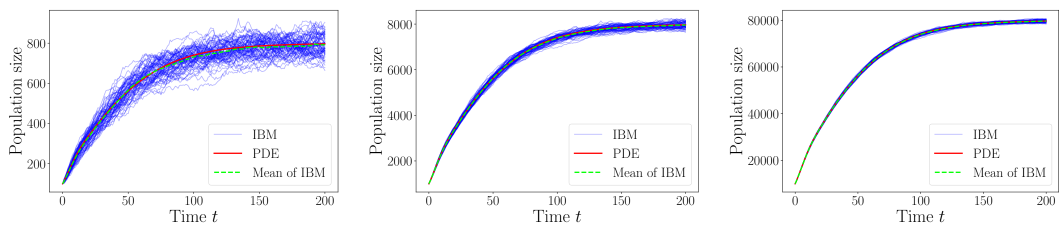
$$\begin{cases} \partial_t u_t(x) + \partial_x \Big(g(x,R)u_t(x)\Big) = b(x+x_0)u_t(x+x_0) - (b(x)+d(x))u_t(x), \\ \int_{\mathbb{R}_+^*} b(y)u_t(y)\mathrm{d}y = \Big(u_t(x_0+) - u_t(x_0-)\Big)g(x_0, R). \end{cases}$$

We then search for particular solutions to the system with fixed resources R, called *decorrelated* solutions, of the form $u : (t,x) \mapsto u_R e^{\Lambda_R t}, R)$, with $u_R > 0$, $\Lambda_R \in \mathbb{R}$. These solutions are associated to an eigenvalue problem for which we conjecture the existence of solutions. In particular, for $\Lambda_R = 0$, this corresponds to non-trivial equilibria for our system of individuals interacting with a resource. In the allometric case, we show the uniqueness of the value for the resource R_{eq} at such an equilibrium, assuming that it exists. First, this allows us to formulate an explicit criterion for validating numerical schemes used to simulate the PDE system (2.4.6)-(2.4.7).

> **Criterion:** *A numerical approximation scheme used to simulate the time evolution of the pair $(u_t, R_t^*)_t$ is considered inaccurate if it satisfies $R_t^* \nrightarrow R_{\mathrm{eq}}$ when $t \to +\infty$.*

Then, in Corollary III.1.11, we show that among the two sets of allometric coefficients (2.3.2) and (2.3.3) identified in Chapter I, the only one that is compatible with the value R_{eq} of the resource at equilibrium verifies the first relation (2.3.2), which is precisely the one proposed by the Metabolic Theory of Ecology. It remains to prove the existence of such an equilibrium in the allometric case.

In the rest of this chapter, we present numerical simulations illustrating, in particular, the large-population convergence (if there exists a unique measure solution to the system (2.4.6)-(2.4.7)) result of Chapter II in the allometric case as in Figure **??**.



(a) K=100 (b) K=1000 (c) K=10000

Figure 2.5: Temporal evolution of the population size for small (a), medium (b) and large (c) initial population sizes K. The red curve is obtained by numerical resolution of the PDE verified by u_t. The blue curves represent the trajectories of 100 independent stochastic simulations of the individual-based model (IBM). The green dotted curve is the mean value of these stochastic simulations.

Many difficulties arise in the implementation of our algorithms, due to the unbounded jump rates and our specific division mechanism during birth events. We discuss in detail the methods implemented to avoid these hindrances. The calibrations of our algorithms are very specific to the set of allometric parameters we choose, and we therefore do not propose a systematic way to deal with the difficulties mentioned below, but rather lines of reflection to design our numerical schemes so that they respect our validation criterion. For example, we discuss the use of arejction-sampling algorithms [Gil76] for the simulation of the individual-based model in Section III.3. Also, in Section III.4.1, for the construction of numerical approximation schemes for the PDE verified by the density u_t, we propose a specific energy discretization grid to respect the birth mechanism which involves the parameter x_0, inspired by grids proposed in the literature [BDG19, RT24]. Finally, we need to precisely calibrate the time step associated with our numerical approximation scheme according to our specific energy discretization grid.

2.6 Perspectives

The main contribution of this thesis is the design of a general individual-based model with unbounded functional parameters. Individual-based models are applicable to many biological questions, among which we can cite the estimation of cell birth and death rates in the case of an epidemiology model developed to process Cuban data on the AIDS epidemic [CTDA08]; the modeling of a chemostat without substrate input or output [LVP09]; the study of mutation and selection dynamics for a multi-resource chemostat model [CJM14]; the control of spatial propagation of a biofilm on the surface of a liquid [AHDP15]; and a measurement of the effectiveness of an immunotherapy treatment for malignant tumors [BCM+16]. It is possible to adapt our work to carry out similar studies with unbounded jump rates, and with regard to the main motivation of this thesis, to address questions of evolutionary ecology linked to the emergence of allometric relationships in ecosystems.

The models developed in Chapters I and II, with respectively constant or variable resources, can be extended in at least two different directions. First, in the spirit of adaptive dynamics [MGM+95, D+04], we can introduce a mutation mechanism at birth on the allometric coefficient α for example, but this can be done on the whole set of parameters of the model, and study the selection phenomena that result. A possible objective would be to identify the convergence to the set of allometric parameters (2.3.2) proposed by the Metabolic Theory of Écology as an Évolutionarily Stable Strategy [Smi82] for the maintenance of ecosystems as we observe them today. Simultaneously, we can add a level of trophic complexity by introducing a predator species that consumes a prey species, itself interacting with an inert resource as in the framework of our model. The challenge would then be, like in the work of Wickman, Litchman and Klausmeier [WLK24], to propose an allometric structure for the prey population, and to identify emerging allometric relationships for the predator population, generated by the eco-evolutionary dynamics of competition *via* predation. More generally, the ambition would be to understand the correct scaling of typical species sizes between trophic levels in a food web to respect a stable allometric structure. As in Chapter I, this allometric structure should guarantee that individual trajectories are well-defined, as well as the survival of species interacting in the considered food web. All of these biological questionings are the subject of ongoing work with Sylvain Billiard, which will result in a transdisciplinary research article intended for biologists.

Mathematically, many questions remain open and we formulate numerous conjectures in each of the three chapters. In Chapter I, we could study the offspring distribution of the underlying branching process in details to show that the allometric parameter set (2.3.3) allows this law to have infinite expectation (see Conjecture I.2.3). This would distinguish the biological behaviors associated with the allometric relationships (2.3.3) from the behaviors associated with the relationships (2.3.2) proposed by the Metabolic Theory of Ecology, for which the expectation of the offspring distribution of the branching process is always finite. In Chapter II, our tightness result encourages us to study a limiting process $(\mu_t^*, R_t^*)_{t\in[0,T]}$ characterized by (2.4.6)-(2.4.7). First, it is still an open question to assess whether there exists a unique measure solution to this PDE system. Then, our attempts of extension of Theorem II.3.1 reduce to a problem of existence and control of regular solutions to this PDE system (*i.e.* we want to show Assumption II.3). In Chapter III, it would be interesting to demonstrate the existence of solutions to the eigenvalue problem associated to the PDE system with fixed resources (see Conjecture III.1.1), in particular the existence of a non-trivial equilibrium. This would complete the uniqueness result for the value of resource R_{eq} at equilibrium that we give in Corollary III.1.9. Finally, the stability and convergence issues of the numerical schemes used for the simulations have not been addressed, in particular because of the exploratory methods used in our algorithms. This could allow a more precise estimation of the quantity of resources at equilibrium for general allometric parameters. Campillo, Champagnat and Fritsch have indeed shown in the framework of a similar model that a mutant population can invade a chemostat, if and only if the equilibrium resource for the mutant is lower than the equilibrium resource for the resident [CCF17]. This would give rise to a fitness criterion for a mutant population in a chemostat, expressed as a function of the allometric parameters.

3– Notations

Every notation used in this thesis will be introduced at the first moment it is written all along Chapters I, II and III. This section is only used to list them so that the reader can refer to it easily while reading. Importantly, the notations of Chapters I, II and III are independent, although there are some similarities between them because they are related to similar individual-based models.

Parameters of the individual-based model

- The energy of an individual at birth is $x_0 > 0$.
- The function b is the individual birth rate. In the allometric case, it is of the form $b : x > 0 \mapsto C_\beta x^\beta \mathbb{1}_{\{x>x_0\}}$. We possibly specify the dependence on x_0 with the notation b_{x_0}.
- The function d is the individual death rate. In the allometric case, it is of the form $d : x > 0 \mapsto C_\delta x^\delta$.
- The function ℓ is the individual metabolic rate (speed of loss of energy over time). In the allometric case, it is of the form $\ell : x > 0 \mapsto C_\alpha x^\alpha$.
- The function $f : (x, R) \mapsto f(x, R)$ is the individual growth rate, that depends both on the individual energy and the state of the resource R. It is of the form $f(x, R) := \phi(R)\psi(x)$, where ϕ, respectively ψ, is a function related to the availability of resources, respectively the individual speed of consumption of the resource. In the allometric case, ψ is of the form $\psi : x > 0 \mapsto C_\gamma x^\gamma$. We write $C_R := \phi(R)C_\gamma - C_\alpha$.
- The function $g : (x, R) \in \mathbb{R}_+^* \times \mathbb{R}^+ \mapsto g(x, R) := f(x, R) - \ell(x)$ is the individual speed of variation of energy, and can be positive or negative.
- In Chapter II, ς is the speed of renewal of resource.

Deterministic flows

- In Chapter I, the flow describing the evolution of individual energy between random jumps with constant resources R is denoted by $(\xi_0, R, t) \mapsto A_{\xi_0,R}(t)$. For $\xi_0 > 0$, we write $t_{\max}(\xi_0)$ for the deterministic time when $A_{\xi_0,R}(t_{\max}(\xi_0)) = 0$ or $+\infty$ ($t_{\max}(\xi_0)$ is equal to $+\infty$ if this never happens).
- In Chapter II, the flow describing the evolution of both measure-valued population process and amount of resources between random jumps is denoted by $(\mu_0, R_0, t) \mapsto X_t(\mu_0, R_0)$. For a given (μ_0, R_0), we write $t_{\exp}(\mu_0, R_0)$ for the maximal time of existence of the previous flow starting from (μ_0, R_0) at time 0.

Construction of the stochastic processes

- The set $\mathcal{U} := \bigcup_{n\in\mathbb{N}} (\mathbb{N}^*)^{n+1}$ is used to enumerate individuals in the population with the usual Ulam-Harris-Neveu notation.
- For $t \geq 0$, the set $V_t \subseteq \mathcal{U}$ is the set of individuals alive at time t.
- The process $(\xi_t^u)_{t\geq 0}$ describes the evolution of the energy of individual u over time. It is $\mathbb{R}_+^* \times \{\partial\}$-valued, where ∂ is a cemetery state, and for every $t \in [0,T]$, $\xi_t^u = \partial$, if and only if $u \notin V_t$. In Chapter I, it can also take the value $\flat$, which is another cemetery state meaning that the individual energy exploded or reached 0 in finite time. We write $\mathcal{L}$ for the extended generator of this process. The process $(\xi_t^{\mathrm{aux}})_{t\geq 0}$ is an auxiliary process used for the construction of individual trajectories.
- The process $(\mu_t)_{t\geq 0}$ is measure-valued, and describes the evolution of the population over time, with

$$\mu_t := \sum_{u\in V_t} \delta_{\xi_t^u}.$$

  In Chapter II, we define $\mathfrak{L}$ the extended generator of this process.
- In Chapter I, we define a branching process $(\Upsilon_n)_{n\geq 1}$, representing the population size generation by generation in $(\mu_t)_{t\geq 0}$.
- If the resource is constant, it is written R. If it is fluctuating over time, it is written $(R_t)_{t\geq 0}$ and evolves in a compact space denoted as $[0, R_{\max}]$.
- In Chapter I and II, the sequence of jump times used to construct the process of interest is denoted as $(J_n)_{n\geq 0}$. In Chapter I, where we study the individual process, we write $\mathfrak{J} := \sup_{n\geq 0} J_n$. To distinguish in Chapter II, where we study the population process, we write $J_\infty := \sup_{n\geq 0} J_n$.
- For $i \in \mathbb{N}$, E_i is an exponential random variable with parameter 1, and U_i is a uniform random variable on $(0,1)$.
- The notation $\mathcal{N}$ is used for a Poisson point measure, with support $\mathrm{supp}(\mathcal{N})$, and associated compensated measure $\widetilde{\mathcal{N}}$.
- In Chapter II, we define a renormalization of our process $\big((\mu_t^K, R_t^K)_{t\geq 0}\big)_{K\geq 1}$, with a scaling parameter $K \geq 1$. Any accumulation pont of this sequence is written $(\mu_t^*, R_t^*)_{t\geq 0}$. The notation $\mathscr{L}$ is used for the law of a stochastic process.

Good definition and study of our processes

- In Chapter I, we associate to each individual trajectory its time of death T_d, and write T_0, respectively T_∞, for the times when it reaches 0, respectively $+\infty$. These stopping times are possibly finite or infinite.
- For $R \geq 0$, we say that $R \in \mathfrak{R}_0$, if and only if

$$\exists x > 0, \forall y \in]0,x], \quad g(y,R) < 0,$$

  and we say that $R \in \mathfrak{R}_\infty$, if and only if

$$\exists x > 0, \forall y \geq x, \quad g(y,R) > 0.$$

  Also, we define $\Omega_R := \{\xi_0 > 0, \forall x \geq \xi_0, g(x,R) > 0\}$.

- For $x > 0$, we say that $R \in \mathcal{R}_x$ if for all $y \geq x$, $g(y, R) > 0$. We naturally extend this notation to $\mathcal{R}_0 := \{R \geq 0, \forall y > 0, g(y, R) > 0\}$.

- For $R \geq 0$ and $x_0 > 0$, we define the operator $K_{x_0,R}$ such that for any bounded function $f : \mathbb{R}_+^* \to \mathbb{R}$,

$$K_{x_0,R}f : \xi_0 \mapsto \int_0^{t_{\max}(\xi_0,R)} \Bigg[b_{x_0}(A_{\xi_0,R}(u)) \\ e^{-\int_0^u (b_{x_0} + d)(A_{\xi_0,R}(\tau))\mathrm{d}\tau} f(A_{\xi_0,R}(u) - x_0) \Bigg] \mathrm{d}u,$$

  and we write $\mathbf{1}$ for the constant function equal to 1.

- We write N_{x_0,R,ξ_0} for the number of direct offspring of an individual with initial energy ξ_0, characteristic energy x_0, and interacting with the constant resource R during its life (*i.e.* the number of birth events on its trajectory). We write $m_{x_0,R}(\xi_0) := \mathbb{E}(N_{x_0,R,\xi_0})$. For $k \geq 1$, we write $M^k_{x_0,R,\xi_0}$, or simply M^k, for the event $\{N_{x_0,R,\xi_0} \geq k\}$.

- We write S^k for the maximal value reached by the auxiliary process ξ^{aux} before time J_k, which is given by $S^k := \max_{1 \leq i \leq k} \xi^{\text{aux}}_{J_i -}$.

- In Chapter I, for $j \geq 0$, we write $\heartsuit_j := x_0^{\frac{3}{\alpha - \beta}} + j$.

- In Chapter II, we write $\tau_{\exp}$ the stopping time when an individual energy in the population reaches 0 or explodes.

Spaces of functions

We write $\hbar : x \in \mathbb{R}^+ \mapsto x + x^2$, and $\overline{g} : x > 0 \mapsto \sup_{R \in [0, R_{\max}]} |g(x, R)|$. Remark that for $x > 0$, $\overline{g}(x) = \max(\ell(x), \phi(R_{\max})\psi(x) - \ell(x))$. In Chapter II, we define a weight function ω on $\mathbb{R}_+^*$ positive, non-decreasing, continuously differentiable and such that

- $\exists C_g > 0, \forall x > 0, \quad \overline{g}(x)(1 + \omega'(x)) \leq C_g(1 + x + \omega(x))$,
- $\exists C_b > 0, \forall x > 0, \quad b(x)(1 + \hbar(|\omega(x_0) + \omega(x - x_0) - \omega(x)|)) \leq C_b(1 + x + \omega(x))$,
- $\exists C_d > 0, \forall x > 0, \quad d(x)\hbar(\omega(x)) \leq C_d(1 + x + \omega(x))$.

In the following, X, Y are metric spaces, $I \subseteq \mathbb{R}$ an interval, $w : I \to \mathbb{R}_+^*$ is a positive and continuous function and we define general notations, mostly used with the weight function ω.

- We write $\mathfrak{B}_w(I)$ for the set of functions f such that f/w is a bounded function on I.

- We write $\mathcal{C}^1(X)$, respectively $\mathcal{C}^{1,1}(X \times Y)$, for the set of functions defined on X, respectively $X \times Y$, that are differentiable with continuous derivatives. If $\varphi : (t, x) \mapsto \varphi_t(x)$ is a $\mathcal{C}^{1,1}(\mathbb{R}^+ \times \mathbb{R}_+^*)$ function, we write $\partial_t\varphi$ and $\partial_x\varphi$ for the associated partial derivatives, and also for every $t \geq 0$ we define

$$\Phi_t : (R, x) \in \mathbb{R}^+ \times \mathbb{R}_+^* \mapsto \partial_t\varphi(t, x) + g(x, R)\partial_x\varphi(t, x).$$

- We write $\mathcal{C}_c(I)$, respectively $\mathcal{C}_c^\infty(I)$, for the set of functions defined on I that are continuous, respectively infinitely differentiable, and with compact support. If φ is such a function, its support is denoted as $\mathrm{supp}(\varphi)$.

- We write $\mathcal{C}_b(I)$ the space of bounded continous functions on I, and $||.||_\infty$ the usual uniform norm on this space.

- We write $\mathcal{C}_w(I)$ for the space of continuous functions f such that f/w is bounded on I. For $f \in \mathcal{C}_w(\mathbb{R}_+^*)$, we define $||f||_w := ||f/w||_\infty$. It is straightforward to verify that $||.||_w$ is a norm on $\mathcal{C}_w(\mathbb{R}_+^*)$ (it is a weighted norm with respect to $||.||_\infty$). If $w \equiv 1$, the norm $||.||_w$ coincides with $||.||_\infty$.

- We define $\mathcal{C}_w^{1,1}(\mathbb{R}^+ \times I)$ the set of functions $\varphi \in \mathcal{C}^{1,1}(\mathbb{R}^+ \times I)$ such that $\varphi : (t,x) \mapsto w(x)$ or

$$\exists C > 0, \forall x \in I, \quad \sup_{t \in \mathbb{R}^+} \left(|\varphi_t(x)| + |\partial_x \varphi_t(x)| w(x) + |\partial_t \varphi_t(x)| \frac{w(x)}{1 + x + w(x)} \right) \leq C w(x).$$

- For every $T \geq 0$, we define $\mathcal{C}_w^{0,0}([0,T] \times I)$ for the set of continous functions on $[0,T] \times I$ such that

$$\exists C > 0, \forall s \in [0,T], \forall x \in I, \quad |\varphi_s(x)| \leq C w(x).$$

- For every $T \geq 0$, we define $\mathcal{C}_{w,T}^{1,1}(\mathbb{R}^+ \times I)$ the set of functions $\varphi \in \mathcal{C}^{1,1}(\mathbb{R}^+ \times I)$ such that

$$\exists C > 0, \forall x \in I, \quad \sup_{t \in [0,T]} \left(|\varphi_t(x)| \left(1 + d(x)\right) + |\partial_x \varphi_t(x)| w(x) + |\partial_t \varphi_t(x)| \right) \leq C w(x).$$

- For $T \geq 0$, $\mathbb{D}([0,T], X)$ is the Skorokhod space of càdlàg functions from $[0,T]$ to X endowed with its usual topology.

Spaces of measures

In the following, for μ measure on I and f measurable from I to $\mathbb{R}$, we write $\langle \mu, f \rangle := \int_{\mathbb{R}_+^*} f \mathrm{d}\mu$.

- For $t \geq 0$, we write $N_t := \langle \mu_t, 1 \rangle$ the population size, $E_t := \langle \mu_t, \mathrm{Id} \rangle$ the total energy in the population and $\Omega_t := \langle \mu_t, \omega \rangle$, and use similar notations with an exponent K for quantities associated to μ_t^K.

- In Chapter III, for $t \geq 0$, u_t is the density of the measure μ_t^* with respect to Lebesgue measure on $\mathbb{R}_+^*$.

For any metric space (X,d), we write $\mathcal{B}(X,d)$ for the Borel σ-algebra on (X,d), *i.e.* the σ-algebra generated by open sets of (X,d). For $I \subseteq \mathbb{R}$ an interval, we simply write $\mathcal{B}(I)$.

- The set $\mathcal{M}_P(I)$ is the set of finite point measures on I.

- The set $\mathcal{M}_w(I)$ the set of positive measures on $(I, \mathcal{B}(I))$ that verify $\langle \mu, w \rangle < +\infty$.

- The vague, respectively w-weak, topology on $\mathcal{M}_w(I)$ is the finest topology for which the applications $\mu \mapsto \langle \mu, f \rangle$ are continous, with f in $\mathcal{C}_c(I)$, respectively in $\mathcal{C}_w(I)$. We write $(\mathcal{M}_w(I), v)$, respectively $(\mathcal{M}_w(I), \mathrm{w})$, when we endow $\mathcal{M}_w(I)$ with the vague topology, respectively the w-weak topology. Remark that if $w \equiv 1$, the w-weak topology coïncide with the usual weak topology, which is the finest topology for which the applications $\mu \mapsto \langle \mu, f \rangle$ are continous, with $f \in \mathcal{C}_b(I)$. We show in Appendix A.2 that these topologies on $\mathcal{M}_w(I)$ define Polish spaces.

Allometric parameters

- In Chapter I, we define two sets of allometric parameters

$$I_1 := \{(\gamma = \alpha, \delta = \alpha - 1, \beta = \alpha - 1)\}$$

and

$$I_2 := \{(\gamma = \alpha, \delta = \alpha - 1, \beta),\ \beta \geq \alpha - 1 + C_\delta/(C_\gamma - C_\alpha)\}.$$

Chapter I– An individual-based stochastic model with constant resources: constraints verified by allometric coefficients for a supercritical branching process

In this chapter, we design a stochastic individual-based model structured in energy, for single species consuming an external resource, where populations are characterized by a typical energy at birth in $\mathbb{R}_+^*$. The resource is maintained at a fixed level, so we benefit from a branching property at the population level. Thus, we focus on individual trajectories, constructed as Piecewise Deterministic Markov Processes, with random jumps modelling births and deaths in the population; and a continuous and deterministic evolution of energy between jumps. We are mainly interested in the case where metabolic (*i.e.* energy loss for maintenance), growth, birth and death rates depend on the individual energy over time, and follow allometric scalings (*i.e.* power laws). Our goal is to determine in a bottom-up approach what are the possible allometric coefficients (*i.e.* exponents of these power laws) under elementary –and ecologically relevant– constraints, for our model to be valid for the whole spectrum of possible body sizes. The main result of this chapter is Theorem I.2.1. Informally, it states that assuming an allometric coefficient α related to metabolism strongly constrains the range of possible values for the allometric coefficients β, δ, γ, respectively related to birth, death and growth rates. In the eco-evolutionary literature, this allometric structure is very precise and often presented as

$$\beta = \delta = \gamma - 1 = \alpha - 1.$$

We recover Equation (2.3.2), but also identify other possible choices for allometric coefficients. We explore the richness of the different behaviors at the individual and population level, depending on the coefficients, and many directions of research remain open. We further highlight and discuss the precise and minimal ecological mechanisms that are involved in these strong constraints on allometric scalings.

In Section I.1, we describe the general features of our model. At the individual level in Section I.1.1.1, we precise the main mechanisms involved: births, deaths, energy loss and gain. In this paper, individual trajectories $(\xi_t)_{t\geq 0}$ are the main object of interest. In Section I.1.1.2, we give a necessary and sufficient condition for these trajectories to be biologically relevant (this is Theorem I.1.1), and the extended generator (see Definition 5.2. in [Dav84]) of the individual process under this condition. At the population level in Section I.1.1.3, we gather individual trajectories into a measure-valued process μ. In

Section I.1.1.4, we focus on a Galton-Watson process, describing the size of generations of μ. In Section I.1.2, we introduce allometric relationships in the model. In Section I.2, we present the main results of this chapter. First, assuming that the allometric coefficient for metabolism verifies $\alpha \leq 1$ (a lot of papers argue for $\alpha = 0.75$ [Pet86, BGA+04, SDF08]), we state our main theorem in Section I.2.1 (see Theorem I.2.1). We also present our results when $\alpha > 1$ in Section I.2.2, where we obtain similar but weaker constraints on the allometric coefficients in Theorem I.2.4. In Section I.3, we discuss the mathematical aspects and biological interpretations of our modelling and results, and give some perspectives. Section I.4 is devoted to the technical intermediate results, leading to Theorem I.1.1, I.2.1 and I.2.4. The proofs are based on three different ways to construct the individual process. The first one, given in Section I.4.1, uses exponential random variables and is useful for Sections I.4.4 to I.4.8. The second construction highlights a coupling between two processes, distinct by a different value for x_0, and is established in Section I.4.7. Also, we construct two other useful couplings between our process and similar ones with modified birth and death rates, in Section I.4.8. The third construction, in Section I.4.9, uses a Poisson point measure, which brings useful martingale properties. In Section I.5, we present numerical simulations to illustrate our results and conjectures. In Appendix A.1, we give some technical details about the definition of the population process of Section I.1.1.3, and the embedded branching process of Section I.1.1.4. In Appendix A.2, we give details for the proof in Section I.4.9.

I.1 Description of the model

I.1.1 A general setting

We design an individual-based model, structured by a positive trait called *energy*. We study single species characterized by their energy at birth $x_0 > 0$, living in an environment described by a constant amount of resources R. First, we will describe in Section I.1.1.1 the individual dynamics through the process $(\xi_t)_{t\geq 0}$, which is the main process of interest in this chapter. We give in Theorem I.1.1 of Section I.1.1.2 a necessary and sufficient condition for this process to be biologically relevant, and provide its extended generator under this condition in Proposition I.1.3. Then in Section I.1.1.3, we gather individual trajectories into a population process. One standard way to look at it is to construct a measure-valued process [FM04, CFM08], such that at time $t \geq 0$, the population is described by a point measure μ_t on $\mathbb{R}_+^*$. In our context, this measure-valued process will benefit from a *branching property*, which is the case for classical and similar approaches (see [Mar16] or Remark 2.2. in [CCF16]). Finally in Section I.1.1.4, we define a process $(\Upsilon_n)_{n\geq 0}$, representing the population size generation by generation in $(\mu_t)_{t\geq 0}$, and such that $(\Upsilon_n)_{n\geq 1}$ is a Galton-Watson process. This allows us to draw a link between a survival criterion for the population process $(\mu_t)_{t\geq 0}$, and the mean number of birth jumps for the individual process $(\xi_t)_{t\geq 0}$.

I.1.1.1 Individual trajectories

For $x_0, \xi_0 > 0$ and $R \geq 0$, we denote by $(\xi_{t,x_0,R,\xi_0})_{t\geq 0}$ the process describing the evolution of energy over time, for an individual starting from energy ξ_0 at time 0, within a species characterized by x_0, with resources R. We simply write $(\xi_t)_{t\geq 0}$ in the following if there is no ambiguity. An individual trajectory will be deterministic between some random jump times, corresponding to birth or death events. We define two cemetery states $\partial, \flat \notin \mathbb{R}$. If an individual energy reaches ∂ at time t, it means that the associated individual is dead and then $\xi_s = \partial$ for $s \geq t$. If an individual energy reaches $\flat$ at time t, it means that this energy exploded in finite time or reached the value 0 at time t, and then $\xi_s = \flat$ for $s \geq t$. The process ξ takes its values in $\mathbb{R}_+^* \cup \{\partial, \flat\}$. We precise now the general structure and parameters of our model.

- **Birth**
  At time t, an individual with energy $\xi_{t-} \notin \{\partial, \flat\}$ jumps from ξ_{t-} to $\xi_t := \xi_{t-} - x_0$ at rate $b_{x_0}(\xi_{t-})$. This event corresponds to the transfer of a constant amount of energy x_0 to a single offspring born at time t.

  We assume that the birth rate b_{x_0}, defined on $\mathbb{R}_+^*$, is equal to 0 for every $x \leq x_0$, so that no individual with non-positive energy appears during a birth event. Also, we assume that it is equal to some positive and continuous function $\tilde{b}$ (not depending on x_0) for $x > x_0$. Finally, for $x > 0$,

$$b_{x_0}(x) = \mathbb{1}_{x > x_0} \tilde{b}(x).$$

- **Death**
  At time t, every individual with energy $\xi_{t-} \notin \{\partial, \flat\}$ dies at rate $d(\xi_{t-})$. The energy then jumps from ξ_{t-} to $\xi_t := \partial$, and we set $\xi_s := \partial$ for all $s \geq t$. The function d is defined on $\mathbb{R}_+^*$, and assumed to be positive and continuous.

- **Energy loss and resource consumption**

  1. Every individual with energy $\xi_t \notin \{\partial, \flat\}$ loses energy over time at rate $\ell(\xi_t)$. The function ℓ is defined on $\mathbb{R}_+^*$ and assumed to be positive, increasing and continuous. This is the function related to metabolism, that is the loss of energy for maintenance, in our model. The metabolism is commonly observed to increase when the mass of an individual increases [WLK24].
  2. In order to balance this energy loss, an individual with energy $\xi_t \notin \{\partial, \flat\}$ consumes the resource at rate $f(\xi_t, R)$. Importantly, we suppose that f is of the form $f(\xi, R) := \phi(R)\psi(\xi)$, where ϕ is a continuous, non-negative, and increasing function on $\mathbb{R}^+$. Also, we assume that ψ is a continuous, positive and increasing function on $\mathbb{R}_+^*$.

     The fact that there is no resource consumption without resource imposes $\phi(0) = 0$, and for a fixed energy x, we assume that the resource consumption is bounded even if there is an infinite amount of resource, which can always be reformulated as $\lim_{R \to +\infty} \phi(R) = 1$. The shape of ϕ is typically the one for the functional response in evolutionary ecology: we can think of Holling type II or III functional responses [YI92]. The function ψ accounts for the conversion of resource into energy by the individual, which is why it is positive and increasing with energy, as for ℓ.

  For every $x > 0$ and $R \geq 0$, we write $g(x, R) := f(x, R) - \ell(x)$. We suppose that for every $R \geq 0$, $g(., R)$ is $\mathcal{C}^1$ on $\mathbb{R}_+^*$, so in particular it is locally Lipschitz continuous. Between two random jump times (due to birth or death events), the energy evolves according to the following equation:

$$\frac{\mathrm{d}\xi_t}{\mathrm{d}t} = g(\xi_t, R). \tag{I.1.1}$$

The regularity of g ensures that Equation (I.1.1) admits a unique positive local solution, starting from any positive energy ξ_0. This solution is denoted as $A_{\xi_0,R} : t \mapsto A_{\xi_0,R}(t)$ or simply $A_{\xi_0}(.)$ if there is no ambiguity on R. It is defined on an interval $[0, t_{\max}(\xi_0, R)[$, where $t_{\max}(\xi_0, R)$, or simply $t_{\max}(\xi_0)$, is the deterministic time when it reaches 0 or $+\infty$ ($t_{\max}(\xi_0)$ is equal to $+\infty$ if this never happens). If this happens between two random jumps at some time t, then the energy jumps from ξ_{t-} to $\xi_t := \flat$, and we set $\xi_s = \flat$ for all $s \geq t$. Classical arguments show that the flow $(\xi_0, t) \mapsto A_{\xi_0}(t)$ is $\mathcal{C}^{1,1}$ on $[0, t_{\max}(\xi_0)[$, meaning that it is differentiable with continuous derivatives in both variables (ξ_0, t).

In the following, we refer to all the previous assumptions for the model as the 'general setting of Section I.1.1.1'. We define formally the process $(\xi_t)_{t\geq 0}$ in Section I.4.1, using an iterative construction of jump times. Also, we give constructions that are equivalent in law (in the sense that they have same distribution of sample paths) in Sections I.4.7 and I.4.9. In the following, we denote by $\mathbb{P}_{x_0,R,\xi_0}$ the law associated to the individual process $(\xi_{t,x_0,R,\xi_0})_{t\geq 0}$. If there is no ambiguity on some of these parameters, we will allow ourselves to lighten the notations into $\mathbb{P}_{\xi_0}$ for example. The associated expectations are naturally written as $\mathbb{E}_{x_0,R,\xi_0}$, or simply $\mathbb{E}$.

I.1.1.2 Necessary and sufficient condition for a biologically relevant individual process

We consider the process $(\xi_{t,x_0,R,\xi_0})_{t\geq 0}$ defined in Section I.1.1.1, and formally constructed in Section I.4.1. We write respectively T_{0,x_0,R,ξ_0} and T_{∞,x_0,R,ξ_0}, or simply T_0 and T_∞, for the random times when the process $(\xi_{t,x_0,R,\xi_0})_{t\geq 0}$ reaches respectively 0 and $+\infty$. Also, we denote by T_{d,x_0,R,ξ_0}, or simply T_d, the random time when our process reaches ∂ (*i.e.* the time of death of the individual). All these hitting times may be finite or infinite. We say that a trajectory $(\xi_t)_{t\geq 0}$ is biologically relevant, if and only if the following event occurs:

$$\{T_d < T_0 \wedge T_\infty\}.$$

Remark that this event is equivalent to $\{T_d < +\infty\}$, because if the time of death is finite, it necessarily means that the process did not reach 0 or $+\infty$ before the death. In other terms, $(\xi_t)_{t\geq 0}$ is not biologically relevant if it reaches $\flat$ before ∂, or if $(\xi_t)_{t\geq 0}$ never reaches ∂ (*i.e.* the individual never dies). Let us state a first situation where the trajectories are biologically relevant. For $R \geq 0$, we say that $R \in \mathfrak{R}_0$, if and only if

$$\exists x > 0, \forall y \in]0, x], \quad g(y, R) < 0,$$

and we say that $R \in \mathfrak{R}_\infty$, if and only if

$$\exists x > 0, \forall y \geq x, \quad g(y, R) > 0.$$

Remark that under the general setting of Section I.1.1.1, we always have $0 \in \mathfrak{R}_0$, and both $\mathfrak{R}_0$ and $\mathfrak{R}_\infty$ are intervals when they are non-empty. We will see in Lemma I.4.6 in Section I.4.5 that if $R \notin \mathfrak{R}_0 \cup \mathfrak{R}_\infty$, then the process is almost surely biologically relevant for every x_0, ξ_0. In the following, we make additional assumptions for our individual trajectories to be biologically relevant almost surely for the remaining values of R (see Theorem I.1.1 below). Let us introduce some notations for this purpose. For $R \geq 0$ and $x_0 > 0$, we define the operator $K_{x_0,R}$ such that for any bounded function $f : \mathbb{R}_+^* \to \mathbb{R}$,

$$K_{x_0,R}f : \xi_0 \mapsto \int_0^{t_{\max}(\xi_0,R)} b_{x_0}(A_{\xi_0,R}(u)) e^{-\int_0^u (b_{x_0} + d)(A_{\xi_0,R}(\tau))\mathrm{d}\tau} f(A_{\xi_0,R}(u) - x_0)\mathrm{d}u. \tag{I.1.2}$$

We write $\mathbf{1}$ for the constant function equal to 1. Remark that $K_{x_0,R}$ is linear, positive and $K_{x_0,R}\mathbf{1} \leq \mathbf{1}$. For any bounded function f and $k \geq 1$, we write $K^k_{x_0,R}f$ for the operator $K_{x_0,R}$ applied to $K^{k-1}_{x_0,R}f$, and $K^0_{x_0,R}f = f$. Remark that $K_{x_0,R}\mathbf{1}(\xi_0)$ corresponds to the probability that, starting from energy ξ_0 with characteristic energy x_0 and resource R, the next jump of the process is a birth event (see Lemma I.4.7 of Section I.4.5 for a proof).

Assumption I.1.1 (Individual energy avoids 0). *For all $x_0 > 0$,*

$$\int_0^{x_0} \frac{d}{\ell}(x)\mathrm{d}x = +\infty. \tag{I.1.3}$$

Assumption I.1.2 (Individual energy avoids $+\infty$). *For all $x_0 > 0$, for all $R \in \mathfrak{R}_\infty$, for all $\xi_0 > 0$ such that $g(x,R) > 0$ for $x \in [\xi_0, +\infty[$,*

$$K^k_{x_0,R}\mathbf{1}(\xi_0) \xrightarrow[k\to+\infty]{} 0 \quad \text{and} \quad \int_{\xi_0}^{+\infty} \frac{(b_{x_0}+d)(x)}{g(x,R)}\mathrm{d}x = +\infty. \tag{I.1.4}$$

Remark: If R and ξ_0 are chosen like in Assumption I.1.2, we can use the change of variables $x = A_{\xi_0,R}(u)$ and write for any bounded function $f : \mathbb{R}_+^* \to \mathbb{R}$ and such ξ_0, R,

$$K_{x_0,R}f(\xi_0) = \int_{\xi_0}^{+\infty} \frac{b_{x_0}(x)}{g(x,R)} e^{-\int_{\xi_0}^{x} \frac{(b_{x_0}+d)(\tau)}{g(\tau,R)}\mathrm{d}\tau} f(x-x_0)\mathrm{d}x.$$

We precise in the following theorem how these assumptions lead to a process ξ with almost surely biologically relevant paths.

Theorem I.1.1. *Under the general setting of Section I.1.1.1, we have*

$$\text{Assumptions I.1.1 and I.1.2} \Leftrightarrow (\forall x_0 > 0, \forall \xi_0 > 0, \forall R \geq 0, \quad \mathbb{P}_{x_0,R,\xi_0}(T_d < +\infty) = 1).$$

Remark: Theorem I.1.1 states that Assumptions I.1.1 and I.1.2 are necessary and sufficient to obtain a biologically relevant individual process for every $R \geq 0$ in our setting. More precisely, Assumption I.1.1 almost surely prevents individual energy from reaching 0 (we only need a condition in the worst possible case, that is $R = 0$). We give an equivalent formulation in Proposition I.4.4 of Section I.4.4. Assumption I.1.2 almost surely prevents individual energy from reaching $+\infty$ (in the case where it would be possible, that is $R \in \mathfrak{R}_\infty$). We give an equivalent formulation, under Assumption I.1.1, in Proposition I.4.11 of Section I.4.5. To prove Theorem I.1.1, we will remark at the end of Section I.4.5 that it is the combination of Proposition I.4.4, Lemma I.4.6 and Proposition I.4.11.

Finally, we provide the extended generator (see Definition 5.2. in [Dav84]) of our process. We denote by $\mathcal{C}^1(\mathbb{R}_+^* \cup \{\partial\}, \mathbb{R})$ the space of functions in $\mathcal{C}^1(\mathbb{R}_+^*, \mathbb{R})$ that take some real value at ∂. We define the operator $\mathcal{L}$ such that for every $\varphi \in \mathcal{C}^1(\mathbb{R}_+^* \cup \{\partial\}, \mathbb{R})$,

$$\mathcal{L}\varphi : x \mapsto \begin{cases} g(x,R)\varphi'(x) + b_{x_0}(x)\left(\varphi(x-x_0) - \varphi(x)\right) + d(x)(\varphi(\partial) - \varphi(x)) & \text{if } x \neq \partial, \\ \varphi(\partial) & \text{if } x = \partial. \end{cases} \tag{I.1.5}$$

We write $\mathcal{D}$ for the subset of $\mathcal{C}^1(\mathbb{R}_+^* \cup \{\partial\}, \mathbb{R})$, such that $\varphi \in \mathcal{D}$, if and only if φ and $\mathcal{L}\varphi$ are bounded on $\mathbb{R}_+^* \cup \{\partial\}$. Under Assumptions I.1.1 and I.1.2, thanks to Theorem I.1.1, the process $(\xi_t)_{t\geq 0}$ almost surely avoids $\flat$, so we can almost surely write $\varphi(\xi_t)$ and $\mathcal{L}\varphi(\xi_t)$ for every $t \geq 0$ and $\varphi \in \mathcal{C}^1(\mathbb{R}_+^* \cup \{\partial\}, \mathbb{R})$.

Proposition I.1.3. *Under the general setting of Section I.1.1.1, and under Assumptions I.1.1 and I.1.2, the individual process $(\xi_{t,x_0,R,\xi_0})_{t\geq 0}$ (formally defined in Section I.4.1) is a Piecewise Deterministic Markov Process (PDMP) with extended generator $\mathcal{L}$, where the domain $D(\mathcal{L})$ of this generator contains $\mathcal{D}$.*

In particular, it means that for every $\varphi \in \mathcal{D}$, the process $(\mathfrak{M}_t)_{t\geq 0}$ defined for $t \geq 0$ by

$$\mathfrak{M}_t := \varphi(\xi_t) - \varphi(\xi_0) - \int_0^t \mathcal{L}\varphi(\xi_s)\mathrm{d}s$$

is a martingale.

Proof. We refer to the formalization of PDMPs by Davis in [Dav84]. Under Assumptions I.1.1 and I.1.2, our process almost surely does not explode or touch 0 in finite time. Furthermore, Equation (I.1.1) admits a unique positive local solution, starting from any positive energy ξ_0 thanks to the regularity of g. Jump rates b_{x_0} and d are $L^1_{\text{loc}}(\mathbb{R}_+^*)$, hence classical techniques using Gronwall inequality show that we verify Assumption 3.1. of [Dav84]. Thus, our result is a consequence of Theorem 5.5. in [Dav84], which characterizes the full domain $D(\mathcal{L})$. The fact that $\mathcal{D} \subseteq D(\mathcal{L})$ can be checked using classical arguments. Indeed, Theorem 5.5. in [Dav84] immediately entails that $\mathcal{C}^\infty$ functions with compact support are in $D(\mathcal{L})$, and we then use localisation and approximation by these test functions. □

Remark: In Theorem 5.5. of [Dav84], the domain $D(\mathcal{L})$ is entirely characterized. We only presented here a partial result on $\mathcal{D}$, for the sake of simplicity.

I.1.1.3 The measure-valued population process

In the previous section, we described an individual process ξ for which we can interpret random jumps as birth or death events. Our goal here is to define a measure-valued population process $(\mu_t)_t$ based on these individual trajectories. In the following, we fix $x_0 > 0$, $R \geq 0$, and we work under the general setting of Section I.1.1.1. To enumerate individuals in the population, we use the Ulam-Harris-Neveu notation. We define

$$\mathcal{U} := \bigcup_{n\in\mathbb{N}} (\mathbb{N}^*)^{n+1}.$$

Over time, every individual will have an index of the form $u := u_0...u_n$ with some $n \geq 0$, and some positive integers $u_0, ..., u_n$, *i.e.* some $u \in \mathcal{U}$. The *generation* of u is $|u| := n$. We denote as V_t the set of indices of individuals alive at time t (*i.e.* individuals born before t, and whose energy is not in $\{\partial, \flat\}$ at time t). The size of the population at time t is then naturally $\text{Card}(V_t)$.

Let $\mathcal{M}_P$ be the set of finite point measures on $\mathbb{R}_+^*$. At time $t = 0$, we pick a random variable $\mu_0 \in \mathcal{M}_P$. Then, there exists a random variable $\mathcal{C}_0 \in \mathbb{N}$ and a random vector $(\xi_0^1, ..., \xi_0^{\mathcal{C}_0}) \in (\mathbb{R}_+^*)^{\mathcal{C}_0}$ such that

$$\mu_0 = \sum_{u\in V_0} \delta_{\xi_0^u},$$

with $V_0 := \{1, ..., \mathcal{C}_0\}$. Initial individuals have single labels $u \in V_0$, their trajectories are independent conditionally to μ_0, distributed as the process ξ of Section I.1.1.1 started from ξ_0^u at time $\tau_u := 0$, and are denoted by $(\xi_t^{u,\tau_u})_{t\geq 0}$.

Then, we define a family of i.i.d. processes $(\xi^u)_{u\in\mathcal{U}\setminus\mathbb{N}^*}$, such that for every $u \in \mathcal{U} \setminus \mathbb{N}^*$, ξ^u is distributed as the process ξ of Section I.1.1.1 started from x_0 (which corresponds to the fixed amount of energy transferred to offspring). Also, these processes are taken independent from the initial processes $(\xi^{u,0})_{u\in V_0}$. We also define, for every $\tau > 0$ and $u \in \mathcal{U} \setminus \mathbb{N}^*$, the shifted process $(\xi_t^{u,\tau})_{t\geq\tau}$ with $\xi_t^{u,\tau} := \xi_{t-\tau}^u$ for every $t \geq \tau$. When a birth event occurs at some random time τ among one of the individuals in $V_{\tau-}$, we create a new individual trajectory. We index the offspring in the following manner: if the parent has the label $u := u_0u_1...u_n$ and has already given birth to k children for $k \geq 0$, the index of the new offspring is taken as $uk := u_0u_1...u_nu_{n+1}$ with $u_{n+1} = k$, and we add the index uk to V_τ. Also, we define $\tau_{uk} := \tau$ and the offspring energy follows the process $\xi^{uk,\tau_{uk}}$. When an individual dies or if its energy reaches $\flat$ at time τ, we remove this individual from V_τ. In this manner, over time, we describe the population with a point measure written as

$$\mu_t = \sum_{u\in V_t} \delta_{\xi_t^{u,\tau_u}},$$

with $V_t \subseteq \mathcal{U}$. Since it relies on the construction of ξ in Section I.4.1, we do not further detail here the construction of μ and postpone it to Appendix A.1.1. In particular, μ_t is well-defined for $t \in [0, \overline{\Theta}[$, where $\overline{\Theta} < +\infty$ if there is an accumulation of jump times (and in that case, $\overline{\Theta}$ is the supremum of the jump times), and $\overline{\Theta} = +\infty$ otherwise. We prove in Appendix A.1.1, Proposition A.1.1, that under Assumptions I.1.1 and I.1.2, then $\overline{\Theta} = +\infty$ almost surely for every random variable $\mu_0 \in \mathcal{M}_P$. The study of such measure-valued processes is standard, the reader can refer to [Tra08] for an age-structured measure-valued process; and for growth-fragmentation models, to [CCF16] in an evolutionary and ecological context, or to [TBV22] for the spatial spreading of a filamentous fungus.

Finally, we denote as $\mathbb{Q}_{\mu_0,x_0,R}$ the law associated to the population process μ, with initial condition μ_0, offspring energy x_0 and resource R. We define the event $\mathfrak{B} := \{\exists t \geq 0, \exists u \in V_t, u \notin V_0\}$, *i.e.* there is at least one birth event in the population. For any $x_0 > 0$, $R \geq 0$, we also define $\mathcal{M}_{P,x_0,R} := \{\mu_0 \in \mathcal{M}_P, \mathbb{Q}_{\mu_0,x_0,R}(\mathfrak{B}) > 0\}$. Remark that $\mathcal{M}_{P,x_0,R} \neq \varnothing$, because $\delta_{2x_0} \in \mathcal{M}_{P,x_0,R}$ for example. We will investigate in this chapter under which conditions on the model parameters the following assumption holds true.

Assumption I.1.4 (Supercriticality of the population process). *For every $x_0 > 0$, there exists $R_0 > 0$, such that for all $R > R_0$, for every $\mu_0 \in \mathcal{M}_{P,x_0,R}$,*

$$\left(\mathbb{Q}_{\mu_0,x_0,R}(\overline{\Theta} = +\infty) = 1\right) \textit{ and } \left(\mathbb{Q}_{\mu_0,x_0,R}(\forall t \geq 0,\ V_t \neq \varnothing) > 0\right).$$

The first condition $\mathbb{Q}_{\mu_0,x_0,R}(\overline{\Theta} = +\infty) = 1$ ensures that the population process is almost surely well-defined for all $t \geq 0$. On this event, Assumption I.1.4 states that for any studied species, there exists an amount of resources from which the population can survive indefinitely with positive probability. Assumption I.1.4 could be verified if at least one of the individuals in the population never dies. Thanks to Theorem I.1.1, we will not consider this case if we work under Assumptions I.1.1 and I.1.2. Furthermore, Proposition A.1.1 in Appendix A.1.1 implies that under Assumptions I.1.1 and I.1.2, $\mathbb{Q}_{\mu_0,x_0,R}(\overline{\Theta} = +\infty) = 1$ for every μ_0, x_0, R. We want to work under Assumption I.1.4, to model the fact that living conditions can be enforced in laboratory studies, which allow populations of bacteria or algae to last as long as we want with high probability, if we maintain a sufficient level of resources [MM19]. We insist here on the fact that it is this particular choice of a fixed value for R that fully supports our assumption. If there were competitive dynamics on R, the birth and death process could lead to almost sure extinction, and the best we can do in this case is to study existence and/or uniqueness of quasi-stationary distributions [CMMSM11].

I.1.1.4 The generation process

Finally, we define a process $(\Upsilon_n)_{n\geq 0}$ describing the size of generations of μ, taking values in $\mathbb{N}\cup\{+\infty\}$. We refer the reader to Corollary 2 in [Don72] for a general description of the embedding of a generation process into a continuous in time population process. For every $n \geq 0$, we define the random set

$$G_n := \{u \in \mathcal{U}, |u| = n, \exists t \geq 0, u \in V_t\},$$

which contains all the individuals of the n-th generation and let

$$\Upsilon_n := \mathrm{Card}(G_n).$$

With this definition, Υ could *a priori* be infinite at some point, but as soon as it should represent the generation sizes of a population, we want Υ_n to be almost surely finite for every $n \geq 0$, so we need the following assumption. For every $x_0, \xi_0 > 0$ and $R \geq 0$, we write N_{x_0,R,ξ_0} for the number of direct offspring of an individual following the process $(\xi_{t,x_0,R,\xi_0})_{t\geq 0}$ during its life (*i.e.* the number of birth events on its trajectory). We write ν_{x_0,R,ξ_0} for the law of N_{x_0,R,ξ_0}.

Assumption I.1.5. $\forall x_0 > 0, \forall R \geq 0, \forall \xi_0 > 0, \quad \mathbb{P}(N_{x_0,R,\xi_0} < +\infty) = 1.$

We will see in Corollary I.4.9 of Section I.4.5 that Assumption I.1.2 implies Assumption I.1.5.

Proposition I.1.6. *Under Assumption I.1.5,* $(\Upsilon_n)_{n\geq 1}$ *is a Galton-Watson process with offspring distribution* ν_{x_0,R,x_0}.

The proof of this fact is developed in Appendix A.1.2. The Galton-Watson process $(\Upsilon_n)_{n\geq 1}$ is either subcritical, critical, or supercritical. We use classical results on Galton-Watson processes to infer the following equivalent formulation of Assumption I.1.4. Let us write $m_{x_0,R}(\xi_0) := \mathbb{E}(N_{x_0,R,\xi_0})$, or simply $m_{x_0}(\xi_0)$.

Proposition I.1.7. *Under the general setting of Section I.1.1.1, under Assumptions I.1.1 and I.1.2, we have that*

$$\textit{Assumption I.1.4} \Leftrightarrow (\forall x_0 > 0, \exists R_0 > 0, \forall R > R_0, \quad m_{x_0,R}(x_0) > 1).$$

The proof of Proposition I.1.7 is given in Appendix A.1.2. Finally, we state an additional assumption.

Assumption I.1.8. $\forall x_0 > 0, \exists R_0 > 0, \forall R > R_0, \quad g(x_0, R) > 0.$

In Assumption I.1.8, we assume that for every species characterized by x_0, there exists an amount of resource R_0 that allows individuals to grow. This assumption seems very natural if we think of our model as a general model able to allow the survival of any species within a broad range of characteristic energy x_0. We do not want that for some x_0, individuals can not grow, no matter the available resources. Thanks to the definition of the generation process Υ, we can show that this last assumption is an immediate consequence of Assumptions I.1.1, I.1.2 and I.1.4.

Lemma I.1.9. *Under the general setting of Section I.1.1.1,*

$$(\textit{Assumptions I.1.1, I.1.2 and I.1.4}) \Rightarrow \textit{Assumption I.1.8}.$$

Proof. Assume that Assumptions I.1.1 and I.1.2 hold true, and that Assumption I.1.8 is not verified. Considering Equation (I.1.1), there exists x_0 such that no individual starting with this characteristic energy can reach higher energies, which leads to a birth rate equal to 0. Then, for any $R \geq 0$, $m_{x_0,R}(x_0) = 0$ and Assumption I.1.4 is not verified by Proposition I.1.7. □

Assumptions I.1.1, I.1.2 and I.1.4 are the main assumptions of this paper. They imply Assumptions I.1.5 and I.1.8, and they allow us to model biologically relevant processes, both at the individual and population levels. We will now define an allometric setting, and our motivation in the rest of this chapter will be to determine the allometric coefficients verifying Assumptions I.1.1, I.1.2 and I.1.4.

I.1.2 The allometric setting

In this section, we highlight a specific choice for the functions $\tilde{b}$, d, ℓ, f. The parameters R and x_0 remain general constants, but we assume here allometric shapes on the four remaining functional parameters. More precisely, we impose:

1. $\tilde{b}(x) := C_\beta x^\beta$, so that $b_{x_0}(x) := \mathbb{1}_{x>x_0} C_\beta x^\beta$,
2. $d(x) := C_\delta x^\delta$,
3. $\ell(x) := C_\alpha x^\alpha$,
4. $f(x, R) := \phi(R) C_\gamma x^\gamma$ (*i.e.* $\psi(x) = C_\gamma x^\gamma$),

with $(\beta, \delta) \in \mathbb{R}$ and $(C_\beta, C_\delta, C_\alpha, C_\gamma, \alpha, \gamma) \in \mathbb{R}_+^*$. Recall that ℓ and ψ are increasing, which is why α and γ are positive. In the following, we refer to all these specific assumptions for the model as the 'allometric setting of Section I.1.2'.

I.2 Main results

Now that our model is fully described, let us present the main results of this paper. First, in Section I.2.1, we work under the allometric setting of Section I.1.2 with $0 < \alpha \leq 1$, and give in Theorem I.2.1 necessary conditions on the allometric coefficients β, δ, γ in order to verify Assumptions I.1.1, I.1.2 and I.1.4. Recall that these assumptions express conditions that need to be valid for every $x_0 > 0$. We give a partial result for the converse implication in Proposition I.2.2. Finally, we present our conjectures about a necessary and sufficient condition on the allometric coefficients to verify Assumptions I.1.1, I.1.2 and I.1.4. We also present our results in the case $\alpha > 1$ in Section I.2.2. The details of the proofs are given in Section I.4. Even if our aim is to have a biological interpretation for precise allometric relationships, most of our intermediate results in Section I.4 are presented and valid under the general setting of Section I.1.1.1.

I.2.1 Allometric constraints in the case $\alpha \leq 1$

For some $0 < \alpha \leq 1$, we introduce two sets of allometric coefficients. The first one is a singleton, accounting for the usual allometric relationships highlighted in (2.3.2):

$$I_1 := \{(\gamma = \alpha, \delta = \alpha - 1, \beta = \alpha - 1)\}.$$

The second one contains a more complex condition on the allometric coefficient β of the birth rate, and is well-defined if $C_\gamma > C_\alpha$:

$$I_2 := \{(\gamma = \alpha, \delta = \alpha - 1, \beta),\ \beta \geq \alpha - 1 + C_\delta/(C_\gamma - C_\alpha)\}.$$

Theorem I.2.1. *Let $0 < \alpha \leq 1$. Under the allometric setting of Section I.1.2, under Assumptions I.1.1 and I.1.2 (biologically relevant model for every $x_0 > 0$) and Assumption I.1.4 (supercriticality for every $x_0 > 0$), we have either*

$$(\gamma, \delta, \beta) \in I_1, \quad C_\gamma > C_\alpha, \quad C_\beta > C_\delta \tag{I.2.6}$$

or

$$(\gamma, \delta, \beta) \in I_2, \quad C_\gamma > C_\alpha, \quad C_\delta \leq C_\gamma - C_\alpha. \tag{I.2.7}$$

Remark: In Section I.4.2, we give a detailed list of the constraints on the allometric coefficients implied by Assumptions I.1.1, I.1.2 and I.1.4, proved in the rest of Section I.4. These constraints together lead to Theorem I.2.1. The reader can visualize the restrictions on allometric coefficients thanks to Figure I.1. Note that the conditions on C_γ, C_α, C_β and C_δ are not represented on this graph. In the following, when the allometric coefficients are in I_1 (respectively I_2), we refer to it as 'the I_1 (respectively I_2) case'.

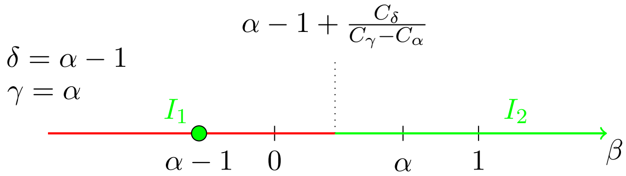


Figure I.1: Visual representation of Theorem I.2.1. The green dot represents the singleton I_1, and the green line represents the set I_2. The non-admissible allometric coefficients β for our model with $\alpha \leq 1$ are highlighted in red (and also, every (γ, δ, β) with $\delta \neq \alpha - 1$ or $\gamma \neq \alpha$ is non-admissible).

An immediate consequence of Theorem I.2.1 is the following.

Corollary I.2.1. *Under the allometric setting of Section I.1.2 and Assumptions I.1.1, I.1.2 and I.1.4, if $C_\delta > C_\gamma - C_\alpha$, then*

$$(\gamma, \delta, \beta) \in I_1, \quad C_\gamma > C_\alpha, \quad C_\beta > C_\delta.$$

Hence, we recover in this specific case the usual allometric relationships highlighted by the Metabolic Theory of Ecology [MM19], but it is supported by clear biological assumptions. We discuss more extensively the consequences of Theorem I.2.1 in Section I.3. Finally, we give a partial result for the converse of Theorem I.2.1.

Proposition I.2.2. *Under the allometric setting of Section I.1.2, if*

$$(\gamma, \delta, \beta) \in I_1,\ C_\gamma > C_\alpha,\ C_\beta > (e-1)C_\delta,\ C_\beta + C_\delta < C_\gamma - C_\alpha, \tag{I.2.8}$$

then Assumptions I.1.1, I.1.2 and I.1.4 hold true.

We give the proof of this fact in Section I.4.10. Remark that (I.2.8) ⇒ (I.2.6), but we will show in Proposition I.4.42 of Section I.4.10 that (I.2.6) is not sufficient to obtain Assumption I.1.4. Also, it is still an open question to know if (I.2.7) is sufficient to verify Assumption I.1.4. We present detailed numerical simulations in Section I.5, that lead to the following conjectures about a necessary and sufficient condition to verify Assumptions I.1.1, I.1.2 and I.1.4, and the behavior of $m_{x_0,R}(\xi_0)$, in the I_2 case.

Conjecture I.2.2. *Let $0 < \alpha \leq 1$. Under the allometric setting of Section I.1.2, we have*

$$(\textit{Assumptions I.1.1, I.1.2 and I.1.4}) \Leftrightarrow ((\text{I.2.7}) \text{ or } (\text{I.2.9})),$$

where (I.2.7) is already presented in Theorem I.2.1 (it is the I_2 case), and the second condition is

$$(\gamma, \delta, \beta) \in I_1, \quad C_\gamma > C_\alpha, \quad C_\delta < C_\gamma - C_\alpha, \quad C_\beta > \Xi\left(\frac{C_\delta}{C_\gamma - C_\alpha}\right) C_\delta, \tag{I.2.9}$$

with $\Xi :]0,1[\to]1,+\infty[$ a convex increasing function.

Conjecture I.2.3. *Let $0 < \alpha \leq 1$. Under the allometric setting of Section I.1.2, if (I.2.7) (in the I_2 case),*

$$\forall x_0 > 0, \forall R \geq 0, \forall \xi_0 > 0, \quad m_{x_0,R}(\xi_0) = +\infty.$$

Remark: Our last conjecture expresses that in the I_2 case, the average number of offspring starting from any x_0, R, ξ_0 is infinite, which is indeed sufficient to verify Assumption I.1.4 (supercriticality of the population process). This is very different from the I_1 case, where we can show that $m_{x_0,R}(\xi_0)$ is always finite (see for example the proof of Proposition I.4.18).

I.2.2 Allometric constraints in the case $\alpha > 1$

We can adapt most of our reasoning from the case $\alpha \leq 1$, and obtain the following theorem.

Theorem I.2.4. *Let $\alpha > 1$. Under the allometric setting of Section I.1.2, under Assumptions I.1.1, I.1.2 and I.1.4, we have:*

1. $\gamma = \alpha,\ C_\gamma > C_\alpha$.
2. $\delta \leq \alpha - 1 \leq \beta$.
3. *If $\beta = \delta = \alpha - 1$, then $C_\beta > C_\delta$.*
4. *If ($\delta = \alpha - 1$ and $\beta > \alpha - 1$), then $\beta \geq \alpha - 1 + \dfrac{C_\delta}{C_\gamma - C_\alpha}$.*
5. *If $\beta > \alpha$, then $\delta \geq \alpha - 1$. Moreover, if $\beta > \alpha$ and $\delta = \alpha - 1$, then $C_\delta \leq C_\gamma - C_\alpha$.*

Remark: All our results until Section I.4.9 are either valid under the general setting of Section I.1.1.1, or the computations can be adapted to the allometric setting of Section I.1.2 with $\alpha > 1$. This is why in the following, we consider that the proofs for every point of Section I.4.2 give the similar points in Theorem I.2.4. However, we cannot use the conclusions of Section I.4.9 (see the remark before Lemma I.4.26). This explains the weaker conditions obtained in Theorem I.2.4, when we compare it to Theorem I.2.1. Still, we think it is important to present our conclusions in that case. Indeed, DeLong and al. measured that metabolic rate scales with body mass superlinearly (*i.e.* $\alpha > 1$) for prokaryotes [DOM+10]. Malerba and Marshall also highlighted this superlinear scaling within a population of green algae [MM19]. We visualize the restrictions on allometric coefficients in the case $\alpha > 1$ thanks to Figure I.2.

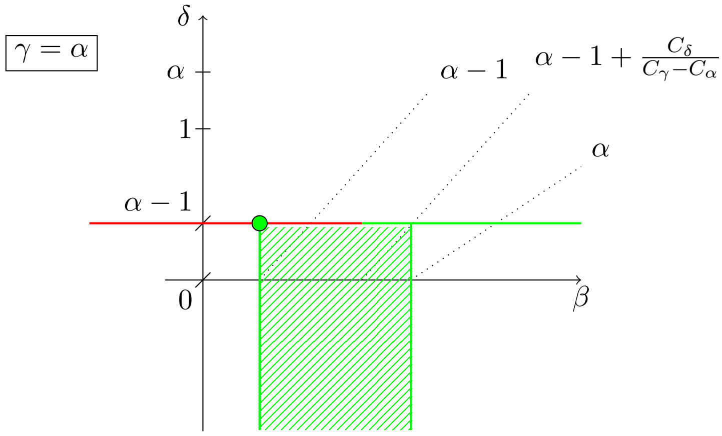


Figure I.2: Visual representation of Theorem I.2.4. All the coefficients (γ, δ, β) verifying one of the following conditions are non-admissible: 1) $\gamma \neq \alpha$; 2) $\delta \neq \alpha - 1$, except for the ones in the hatched area and its border; 3) $\gamma = \alpha$, $\delta = \alpha - 1$ and β on the red line (except for the green dot).

I.3 Discussion

Before going into the mathematical details, we discuss the biological interpretation of our assumptions in Section I.3.1, and Section I.3.2 gives our general conclusions and perspectives.

I.3.1 Revisiting dimensional arguments of the Metabolic Theory of Ecology

In Assumption I.1.2, we emphasize the ratio $(b_{x_0} + d)/g(., R)$. The numerator accounts for the influence of birth and death rates, whereas the denominator represents the speed of acquisition or loss of energy by the individual in environment R. A classical idea is that $(b_{x_0} + d)/g(., R)$ can be expressed as the inverse of an energy, because one can view b_{x_0} and d as dimensionless rates and g as a gain of energy per unit of time [SDF08], [MM19]. Usually, this is a justification for the choice $\beta = \delta = \alpha - 1$. We underline here that this choice for the allometric coefficients is only a sufficient condition to obtain a biologically relevant individual process.

Precisely, Assumption I.1.1 and Assumption I.1.2 indicate that the ratio $(b_{x_0} + d)/g(., R)$

has divergent integrals, respectively near 0 and $+\infty$. One possibility is to have the same allometric coefficient for b_{x_0} and d, and in that case we necessarily have $\beta = \delta = \alpha - 1$, so that the previous ratio is of order $1/x$. But there are plenty of other possibilities to obtain this divergent integral: for example, it happens when $\beta > \alpha - 1$ and $\delta = \alpha - 1$, and we see in Figure I.1 that the green area allows this kind of other reasonable choice. Hence, the choice $\beta = \delta = \alpha - 1$ is sufficient but not necessary to verify Assumptions I.1.1 and I.1.2. This is why we need an assumption about the population, which is Assumption I.1.4, to be more precise about the interspecific allometric constraints. This last assumption goes beyond the considerations about the ratio $(b_{x_0} + d)/g(., R)$, so beyond the classical dimensional argument.

I.3.2 General conclusion and perspectives

To put it in a nutshell, here are the assumptions we can make for the conception of our model:

- The functions ruling energetic dynamics and the underlying birth and death process have allometric shapes.
- An individual almost surely never reaches energy 0 before dying.
- An individual almost surely never reaches energy $+\infty$ before dying. Also, an individual has almost surely a finite lifetime.
- Any population, characterized by the energy at birth x_0, survives with positive probability, at least for a sufficient amount of resources.

The way we put it here highlights that this is really the first allometric assumption that is restrictive and leads to the conclusions of Theorem I.2.1. The other assumptions are strikingly basic, even if we underline that what we call a death event only accounts for intrinsic causes of death (*i.e.* individuals only die because of the energy loss due to metabolism, and not competition factors). Notice that the previous assumptions also imply the following biologically relevant and basic features:

- An individual has almost surely a finite number of direct offspring.
- An individual can always feed himself and grow if there are enough resources in the environment.

Recall here that a crucial aspect of our reasoning is that we want our model to respect the previous constraints for any kind of living species, tiny or gigantic. Thus, it is certainly adapted for the study of allometries involving several species, in the spirit of the guiding work of [Pet86] where the linear regressions are performed on a broad range of living species (nearly 20 orders of magnitude of body mass). The constraints we obtain are interspecific, and not intraspecific. One can refer to [KW97] or [MM19] to understand this gap between interspecific allometries and within-species dynamics. An interesting opening would be to study the evolution of the allometric coefficient α, adding a mutation mechanism in our previous setting. In other terms, we could adopt an evolutive point of view in line with the philosophy of adaptive dynamics [MGM+95, Die03].

Our feeling is that the main phenomenon at stake here is the universality of α along x_0. It is eventually not a surprise that we get close to the results of the Metabolic Theory of Ecology. The simple case considered in this paper is meant to give an insight about what kind of mechanisms, other than competition for resources or trait evolution, are important to balance the energetic dynamics. To aim for the complex reality of an entire food web, one would add interaction between individuals and/or species and a dynamic evolution of the resource (see [LL05] or [FBC21] for a general approach). We leave this for future work.

I.4 Construction of the process and proofs

In Section I.4.1, we construct the individual process highlighted in Section I.1.1.1. In Section I.4.2, we give the different steps of the proof of Theorem I.2.1. Even if we chose to highlight the allometric consequences of our reasoning in Section I.2, the results presented in the following are valid under the general setting of Section I.1.1.1, so their interpretation goes beyond the allometric setting (except for Sections I.4.8 and I.4.9).

I.4.1 Construction of the individual process $(\xi_t)_{t\geq 0}$

Let us fix x_0, R, ξ_0. Recall that $A : (\xi_0, t) \mapsto A_{\xi_0}(t)$ is the homogeneous-in-time flow associated to Equation (I.1.1) (R is implicit in the notations). For $\xi_0 > 0$, $A_{\xi_0}(.)$ is defined on $[0, t_{\max}(\xi_0)[$, where $t_{\max}(\xi_0)$ is the deterministic time when $A_{\xi_0}(t_{\max}(\xi_0)) = 0$ or $+\infty$ ($t_{\max}(\xi_0)$ is equal to $+\infty$ if this never happens). One way to construct $(\xi_t)_{t\geq 0}$ is to use two ingredients: 1) we first construct an auxiliary process $(\xi_t^{\text{aux}})_{t\geq 0}$ starting from $\xi_0^{\text{aux}} = \xi_0$ with birth events at rate $(b_{x_0} + d)\mathbb{1}_{\{\xi_t^{\text{aux}} > x_0\}}$ and death event at rate $d\mathbb{1}_{\{\xi_t^{\text{aux}} \leq x_0\}}$, and then 2) kill this process at a random time to obtain the appropriate birth and death rates b_{x_0} and d. This construction of the process ξ is made in the spirit of a Gillespie algorithm [Gil76]. Although this construction may not seem the most natural, it will be convenient for couplings in the proofs in the following. We insist on the fact that for this construction, we only work under the general setting of Section I.1.1.1, and in particular without Assumptions I.1.1 and I.1.2. This is essential if we want Theorem I.1.1 to be meaningful.

- Let $(E_i)_{i\geq 1}$ be i.i.d. random variables with exponential laws of parameter 1. First, if

$$\int_0^{t_{\max}(\xi_0)} (b_{x_0} + d)(A_{\xi_0}(s))\mathrm{d}s \leq E_1,$$

we set $J_1 := +\infty$. In that case, we set $\xi_t^{\text{aux}} = A_{\xi_0}(t)$ for $t \in [0, t_{\max}(\xi_0)[$ and $\xi_t^{\text{aux}} = \flat$ for $t \geq t_{\max}(\xi_0)$. Recall that reaching $\flat$ means that the individual energy exploded or reached 0 before time t. Otherwise, as $(b_{x_0} + d)$ is a positive function, we can define the first time of jump as

$$J_1 := \inf\left\{t \in [0, t_{\max}(\xi_0)[, \int_0^t (b_{x_0} + d)(A_{\xi_0}(s))\mathrm{d}s = E_1\right\}.$$

In that case, $A_{\xi_0}(J_1)$ is well-defined, and we set $\xi_t^{\text{aux}} = A_{\xi_0}(t)$ for $t \in [0, J_1[$, and

$$\xi_{J_1}^{\text{aux}} = (A_{\xi_0}(J_1) - x_0)\,\mathbb{1}_{\{A_{\xi_0}(J_1) > x_0\}} + \partial\mathbb{1}_{\{A_{\xi_0}(J_1) \leq x_0\}}.$$

Remark that we need to check if $A_{\xi_0}(J_1) \leq x_0$. If this occurs, the jump is necessarily a death jump, as soon as $b_{x_0} \equiv 0$ on $]0, x_0]$, and then we set $\xi_t^{\text{aux}} = \partial$ for $t \geq J_1$. We can derive the distribution function of J_1. For every $u \in]0, t_{\max}(\xi_0)[$:

$$\mathbb{P}_{x_0,R,\xi_0}(J_1 < u) = \int_0^u (b_{x_0} + d)(A_{\xi_0}(t))e^{-\int_0^t (b_{x_0} + d)(A_{\xi_0}(s))\mathrm{d}s}\mathrm{d}t, \qquad \text{(I.4.10)}$$

and $\mathbb{P}_{x_0,R,\xi_0}(J_1 = +\infty) = 1 - \mathbb{P}_{x_0,R,\xi_0}(J_1 < t_{\max}(\xi_0))$.

Now, let us suppose that we defined J_n and $\xi^{\text{aux}}_{J_n}$ for some $n \geq 1$. If $J_n = +\infty$, or $J_n < +\infty$ and $\xi^{\text{aux}}_{J_n} = \partial$, we simply set $J_{n+1} := +\infty$. In that case, ξ^{aux}_t is already defined for all $t \geq 0$, because it already reached $\flat$ or ∂. Now, we suppose that $J_n < +\infty$ and $\xi^{\text{aux}}_{J_n} \neq \partial$, and we distinguish between two remaining cases. First, if

$$\int_{J_n}^{J_n + t_{\max}(\xi^{\text{aux}}_{J_n})} (b_{x_0} + d)(A_{\xi^{\text{aux}}_{J_n}}(s - J_n))\mathrm{d}s \leq E_{n+1},$$

we again set $J_{n+1} := +\infty$, $\xi^{\text{aux}}_t = A_{\xi^{\text{aux}}_{J_n}}(t - J_n)$ for $t \in [J_n, J_n + t_{\max}(\xi^{\text{aux}}_{J_n})[$ and $\xi^{\text{aux}}_t = \flat$ for $t \geq J_n + t_{\max}(\xi^{\text{aux}}_{J_n})$. Otherwise, we can define the $(n+1)$-th time of jump as

$$J_{n+1} := \inf\left\{ t \in [J_n, J_n + t_{\max}(\xi^{\text{aux}}_{J_n})[, \int_{J_n}^{t} (b_{x_0} + d)(A_{\xi^{\text{aux}}_{J_n}}(s - J_n))\mathrm{d}s = E_{n+1} \right\}.$$

In that last case, we set $\xi^{\text{aux}}_t = A_{\xi^{\text{aux}}_{J_n}}(t - J_n)$ for $t \in [J_n, J_{n+1}[$, and

$$\xi^{\text{aux}}_{J_{n+1}} = \left(A_{\xi^{\text{aux}}_{J_n}}(J_{n+1} - J_n) - x_0 \right) \mathbb{1}_{\{A_{\xi^{\text{aux}}_{J_n}}(J_{n+1} - J_n) > x_0\}} + \partial \mathbb{1}_{\{A_{\xi^{\text{aux}}_{J_n}}(J_{n+1} - J_n) \leq x_0\}}.$$

If $\xi^{\text{aux}}_{J_{n+1}} = \partial$, we then set $\xi^{\text{aux}}_t = \partial$ for $t \geq J_{n+1}$. Eventually, it is possible, in the general setting of Section I.1.1.1 and without further assumptions, that $\mathfrak{J} := \sup_{n \in \mathbb{N}^*} J_n < +\infty$ (*i.e.* we have an accumulation of jump times). In that final case, we set $\xi^{\text{aux}}_t = \flat$ for $t \geq \mathfrak{J}$.

- Then, we define a sequence $(U_i)_{i\geq 1}$ of i.i.d. random variables with uniform laws on $[0, 1]$, independent of $(E_i)_{i\geq 1}$. We define the time of death T_d as

$$T_d := \inf\left\{ J_i,\ \xi^{\text{aux}}_{J_i} \neq \flat \text{ and } U_i \leq \frac{d}{b_{x_0} + d}\left(\xi^{\text{aux}}_{J_i -}\right) \right\},$$

with the convention $\inf(\varnothing) = +\infty$. We also define $T_0 := \inf\{t \geq 0, \xi^{\text{aux}}_t = \flat, \xi^{\text{aux}}_s \xrightarrow[s \to t-]{} 0\}$ and $T_\infty := \inf\{t \geq 0, \xi^{\text{aux}}_t = \flat, \xi^{\text{aux}}_s \xrightarrow[s \to t-]{} +\infty\}$. Remark that $\{T_d < T_0 \wedge T_\infty\} \subseteq \{\mathfrak{J} = +\infty\}$, so if our process is biologically relevant as defined in Section I.1.1.2, there is no accumulation of jump times. We then define our final process ξ for $t \in \mathbb{R}^+$ as

$$\xi_t := \xi^{\text{aux}}_t \mathbb{1}_{\{t < T_d\}} + \partial \mathbb{1}_{\{t \geq T_d\}}.$$

The reader can check that if for some $n \in \mathbb{N}$, the condition $\{A_{\xi^{\text{aux}}_{J_n}}(J_{n+1} - J_n) \leq x_0\}$ is verified, then we have $\frac{d}{b_{x_0}+d}\left(\xi^{\text{aux}}_{J_n -}\right) = 1 \geq U_n$, so that our jumps to ∂ for ξ^{aux} are consistent with the definition of T_d. These jumps for ξ^{aux} to ∂ are necessary to keep a well-defined positive (when it is not ∂ or $\flat$) process ξ^{aux}. They account for deaths of individuals with energy smaller than x_0, whereas the definition of T_d also includes deaths with an energy higher than x_0. Finally, birth times for the process ξ are exactly times J_i such that $J_i < +\infty$ and $J_i \neq T_d$, so the number of direct offspring of an individual during its life, is defined as

$$N_{x_0,R,\xi_0} := \sup\{i \geq 0, J_i < T_d\}. \tag{I.4.11}$$

I.4.2 Sketch of the proof of Theorem I.1.1 and Theorem I.2.1

We prove Theorem I.1.1 by splitting the different possible situations depending on the resource R. We consider the case $R \in \mathfrak{R}_0 \setminus \mathfrak{R}_\infty$ in Proposition I.4.4 of Section I.4.4, the case $R \notin \mathfrak{R}_0 \cup \mathfrak{R}_\infty$ in Lemma I.4.6 of Section I.4.5, and conclude for $R \in \mathfrak{R}_\infty$ in Proposition I.4.11 of Section I.4.5. Also, we divide the proof of Theorem I.2.1 in eight distinct points. Let $0 < \alpha \leq 1$. Under the allometric setting of Section I.1.2, we have:

1. Assumption I.1.8 $\Leftrightarrow$ ($\gamma = \alpha$ and $C_\gamma > C_\alpha$).
2. Assumption I.1.1 $\Leftrightarrow \delta \leq \alpha - 1$.
3. Assumptions I.1.2 and I.1.8 $\Rightarrow$ (Assumption I.1.5 and $\max(\beta, \delta) \geq \alpha - 1$).
4. (Assumptions I.1.1, I.1.2, I.1.4 and $\max(\beta, \delta) \geq \alpha - 1$) $\Rightarrow$ ($\beta \geq \alpha - 1$, and if $\beta = \delta = \alpha - 1$, then $C_\beta > C_\delta$).
5. (Assumptions I.1.2, I.1.4 and $\delta = \alpha - 1$) $\Rightarrow \left(\text{if } \beta > \alpha - 1, \text{ then } \beta \geq \alpha - 1 + \dfrac{C_\delta}{C_\gamma - C_\alpha}\right)$.
6. Assumptions I.1.1, I.1.2, I.1.4 $\Rightarrow$ (if $\beta > \alpha$, then $\delta \geq \alpha - 1$).
7. (Assumptions I.1.2, I.1.4 and $\delta = \alpha - 1$) $\Rightarrow$ (if $\beta > \alpha - 1$, then $C_\delta \leq C_\gamma - C_\alpha$).
8. Assumptions I.1.2 and I.1.8 $\Rightarrow$ (if $\beta \leq \alpha$, then $\delta \geq \alpha - 1$).

Remark: Theorem I.2.1 is a consequence of points **1** to **8**. Thanks to Lemma I.1.9 and point **3**, we know that in Theorem I.2.1, we work in fact under Assumptions I.1.1 to I.1.8, and $\max(\beta, \delta) \geq \alpha - 1$. Then, $\gamma = \alpha$ and $C_\gamma > C_\alpha$ because of **1**. Also, $\delta = \alpha - 1$ because of **2, 6, 8**. Finally, in Theorem I.2.1, the remaining conditions on $\beta, C_\beta, C_\delta, C_\gamma, C_\alpha$ are the combination of **4, 5, 7**.

For the previously enumerated results, we precise where the reader can find the associated proofs, with some insight on the mathematical tools involved. Except for points **6, 7, 8**, the following results are stated and valid under the general setting of Section I.1.1.1.

1. See Proposition I.4.1 in Section I.4.3, which is purely deterministic.
2. Under the allometric setting of Section I.1.2, Equation (I.1.3) is

   $$\forall x_0 > 0, \int_0^{x_0} \frac{C_\delta}{C_\alpha} x^{\delta - \alpha} \mathrm{d}x = +\infty,$$

   which is equivalent to $\delta \leq \alpha - 1$.
3. See Corollary I.4.9 in Section I.4.5. The proof is based on an operator point of view on our probabilistic questioning.
4. See Proposition I.4.12 in Section I.4.6, which uses basic probability calculus.
5. See Proposition I.4.18 in Section I.4.7.2. The proof is based on Proposition I.4.16 in Section I.4.7, which uses a useful coupling to compare different individual trajectories.
6. This will be proven in Sections I.4.8.1 and I.4.8.2. We define several couplings of our individual process with simpler processes, and also track the maximal energy reached by an individual.

7. We apply the reasoning of **6.** in the case $\delta = \alpha - 1$. This constitutes Corollary I.4.24 in Section I.4.8.3.

8. We use a representation of the individual process $(\xi_t)_{t\geq 0}$ with Poisson point measures, and martingale techniques. We were inspired by the concept of asymptotic pseudotrajectory developed by Benaïm and Hirsch [Ben99]. The complete proof of this last point is developed in Section I.4.9.

I.4.3 Proof of 1. in Section I.4.2

We begin with a simple result that illustrates perfectly our way of reasoning. Also, this section is particular in the sense that our considerations here are purely deterministic. Assumption I.1.8 translates into a condition on functions ψ and ℓ, ruling the energy dynamics in Equation (I.1.1).

Proposition I.4.1. *Under the general setting of Section I.1.1.1, Assumption I.1.8 is equivalent to*

$$\forall x > 0, \quad \psi(x) > \ell(x). \tag{I.4.12}$$

Under the allometric setting of Section I.1.2, (I.4.12) *is equivalent to*

$$\gamma = \alpha \text{ and } C_\gamma > C_\alpha.$$

Proof. Let $x > 0$, we have $g(x, R) = \phi(R)\psi(x) - \ell(x)$. It is straightforward to verify that Assumption I.1.8 is equivalent to (I.4.12), because ϕ is non-negative, non-decreasing and $\phi(R) \to 1$ when $R \to +\infty$.
Under the allometric setting of Section I.1.2, by letting $x \to +\infty$, we see that (I.4.12) cannot be verified unless $\gamma > \alpha \quad \text{or} \quad (\gamma = \alpha \text{ and } C_\gamma > C_\alpha)$.
In the same vein, if we consider now $x \to 0$, we also obtain regarding (I.4.12) that $\gamma < \alpha \quad \text{or} \quad (\gamma = \alpha \text{ and } C_\gamma > C_\alpha)$, which ends the direct implication. It is easy to see that the converse also holds true. □

Remark: This result gives point **1.** of Section I.4.2. We immediately observe one of the major feature of the allometric setting: power functions of the form $x \mapsto x^\kappa$ have an antagonistic behavior near 0 and near $+\infty$, depending on the sign of κ. In order to enforce general principles valid for any x_0, one has to choose precise allometric coefficients.

Under the allometric setting of Section I.1.2, an immediate consequence of Proposition I.4.1 is that, under Assumption I.1.8, Equation (I.1.1) becomes

$$\frac{\mathrm{d}\xi_t}{\mathrm{d}t} = C_R \xi_t^\alpha, \tag{I.4.13}$$

with $C_R := \phi(R)C_\gamma - C_\alpha \leq C_\gamma - C_\alpha$. Remark that if $0 < \alpha \leq 1$, for every (R, ξ_0), the solution of Equation (I.1.1) starting from ξ_0 with avalaible resources R cannot explode in finite time, but can possibly reach 0 depending on the value of R. In the case $\alpha > 1$, a solution of (I.4.13) with $C_R > 0$ can reach $+\infty$ in finite time, but not 0. These situations have been taken into account in the construction of the individual process $(\xi_t)_{t\geq 0}$ in Section I.4.1.

I.4.4 Equivalent formulations of Assumption I.1.1

In the following, we give probabilistic interpretations of Assumption I.1.1.

Lemma I.4.2. *Under the general setting of Section I.1.1.1, Assumption I.1.1 is equivalent to*

$$\forall x_0 > 0, \forall \xi_0 > 0, \quad \mathbb{P}_{x_0,R=0,\xi_0}(T_d < +\infty) = 1. \tag{I.4.14}$$

Proof. First, let us show that Assumption I.1.1 is equivalent to the weaker condition

$$\mathbb{P}_{x_0,R=0,\xi_0=x_0}(T_d < +\infty) = 1 \tag{I.4.15}$$

for every $x_0 > 0$. We fix x_0, $R = 0$ and $\xi_0 = x_0$ and investigate under which condition (I.4.15) holds true. In this situation, the energy only decreases over time, and the only random event possibly occurring is a death, so from the construction of Section I.4.1, $\{T_d < +\infty\} = \{J_1 < t_{\max}(x_0, 0)\}$ (with $t_{\max}(x_0, 0)$ possibly equal to $+\infty$). We use (I.4.10) with $(b_{x_0} + d) \equiv d$ on $]0, x_0]$ to obtain

$$\begin{aligned}\mathbb{P}_{x_0,0,x_0}(T_d < +\infty) &= \mathbb{P}_{x_0,0,x_0}(J_1 < t_{\max}(x_0, 0))\\ &= \int_0^{t_{\max}(x_0,0)} d(A_{x_0,0}(s))e^{-\int_0^s d(A_{x_0,0}(\tau))\mathrm{d}\tau} \mathrm{d}s\\ &= 1 - e^{-\int_0^{t_{\max}(x_0,0)} d(A_{x_0,0}(\tau))\mathrm{d}\tau}.\end{aligned}$$

Thus, (I.4.15) is equivalent to ($e^{-\int_0^{t_{\max}(x_0,0)} d(A_{x_0,0}(\tau))\mathrm{d}\tau} = 0$ for all $x_0 > 0$). The function ℓ is positive increasing and $A_{x_0,0}(.)$ is the solution of (I.1.1) with $A_{x_0,0}(0) = x_0$ and $R = 0$ (so $g(x, R) = -\ell(x)$). Hence, $A_{x_0,0}(t)$ goes to 0 when t goes to $t_{\max}(x_0, 0)$, and we can perform the change of variables $u = A_{x_0,0}(\tau)$, which gives exactly Assumption I.1.1. Now, we suppose that (I.4.15) holds true and we show (I.4.14). The same reasoning as before still applies when $\xi_0 \le x_0$, because there are no birth events (we just replace $t_{\max}(x_0, 0)$ by $t_{\max}(\xi_0, 0)$). If $\xi_0 > x_0$, we define the random time

$$\tau_{x_0,\xi_0} := \inf\{t \in [0, T_d[, \ \xi_{t,x_0,0,\xi_0} \le x_0\},$$

with the convention $\inf(\varnothing) = +\infty$. First if $\tau_{x_0,\xi_0} = +\infty$, then $T_d < T_0 = +\infty$ because before T_d, the energy is in $[x_0, \xi_0]$ and the death rate is positive continuous, so lower bounded by a positive constant, on this segment. Else if $\tau_{x_0,\xi_0} < +\infty$, by Markov property, we start at time τ_{x_0,ξ_0} from a new initial condition $\xi_1 \le x_0$, for which the result is already proven. □

In fact, Assumption I.1.1 also ensures that individual trajectories almost surely avoid 0 for every $x_0 > 0$, $\xi_0 > 0$, and any amount of resources $R \ge 0$.

Proposition I.4.3. *Under the general setting of Section I.1.1.1 and under Assumption I.1.1, we have*

$$\forall x_0 > 0, \forall \xi_0 > 0, \forall R \ge 0, \quad \mathbb{P}_{x_0,R,\xi_0}(T_0 = +\infty) = 1.$$

Proof. Let $x_0, \xi_0 > 0$ and $R \geq 0$ and assume by contradiction that $\mathbb{P}_{x_0,R,\xi_0}(T_0 < +\infty) > 0$. From the construction of Section I.4.1, on the event $\{T_0 < +\infty\}$, there exists a family $(E_i)_{i\geq 1}$ of i.i.d. exponential random variables with parameter 1, a family $(U_i)_{i\geq 1}$ of i.i.d. uniform random variables on $[0,1]$ and $n \in \mathbb{N}^*$ such that $J_{n-1} < +\infty$ and

$$\int_0^{t_{\max}(\xi^{\text{aux}}_{J_{n-1}},R)} (b_{x_0} + d)(A_{\xi^{\text{aux}}_{J_{n-1}},R}(s))\mathrm{d}s \leq E_n,$$

with the convention $J_0 = 0$. Moreover, as $T_0 < +\infty$, we necessarily have

$$A_{\xi^{\text{aux}}_{J_{n-1}},R}(s) \xrightarrow[s\to t_{\max}(\xi^{\text{aux}}_{J_{n-1}},R)]{} 0,$$

meaning that the flow $s \mapsto A_{\xi^{\text{aux}}_{J_{n-1}},R}(s)$ is decreasing and $g(x,R) < 0$ for every $x \in]0, \xi^{\text{aux}}_{J_{n-1},R}]$, according to Equation (I.1.1). The change of variables $x = A_{\xi^{\text{aux}}_{J_{n-1}},R}(s)$ is then licit and we have

$$\int_0^{\xi^{\text{aux}}_{J_{n-1}}} \frac{(b_{x_0} + d)(x)}{-g(x,R)}\mathrm{d}x \leq E_n < +\infty.$$

As g is increasing in R and $b_{x_0} \equiv 0$ on $]0, x_0]$, this leads to

$$\int_0^{\xi^{\text{aux}}_{J_{n-1}}\wedge x_0} \frac{d(x)}{-g(x,0)}\mathrm{d}x < +\infty,$$

which contradicts Assumption I.1.1 (because $g(x,0) \equiv -\ell$) and concludes the proof. $\square$

Finally, we can give a stronger equivalent formulation of Assumption I.1.1.

Proposition I.4.4. *Under the general setting of Section I.1.1.1, Assumption I.1.1 is equivalent to*

$$\forall x_0 > 0, \forall \xi_0 > 0, \forall R \in \mathfrak{R}_0 \setminus \mathfrak{R}_\infty, \quad \mathbb{P}_{x_0,R,\xi_0}(T_d < +\infty) = 1.$$

Proof. First, remark that $0 \in \mathfrak{R}_0 \setminus \mathfrak{R}_\infty$, so by Lemma I.4.2, we only have to show that Assumption I.1.1 implies $\mathbb{P}_{x_0,R,\xi_0}(T_d < +\infty) = 1$ for $R > 0$ in $\mathfrak{R}_0 \setminus \mathfrak{R}_\infty$ and any $x_0 > 0$, $\xi_0 > 0$ that we fix for now. Because $R \notin \mathfrak{R}_\infty$, there exists $x_2 > \xi_0$ such that $g(x_2, R) \leq 0$. Then, $T_\infty = +\infty$, and for $t < T_d \wedge T_0$, $\xi_{t,x_0,R,\xi_0} \leq x_2$. Also, by Proposition I.4.3, we also have $T_0 = +\infty$ almost surely. However, $R \in \mathfrak{R}_0$, so there exists $x_1 > 0$, such that $g(y,R) < 0$ for $y \leq x_1$. We define the random time

$$\tau := \inf\{t \in [0, T_d[,\ \xi_{t,x_0,R,\xi_0} \leq x_1 \wedge x_0\},$$

with the convention $\inf(\varnothing) = +\infty$. First, if $\tau = +\infty$, we have $\xi_{t,x_0,R,\xi_0} \in [x_1 \wedge x_0, x_2]$ before T_d and the death rate is positive continuous on this segment so $T_d < +\infty$. Else if $\tau < +\infty$, by Markov property, we start at time τ from a new initial condition $\xi_1 \leq x_1 \wedge x_0$. As $\frac{d(x)}{-g(x,R)} > \frac{d(x)}{\ell(x)}$ for $x \leq \xi_1$, the same reasoning as Lemma I.4.2 applies, from the initial condition ξ_1 with resource R, so $T_d < +\infty$. $\square$

I.4.5 Proof of 3. in Section I.4.2 and consequences of Assumption I.1.2

First, we prove intermediate lemmas that lead to Corollary I.4.9, which entails point **3.** in Section I.4.2. Then, we will give in Proposition I.4.11 an equivalent formulation of Assumption I.1.2 under Assumption I.1.1, which completes the proof of Theorem I.1.1 with Proposition I.4.4 and Lemma I.4.6. We begin with a technical lemma on $\mathfrak{J}$, the supremum of jump times of ξ^{aux} defined in Section I.4.1.

Lemma I.4.5. *Under the general setting of Section I.1.1.1, for every x_0, R, ξ_0, we have almost surely*

$$\{\mathfrak{J} < +\infty\} \subseteq \{\mathfrak{J} = T_\infty\}.$$

Proof. We fix x_0, R, ξ_0 in the following. Remark that

$$\{\mathfrak{J} < +\infty\} \subseteq \left\{\forall n \in \mathbb{N}^*, \xi^{\text{aux}}_{J_n} \notin \{\partial, \flat\}\right\} \subseteq \{\forall n \in \mathbb{N}^*,\ J_n < T_0 \wedge T_\infty \wedge T_d\}.$$

Thus, on the event $\{\mathfrak{J} < +\infty\}$, the only possible random jumps on the finite time interval $[0, \mathfrak{J}[$ are birth events, and there is an infinite amount of such birth events. Remark that the event $\{\mathfrak{J} < T_\infty\}$ is equal to

$$\left\{\exists M > 0, \forall t \in [0, \mathfrak{J}[, \exists s \in]t, \mathfrak{J}[, \xi^{\text{aux}}_s \leq M\right\}, \tag{I.4.16}$$

so we suppose by contradiction that $\{\mathfrak{J} < +\infty\}$ and (I.4.16) occur with positive probability and we work on these events in the following. We define $\tau := \sup\{t \in [0, \mathfrak{J}[, \xi^{\text{aux}}_t > M+1\}$, and distinguish between two cases. First, if $\tau < \mathfrak{J}$, it means that there is an infinite number of birth events and $\xi^{\text{aux}}_t \leq M+1$ on the finite time interval $]\tau, \mathfrak{J}[$. This happens with probability 0, because the birth rate b_{x_0} is bounded on $[0, M+1]$. Else if $\tau = \mathfrak{J}$, by (I.4.16), we can define two random sequences $(s_k)_{k\in\mathbb{N}}$ and $(t_k)_{k\in\mathbb{N}}$ such that for all $k \in \mathbb{N}$, $0 \leq s_k < t_k < s_{k+1} < \mathfrak{J}$, with $\xi^{\text{aux}}_{s_k} \leq M$ and $\xi^{\text{aux}}_{t_k} > M+1$. By definition of the process, the only way ξ^{aux} can increase is deterministic, between random jumps. Hence, there exists a deterministic time $\sigma > 0$ such that for all $k \geq \mathbb{N}$, $t_k - s_k \geq \sigma$. This is a contradiction and concludes the proof, because we would then have

$$+\infty = \sum_{k\geq 0} \sigma \leq \sum_{k\geq 0} (t_k - s_k) \leq \sum_{k\geq 0} (s_{k+1} - s_k) \leq \mathfrak{J} < +\infty.$$

□

For $R \geq 0$, we define $\Omega_R := \{\xi_0 > 0, \forall x \geq \xi_0, g(x, R) > 0\}$, and notice that $R \in \mathfrak{R}_\infty$, if and only if $\Omega_R \neq \varnothing$.

Lemma I.4.6. *Under the general setting of Section I.1.1.1, we have*

$$\forall x_0 > 0, \forall R \notin \mathfrak{R}_0 \cup \mathfrak{R}_\infty, \forall \xi_0 > 0, \quad \mathbb{P}_{x_0,R,\xi_0}(T_d < +\infty) = 1,$$

and

$$\forall x_0 > 0, \forall R \in \mathfrak{R}_\infty \setminus \mathfrak{R}_0, \forall \xi_0 \notin \Omega_R, \quad \mathbb{P}_{x_0,R,\xi_0}(T_d < +\infty) = 1.$$

Proof. Let $\xi_0 > 0$, $x_0 > 0$ and $R \notin \mathfrak{R}_0 \cup \mathfrak{R}_\infty$, then

$$\forall x > 0, \exists y \in]0, x], \quad g(y, R) \geq 0,$$

and

$$\forall x > 0, \exists y \geq x, \quad g(y, R) \leq 0.$$

In that case, according to (I.1.1), we have $T_0 = T_\infty = +\infty$. In particular, $\mathfrak{J} = +\infty$ almost surely by Lemma I.4.5, because if $\mathfrak{J} < +\infty$, we would have $\mathfrak{J} = T_\infty < +\infty$. We can find $\overline{y} \geq x_0 \vee \xi_0$ such that $g(\overline{y}, R) \leq 0$, so before T_d, we have $\xi_t \leq \overline{y}$. If $R \in \mathfrak{R}_\infty \setminus \mathfrak{R}_0$ and $\xi_0 \notin \Omega_R$, we verify that we can work in the exact same setting. Now, suppose by contradiction that $\{T_d = +\infty\}$ occurs with positive probability, and work on this event. We define

$$\underline{\xi} := \inf_{t \geq 0} \xi_t.$$

First, if $\underline{\xi} > 0$, the process stays in $[\underline{\xi}, \overline{y}]$ over time. Otherwise, $\underline{\xi} = 0$, and the only way this could happen with $R \notin \mathfrak{R}_0$ is that there is an infinite number of birth events. Hence, we can construct two random sequences $(s_k)_{k \in \mathbb{N}}$ and $(t_k)_{k \in \mathbb{N}}$ such that for all $k \in \mathbb{N}$, $0 \leq s_k < t_k < s_{k+1}$, with $\xi^{\text{aux}}_{s_k} \leq x_0/2$ and $\xi^{\text{aux}}_{t_k} > x_0$. As in the proof of Lemma I.4.5, this implies that the process stays almost surely an infinite amount of time in the segment $[x_0/2, x_0]$. In both cases, the process stays an infinite amount of time in a segment, where the death rate d is lower-bounded by a positive constant, so the time of death T_d would be almost surely finite, which is a contradiction. □

Recall that $K_{x_0,R}$ is the operator defined by (I.1.2). In the two following lemmas, we give a probabilistic interpretation of $K^k_{x_0,R}\mathbf{1}$ for $k \geq 1$, as the probability that the k first jumps of the process are birth events. For $k \geq 1$, we write $M^k_{x_0,R,\xi_0}$, or simply M^k, for the event $\{N_{x_0,R,\xi_0} \geq k\}$ (*i.e.* an individual starting from ξ_0 with characteristic energy x_0 and resources R has at least k direct offspring during its life), with N_{x_0,R,ξ_0} defined by (I.4.11). We write B_{x_0,R,ξ_0}, or simply B, for the event $\{J_1 < T_d\}$, (*i.e.* the first jump event is a birth). Also, we write D_{x_0,R,ξ_0}, or simply D for the event $\{J_1 = T_d\} \cap \{J_1 < t_{\max}(\xi_0, R)\}$, (*i.e.* the first jump event is a death).

Lemma I.4.7. *Under the general setting of Section I.1.1.1, for every $x_0, \xi_0 > 0$, $R \geq 0$ and $0 < u < t_{\max}(\xi_0)$, we have*

$$\mathbb{P}_{x_0,R,\xi_0}(\{J_1 < u\} \cap B) = \int_0^u b_{x_0}(A_{\xi_0,R}(t)) e^{-\int_0^t (b_{x_0} + d)(A_{\xi_0,R}(s)) \mathrm{d}s} \mathrm{d}t,$$

In particular,

$$\mathbb{P}_{x_0,R,\xi_0}(B) = K_{x_0,R}\mathbf{1}(\xi_0).$$

Proof. From the construction of Section I.4.1, on the event $\{J_1 < u < t_{\max}(\xi_0)\}$, the event B is exactly

$$\left\{ U_1 > \frac{d}{b_{x_0} + d}(A_{\xi_0,R}(J_1)) \right\},$$

with U_1 being a uniform random variable on $[0,1]$ independent of J_1. Hence, conditioning on the value of J_1 and using (I.4.10), we obtain

$$\mathbb{P}_{x_0,R,\xi_0}(\{J_1 < u\} \cap B) = \int_0^u \left[\left(1 - \frac{d}{b_{x_0} + d}(A_{\xi_0,R}(t))\right) (b_{x_0} + d)(A_{\xi_0,R}(t)) e^{-\int_0^t (b_{x_0} + d)(A_{\xi_0,R}(s))\Big] \mathrm{d}s} \mathrm{d}t,$$

which leads to the result. The second equality of the lemma comes from the limit $u \to t_{\max}(\xi_0, R)$ (increasing events on the left-hand side and increasing integral when u increases on the right-hand side). □

Lemma I.4.8. *Under the general setting of Section I.1.1.1, we have*

$$\forall k \geq 1, \quad \forall x_0 > 0, \forall \xi_0 > 0, \forall R \geq 0, \ \mathbb{P}(M^k_{x_0,R,\xi_0}) = K^k_{x_0,R}\mathbf{1}(\xi_0), \tag{I.4.17}$$

so that

$$\textit{Assumption I.1.5} \Leftrightarrow \left(\forall x_0 > 0, \forall R \geq 0, \forall \xi_0 > 0, \ K^k_{x_0,R}\mathbf{1}(\xi_0) \xrightarrow[k \to +\infty]{} 0\right).$$

Proof. We prove (I.4.17) by induction on $k \geq 1$. The base case is given by Lemma I.4.7, because $M^1 = B$. Now we suppose that the result holds true for some $k \geq 1$. We define $(\tilde{\xi}_{\cdot,x_0,R,x})_{x>0}$ a family of processes with same law as $\xi_{\cdot,x_0,R,x}$, but independent from our initial process $\xi_{\cdot,x_0,R,\xi_0}$, and in particular independent from $M^1_{x_0,R,\xi_0}$. We denote by $\tilde{M}^k_{x_0,R,x}$ the associated events, defined like $M^k_{x_0,R,\xi_0}$ for $\xi_{\cdot,x_0,R,\xi_0}$. By Markov property, we have that

$$\mathbb{P}(M^{k+1}_{x_0,R,\xi_0}) = \mathbb{P}(M^1_{x_0,R,\xi_0} \cap \tilde{M}^k_{x_0,R,A_{\xi_0,R}(J_1)-x_0}).$$

Conditioning on the value of J_1 (which is in $]0, t_{\max}(\xi_0)[$ on the event M^1) and using (I.4.10) like in the proof of Lemma I.4.7, we obtain

$$\mathbb{P}(M^{k+1}_{x_0,R,\xi_0}) = \int_0^{t_{\max}(\xi_0,R)} b_{x_0}(A_{\xi_0,R}(t)) e^{-\int_0^t (b_{x_0} + d)(A_{\xi_0,R}(s))\mathrm{d}s} \mathbb{P}(M^k_{x_0,R,A_{\xi_0,R}(t)-x_0})\mathrm{d}t,$$

because $\tilde{\xi}_{\cdot,x_0,R,x}$ and $\xi_{\cdot,x_0,R,x}$ have the same law. This concludes by induction hypothesis. The equivalence of Lemma I.4.8 follows from

$$\{N_{x_0,R,\xi_0} = +\infty\} = \bigcap_{k \geq 1} M^k_{x_0,R,\xi_0}.$$

□

We highlight the following consequence of Lemmas I.4.6 and I.4.8, which implies point **3.** in Section I.4.2.

Corollary I.4.9. *Under the general setting of Section I.1.1.1,*

$$\textit{Assumption I.1.2} \Rightarrow \textit{Assumption I.1.5}.$$

Under the allometric setting of Section I.1.2, we also have

$$(\textit{Assumptions I.1.2 and I.1.8}) \Rightarrow \max(\beta, \delta) \geq \alpha - 1.$$

Proof. For the first point of the corollary, thanks to Lemma I.4.8, the only thing to verify is that for any $x_0 > 0$, $\mathbb{P}(N_{x_0,R,\xi_0} < +\infty) = 1$, for every $R \notin \mathfrak{R}_\infty$, or $R \in \mathfrak{R}_\infty$ and $\xi_0 > 0$ such that there exists $x_2 > \xi_0$, with $g(x_2, R) \leq 0$ (*i.e.* $\xi_0 \notin \Omega_R$). Remark that $R \notin \mathfrak{R}_\infty$ implies the existence of such x_2 for any $\xi_0 > 0$. We work with such parameters in the following, and remark that in both cases, $T_\infty = +\infty$, so $\mathfrak{J} = +\infty$ by Lemma I.4.5. First, if $R \notin \mathfrak{R}_0$, $T_d < +\infty$ by Lemma I.4.6 so $N_{x_0,R,\xi_0} < +\infty$ by definition, see (I.4.11). Then if $R \in \mathfrak{R}_0$, there exists $x_1 > 0$ such that $g(y, R) < 0$ if $y \leq x_1$. We define

$$\tau := \inf\{t \geq 0, \xi_{t,x_0,R,\xi_0} \leq x_0 \wedge x_1\},$$

with the convention $\inf(\varnothing) = +\infty$. We have again two cases. First if $\tau = +\infty$, then ξ_{t,x_0,R,ξ_0} stays in $[x_0 \wedge x_1, x_2]$ before T_d, and the death rate is positive continuous on this segment, so $T_d < +\infty$, hence $N_{x_0,R,\xi_0} < +\infty$. Else if $\tau < +\infty$, there are no possible births after time τ because individual energy remains under x_0, so $N_{x_0,R,\xi_0} < +\infty$.

Under the allometric setting of Section I.1.2 and Assumption I.1.8, according to Proposition I.4.1, $\gamma = \alpha$ and $\mathfrak{R}_\infty \neq \varnothing$. We take R in this set and then for all $y > 0$, $g(y, R) > 0$. The integral condition of Assumption I.1.2 then leads to the second point. □

Now, we give another consequence of Assumption I.1.2.

Proposition I.4.10. *Under the general setting of Section I.1.1.1,*

$$\textit{Assumption I.1.2} \Rightarrow (\forall x_0 > 0, \forall R \in \mathfrak{R}_\infty \setminus \mathfrak{R}_0, \forall \xi_0 > 0, \quad \mathbb{P}_{x_0,R,\xi_0}(T_d < +\infty) = 1).$$

Proof. We fix any $x_0 > 0$, $R \in \mathfrak{R}_\infty \setminus \mathfrak{R}_0$, $\xi_0 > 0$, and we compute $\mathbb{P}_{x_0,R,\xi_0}(T_d < +\infty)$. In the following, we write $p_k : x > 0 \mapsto \mathbb{P}_{x_0,R,x}(\{T_d = J_k\} \cap \{T_d < +\infty\})$. On the event $\{T_d < +\infty\}$, T_d is one of the J_k, so we immediately have

$$\mathbb{P}_{x_0,R,\xi_0}(T_d < +\infty) = \sum_{k \geq 1} p_k(\xi_0). \tag{I.4.18}$$

First, we have from the construction of Section I.4.1 that

$$p_1(\xi_0) = \mathbb{P}_{x_0,R,\xi_0}(T_d = J_1, T_d < +\infty) = \mathbb{P}_{x_0,R,\xi_0}(D) = \mathbb{P}_{x_0,R,\xi_0}(J_1 < t_{\max}(\xi_0)) - \mathbb{P}_{x_0,R,\xi_0}(B),$$

which gives thanks to (I.4.10) and Lemma I.4.7,

$$p_1(\xi_0) = 1 - \sigma_{x_0,R,\xi_0} - K_{x_0,R}\mathbf{1}(\xi_0), \tag{I.4.19}$$

with $\sigma_{x_0,R,\xi_0} := \exp\left(-\int_0^{t_{\max}(\xi_0,R)} (b_{x_0} + d)(A_{\xi_0,R}(\tau))\mathrm{d}\tau\right)$. Then, for $k \geq 1$, we use Markov property as in the proof of Lemma I.4.8, but with the events $\{T_d = J_k\} \cap \{T_d < +\infty\}$ instead of $M^k_{x_0,R,\xi_0}$, to obtain

$$\begin{aligned} p_{k+1}(\xi_0) &= \int_0^{t_{\max}(\xi_0,R)} b_{x_0}(A_{\xi_0,R}(u))e^{-\int_0^u (b_{x_0} + d)(A_{\xi_0,R}(\tau))\mathrm{d}\tau} p_k(A_{\xi_0,R}(u) - x_0)\mathrm{d}u \\ &= K_{x_0,R}p_k(\xi_0), \end{aligned}$$

and this is valid for every $\xi_0 > 0$ and $k \geq 1$, so

$$p_{k+1} = K_{x_0,R} p_k = K^k_{x_0,R} p_1. \tag{I.4.20}$$

Now, thanks to Equations (I.4.18), (I.4.19), (I.4.20), we obtain

$$\begin{aligned}\mathbb{P}_{x_0,R,\xi_0}(T_d < +\infty) &= \sum_{k\geq 1} K^{k-1}_{x_0,R}(\mathbf{1} - \sigma_{x_0,R,.} - K_{x_0,R}\mathbf{1})(\xi_0) \\ &= \sum_{k\geq 1} K^{k-1}_{x_0,R}(\mathbf{1} - K_{x_0,R}\mathbf{1})(\xi_0) - \sum_{k\geq 1} K^{k-1}_{x_0,R}\sigma_{x_0,R,.}(\xi_0) \\ &= 1 - \lim_{k\to+\infty} K^k_{x_0,R}\mathbf{1}(\xi_0) - \sum_{k\geq 1} K^{k-1}_{x_0,R}\sigma_{x_0,R,.}(\xi_0).\end{aligned}$$

We can split the sums from the first line to the second, because one of them is telescopic (and $\lim_{k\to+\infty} K^k_{x_0,R}\mathbf{1}(\xi_0)$ exists as the limit of a decreasing non-negative sequence) and the first line is the wanted probability. Now, if $\xi_0 \in \Omega_R$, then $g(x, R) > 0$ for $x \geq \xi_0$, and we can make the change of variables $x = A_{\xi_0,R}(\tau)$ in the expression of σ_{x_0,R,ξ_0}, with $A_{\xi_0,R}(\tau)$ going to $+\infty$ when τ goes to $t_{\max}(\xi_0, R)$, so

$$\sigma_{x_0,R,\xi_0} := \exp\left(-\int_{\xi_0}^{+\infty} \frac{(b_{x_0} + d)(x)}{g(x, R)} \mathrm{d}x\right).$$

Remark that for $\xi_0 \in \Omega_R$, if $\sigma_{x_0,R,\xi_0} = 0$, then $\sigma_{x_0,R,x} = 0$ for every $x \in \Omega_R$, by continuity of b_{x_0}, d and $g(., R)$. Also, if $\xi_0 \notin \Omega_R$, recall that $R \notin \mathfrak{R}_0$, so $t_{\max}(\xi_0, R) = +\infty$ and $A_{\xi_0,R}(\tau)$ remains in a compact for $\tau \geq 0$, hence $\sigma_{x_0,R,\xi_0} = 0$. This implies that for every $\xi_0 \in \Omega_R$, $(\sigma_{x_0,R,\xi_0} = 0 \Leftrightarrow \sigma_{x_0,R,.} \equiv 0)$. Thus, for $\xi_0 \in \Omega_R$, $(\mathbb{P}_{x_0,R,\xi_0}(T_d = +\infty) = 1)$ is equivalent to $(\sigma_{x_0,R,\xi_0} = 0$ and $\lim_{k\to+\infty} K^k_{x_0,R}\mathbf{1}(\xi_0) = 0)$. Assuming Assumption I.1.2 implies that $(\sigma_{x_0,R,\xi_0} = 0$ and $\lim_{k\to+\infty} K^k_{x_0,R}\mathbf{1}(\xi_0) = 0)$ for every $R \in \mathfrak{R}_\infty \setminus \mathfrak{R}_0$ and $\xi_0 \in \Omega_R$. Also, if $R \in \mathfrak{R}_\infty \setminus \mathfrak{R}_0$ and $\xi_0 \notin \Omega_R$, then $\mathbb{P}_{x_0,R,\xi_0}(T_d = +\infty) = 1$ by Lemma I.4.6, which ends the proof. □

Finally, we complete the proof of Theorem I.1.1.

Proposition I.4.11. *Under the general setting of Section I.1.1.1, and under Assumption I.1.1, we have*

$$\textit{Assumption I.1.2} \Leftrightarrow (\forall x_0 > 0, \forall R \in \mathfrak{R}_\infty, \forall \xi_0 \in \Omega_R, \quad \mathbb{P}_{x_0,R,\xi_0}(T_d < +\infty) = 1).$$

Also, under Assumption I.1.1, for every $x_0 > 0$, $R \in \mathfrak{R}_\infty$ *and* $\xi_0 \notin \Omega_R$, *we have* $\mathbb{P}_{x_0,R,\xi_0}(T_d < +\infty) = 1$.

Proof. We fix $R \in \mathfrak{R}_\infty$ in the following. By Assumption I.1.1 and Proposition I.4.3, we have almost surely $T_0 = +\infty$. We can work on this event and apply the exact same technique as in the proof of Proposition I.4.10 (in particular, we verify that we still have $\sigma_{x_0,R,\xi_0} = 0$ for every $\xi_0 \notin \Omega_R$) to prove that $(\mathbb{P}_{x_0,\xi_0,R}(T_d = +\infty) = 1)$ is equivalent to $(\sigma_{x_0,R,\xi_0} = 0$ and $\lim_{k\to+\infty} K^k_{x_0,R}\mathbf{1}(\xi_0) = 0)$ for every $R \in \mathfrak{R}_\infty$ and $\xi_0 \in \Omega_R$, which concludes for the equivalence. Finally, if $\xi_0 \notin \Omega_R$, it is as if we work with $R \notin \mathfrak{R}_\infty$, and under Assumption I.1.1, we can use the same techniques as in the proofs of Proposition I.4.4 or Lemma I.4.6 to obtain the second part of Proposition I.4.11. □

Remark: What is important for the equivalence in the previous proof is the use of Assumption I.1.1 first to ensure that $T_0 = +\infty$ almost surely, even if $R \in \mathfrak{R}_0$. Also, we need to consider $\xi_0 \in \Omega_R$ to perform the change of variables $x = A_{\xi_0,R}(\tau)$ in the proof of Proposition I.4.10, and to ensure $t_{\max}(\xi_0, R) = +\infty$. At this point, the reader can verify that the combination of Proposition I.4.4, Lemma I.4.6 and Proposition I.4.11 implies Theorem I.1.1.

I.4.6 Proof of 4. in Section I.4.2

A first immediate consequence of Assumption I.1.4 is the following, which entails **4.** in Section I.4.2.

Proposition I.4.12. *Under the general setting of Section I.1.1.1, Assumptions I.1.1, I.1.2 and I.1.4, we have*

$$\forall x > 0, \exists y \geq x, \quad \tilde{b}(y) > d(y). \tag{I.4.21}$$

Under the allometric setting of Section I.1.2, (I.4.21) is equivalent to

$$\beta > \delta \text{ or } (\beta = \delta,\ C_\beta > C_\delta).$$

Proof. Let us suppose that

$$\exists x > 0, \forall y \geq x, \quad \tilde{b}(y) \leq d(y).$$

Then, recall that $b_{x_0} \equiv 0$ on $]0, x_0]$, so for every $x_0 \geq x$,

$$\forall y > 0,\ \frac{b_{x_0}(y)}{b_{x_0}(y) + d(y)} \leq 1/2.$$

But in that case, no matter the choice of R and the instant at which the individual energy jumps, the probability of giving birth instead of dying will be lower than or equal to $1/2$. In other terms, we have

$$\forall x_0 \geq x, \forall R \geq 0,\ m_{x_0,R}(x_0) \leq 1,$$

which contradicts Assumption I.1.4, by Proposition I.1.7. The allometric consequence is straightforward. □

Remark: It is natural to compare the birth and death rates to conclude about the supercriticality of the process. Informally, Proposition I.4.12 says that the birth rate should dominate the death rate, at least in the high energy regime. It seems biologically natural to assume that a well-fed individual is more likely to give birth to offspring than to die.

An important difficulty in the sequel is to conclude for small x_0. It could be possible to have a lower birth rate than the death rate when the energy is low, but higher if the individual energy increases (as it is the case in the allometric setting of Section I.1.2 if $\beta > \delta$). Then, we have to control the probability for individuals to reach the favorable high energies, to know better about the expected number of offspring starting from x_0, which is $m_{x_0,R}(x_0)$. This is the main goal of the next subsections.

I.4.7 Proof of 5. in Section I.4.2

In this section, we first introduce a coupling between the processes $\xi_{\cdot,x_0,R,\xi_0}$ for different values of x_0 and/or ξ_0, in the same environment R. The main result of this section is Proposition I.4.16, valid under the general setting of Section I.1.1.1. Then, we apply the same techniques in the allometric setting of Section I.1.2 and the particular case $\delta = \alpha - 1$, to obtain **5.** in Section I.4.2.

I.4.7.1 Preliminary results

In the following, for $x>0$, we say that $R \in \mathcal{R}_x$ if for all $y \geq x$, $g(y,R)>0$ (note that $R \in \mathfrak{R}_\infty$, if and only if there exists $x>0$ such that $R \in \mathcal{R}_x$). We naturally extend this notation to $\mathcal{R}_0 := \{R \geq 0, \forall y>0, g(y,R)>0\}$ (be careful that $\mathcal{R}_0$ is distinct from the previous notation $\mathfrak{R}_0$). These sets are possibly equal to $\varnothing$ under the general setting I.1.1.1, and in that case most of our following results do not apply. However, the following lemma states that under the allometric setting of Section I.1.2 and Assumption I.1.8, $\mathcal{R}_0 \neq \varnothing$.

Lemma I.4.13. *Under the general setting of Section I.1.1.1, for $x>0$, if there exists $R>0$ such that $R \in \mathcal{R}_x$ then*

$$[R,+\infty[\subseteq \mathcal{R}_x.$$

In addition, under the allometric setting of Section I.1.2 and Assumption I.1.8, there exists $R_0>0$ such that

$$]R_0,+\infty[=\mathcal{R}_0.$$

Proof. The first point comes from the fact that $g(x,.)$ is increasing on $\mathbb{R}_+$ for any $x>0$. The allometric consequence is straightforward by Proposition I.4.1 and considering Equation (I.4.13). □

In the following, we will use the notation B_{x_0,R,ξ_0} from Section I.4.5. In this section, we present our results with the assumption that $d/\tilde{b}$ is non-increasing. In fact, under the allometric setting of Section I.1.2, Assumptions I.1.1, I.1.2 and I.1.4, this assumption is a natural consequence of Proposition I.4.12. The following lemma highlights a useful coupling between the processes $\xi_{.,x_0,R,\xi_0}$ for different values of x_0 and/or ξ_0 and the same well-chosen R.

Lemma I.4.14. *We work under the general setting of Section I.1.1.1. We suppose Assumptions I.1.1 and I.1.2, and that $d/\tilde{b}$ is non-increasing. Let $(x_0,\hat{x}_0,\xi_0,\hat{\xi}_0)>0$ and $R \in \mathcal{R}_{\xi_0 \wedge \hat{\xi}_0}$. Let $\xi_{.,x_0,R,\xi_0}$ and $\xi_{.,\hat{x}_0,R,\hat{\xi}_0}$ be individual trajectories with law described in Section I.4.1, respectively characterized by x_0 and $\hat{x}_0$, and respectively starting from ξ_0 and $\hat{\xi}_0$, in the same environment R. Then there exists a coupling of these random processes such that*

- *If $\max(x_0,\xi_0) \geq \max(\hat{x}_0,\hat{\xi}_0)$, then on the event $B_{x_0,R,\xi_0} \cap B_{\hat{x}_0,R,\hat{\xi}_0}$,*

$$\xi_{J_1^{\hat{x}_0,R,\hat{\xi}_0},\hat{x}_0,R,\hat{\xi}_0} + \hat{x}_0 \leq \xi_{J_1^{x_0,R,\xi_0},x_0,R,\xi_0} + x_0,$$

  where $J_1^{x_0,R,\xi_0}$ and $J_1^{\hat{x}_0,R,\hat{\xi}_0}$ denotes the first time of jump, respectively associated to $\xi_{.,x_0,R,\xi_0}$ and $\xi_{.,\hat{x}_0,R,\hat{\xi}_0}$.

- *If $\xi_0 \geq \hat{\xi}_0$ and $\hat{x}_0 \geq x_0$, then*

$$B_{\hat{x}_0,R,\hat{\xi}_0} \subseteq B_{x_0,R,\xi_0}.$$

Proof. The proof is divided in three steps.

Step 1: construction of a coupling

First, we couple $\xi_{.,x_0,R,\xi_0}$ and $\xi_{.,\hat{x}_0,R,\hat{\xi}_0}$, and more generally define simultaneously all the $\xi_{.,x,R,y}$ for $x > 0$ and $y > 0$. The reader can check that this new definition of the individual process and the one of Section I.4.1 will have same distribution of sample paths. Indeed, instead of constructing jump times at inhomogeneous exponential rate depending on $b_{x_0} + d$, we will separate it into inhomogeneous exponential rates depending on b_{x_0} and d separately, and the fact that the jump times have the same law comes from the usual property for the law of the minimum of independent exponential random variables. The following construction is very similar to the one given in Section I.4.1. We use it because it is convenient for our coupling purposes.

We pick i.i.d. random variables $(F_i)_{i\geq 0}$, following exponential laws with parameter 1, and define auxiliary processes $(\xi^{\text{aux}}_{t,x,R,y})_{t\geq 0}$, with random jumps occuring at the following times.

- First, if
$$\int_0^{t_{\max}(y,R)} b_x(A_{y,R}(s))\mathrm{d}s \leq F_1,$$
we set $J_1^{x,R,y} := +\infty$, $\xi^{\text{aux}}_{t,x,R,y} = A_{y,R}(t)$ for $t \in [0, t_{\max}(\xi_0, R)[$ and $\xi^{\text{aux}}_{t,x,R,y} = \flat$ for $t \geq t_{\max}(\xi_0, R)$. Otherwise, we define the first time of jump as
$$J_1^{x,R,y} := \inf\left\{t \in [0, t_{\max}(y,R)[, \int_0^t b_x(A_{y,R}(s))\mathrm{d}s = F_1\right\},$$
we set $\xi^{\text{aux}}_{t,x,R,y} = A_{y,R}(t)$ for $t \in [0, J_1^{x,R,y}[$, and
$$\xi^{\text{aux}}_{J_1^{x,R,y},x,R,y} = A_{y,R}(J_1^{x,R,y}) - x.$$

- Now, let us suppose that we defined $J_n^{x,R,y}$ for some $n \geq 1$. If $J_n^{x,R,y} = +\infty$, we simply set $J_{n+1}^{x,R,y} := +\infty$ and $\xi^{\text{aux}}_{t,x,R,y}$ is already defined for all $t \geq 0$, it already reached $\flat$. Else if
$$\int_{J_n^{x,R,y}}^{J_n^{x,R,y}+t_{\max}\left(\xi^{\text{aux}}_{J_n^{x,R,y},x,R,y}\right)} b_x\left(A_{\xi^{\text{aux}}_{J_n^{x,R,y},x,R,y}}(s - J_n^{x,R,y})\right)\mathrm{d}s \leq F_{n+1},$$
we also set $J_{n+1}^{x,R,y} := +\infty$, $\xi^{\text{aux}}_{t,x,R,y} = A_{\xi^{\text{aux}}_{J_n^{x,R,y},x,R,y}}(t - J_n^{x,R,y})$ for $t \in [J_n^{x,R,y}, J_n^{x,R,y} + t_{\max}\left(\xi^{\text{aux}}_{J_n^{x,R,y},x,R,y}\right)[$ and $\xi^{\text{aux}}_{t,x,R,y} = \flat$ for $t \geq J_n^{x,R,y} + t_{\max}\left(\xi^{\text{aux}}_{J_n^{x,R,y},x,R,y}\right)$. Otherwise, we can define the $(n+1)$-th time of jump as
$$J_{n+1}^{x,R,y} := \inf\left\{t \in \left[J_n^{x,R,y}, J_n^{x,R,y} + t_{\max}\left(\xi^{\text{aux}}_{J_n^{x,R,y},x,R,y}\right)\right),\right.$$
$$\left.\int_{J_n}^t b_x\left(A_{\xi^{\text{aux}}_{J_n^{x,R,y},x,R,y}}(s - J_n^{x,R,y})\right)\mathrm{d}s = F_{n+1}\right\},$$
we set $\xi^{\text{aux}}_{t,x,R,y} = A_{\xi^{\text{aux}}_{J_n^{x,R,y},x,R,y}}(t - J_n^{x,R,y})$ for $t \in [J_n^{x,R,y}, J_{n+1}^{x,R,y}[$, and
$$\xi^{\text{aux}}_{J_{n+1}^{x,R,y},x,R,y} = A_{\xi^{\text{aux}}_{J_n^{x,R,y},x,R,y}}(J_{n+1}^{x,R,y} - J_n^{x,R,y}) - x.$$
Notice that the jump rate of ξ^{aux} is b_x, so $\xi^{\text{aux}}_{J_{n+1}^{x,R,y},x,R,y} > 0$.

Then, we define the time of death:

$$T_{d,x,R,y} := \inf\left\{t \geq 0, \int_0^t d(\xi^{\text{aux}}_{s,x,R,y})\mathrm{d}s = F_0\right\},$$

and finally set $\xi_{t,x,R,y} := \xi^{\text{aux}}_{t,x,R,y}\mathbb{1}_{\{t<T_{d,x,R,y}\}} + \partial\mathbb{1}_{\{t\geq T_{d,x,R,y}\}}$ for all $t \geq 0$. Under Assumptions I.1.1 and I.1.2, this process is almost surely biologically relevant thanks to Theorem I.1.1.

Step 2: proof of the first point of the lemma.

Our coupling allows us to obtain immediately

$$\int_0^{J_1^{x_0,R,\xi_0}} b_{x_0}(A_{\xi_0}(s))\mathrm{d}s = F_1 = \int_0^{J_1^{\hat{x}_0,R,\hat{\xi}_0}} b_{\hat{x}_0}(A_{\hat{\xi}_0}(s))\mathrm{d}s.$$

Thanks to the choice of R, $g(x,R) > 0$ if $x \geq \xi_0 \wedge \hat{\xi}_0$, so we can use the change of variables $u = A_{\xi_0}(s)$, or $u = A_{\hat{\xi}_0}(s)$ and obtain

$$\int_{\xi_0}^{A_{\xi_0}(J_1^{x_0,R,\xi_0})} \frac{b_{x_0}(u)}{g(u,R)}\mathrm{d}u = \int_{\hat{\xi}_0}^{A_{\hat{\xi}_0}(J_1^{\hat{x}_0,R,\hat{\xi}_0})} \frac{b_{\hat{x}_0}(u)}{g(u,R)}\mathrm{d}u,$$

so

$$\int_{\max(x_0,\xi_0)}^{A_{\xi_0}(J_1^{x_0,R,\xi_0})} \frac{\tilde{b}(u)}{g(u,R)}\mathrm{d}u = \int_{\max(\hat{x}_0,\hat{\xi}_0)}^{A_{\hat{\xi}_0}(J_1^{\hat{x}_0,R,\hat{\xi}_0})} \frac{\tilde{b}(u)}{g(u,R)}\mathrm{d}u.$$

As $\max(x_0,\xi_0) \geq \max(\hat{x}_0,\hat{\xi}_0)$ and $\tilde{b}(.)/g(.,R)$ is positive on the integration intervals, this enforces $A_{\xi_0}(J_1^{x_0,R,\xi_0}) \geq A_{\hat{\xi}_0}(J_1^{\hat{x}_0,R,\hat{\xi}_0})$. Thus, on the event $B_{x_0,R,\xi_0} \cap B_{\hat{x}_0,R,\hat{\xi}_0}$, the first point follows from the above definition of $\xi_{J_1^{x_0,R,\xi_0},x_0,R,\xi_0}$ and $\xi_{J_1^{\hat{x}_0,R,\hat{\xi}_0},\hat{x}_0,R,\hat{\xi}_0}$.

Step 3: proof of the second point of the lemma.

Remark that if we fix $x > 0$ and consider $y \in [\xi_0 \wedge \hat{\xi}_0, +\infty[\mapsto \int_y^{A_y(J_1^{x,R,y})} \frac{b_x(u)}{g(u,R)}\mathrm{d}u$, then this function is constant, equal to F_1. One can check that this function is almost surely differentiable thanks to Theorem I.1.1, the choice of R, and the regularity of the flow, and the previous fact means that its derivative vanishes, so that for every $x > 0$ and $y \geq \xi_0 \wedge \hat{\xi}_0$:

$$\frac{\partial A_y(J_1^{x,R,y})}{\partial y} \times \frac{b_x(A_y(J_1^{x,R,y}))}{g(A_y(J_1^{x,R,y}),R)} - \frac{b_x(y)}{g(y,R)} = 0. \tag{I.4.22}$$

Now, for $x > 0$ and $y \geq \xi_0 \wedge \hat{\xi}_0$, the event $B_{x,R,y}$ translates into

$$\int_0^{J_1^{x,R,y}} d(A_y(s))\mathrm{d}s < F_0,$$

because $A_y(.)$ and $\xi_{.,x,R,y}$ coincide before $J_1^{x,R,y}$. With the change of variables $u = A_y(s)$, we obtain

$$\int_y^{A_y(J_1^{x,R,y})} \frac{d(u)}{g(u,R)}\mathrm{d}u < F_0.$$

Hence, to prove the second point, it suffices to establish that

$$\int_{\xi_0}^{A_{\xi_0}(J_1^{x_0,R,\xi_0})} \frac{d(u)}{g(u,R)}\mathrm{d}u \leq \int_{\hat{\xi}_0}^{A_{\hat{\xi}_0}(J_1^{\hat{x}_0,R,\hat{\xi}_0})} \frac{d(u)}{g(u,R)}\mathrm{d}u.$$

As $\hat{x}_0 \geq x_0$, we have $b_{x_0}(A_{\xi_0}(s)) \geq b_{\hat{x}_0}(A_{\xi_0}(s))$ for $s \geq 0$, so $J_1^{x_0,R,\xi_0} \leq J_1^{\hat{x}_0,R,\xi_0}$. The flow is increasing because $g(.,R)$ is positive so

$$A_{\xi_0}(J_1^{x_0,R,\xi_0}) \leq A_{\xi_0}(J_1^{\hat{x}_0,R,\xi_0}),$$

and $d(.)/g(.,R)$ is positive so it suffices to prove that

$$\int_{\xi_0}^{A_{\xi_0}(J_1^{\hat{x}_0,R,\xi_0})} \frac{d(u)}{g(u,R)}\mathrm{d}u \leq \int_{\hat{\xi}_0}^{A_{\hat{\xi}_0}(J_1^{\hat{x}_0,R,\hat{\xi}_0})} \frac{d(u)}{g(u,R)}\mathrm{d}u.$$

Finally, $\xi_0 \geq \hat{\xi}_0$, so it suffices to show that $y \in [\xi_0 \wedge \hat{\xi}_0, +\infty[\mapsto \displaystyle\int_y^{A_y(J_1^{\hat{x}_0,R,y})} \frac{d(u)}{g(u,R)}du$ is non-increasing. Again, this function is differentiable, so we prove that

$$\forall y \geq \xi_0 \wedge \hat{\xi}_0, \quad \frac{\partial A_y(J_1^{\hat{x}_0,R,y})}{\partial y} \times \frac{d(A_y(J_1^{\hat{x}_0,R,y}))}{g(A_y(J_1^{\hat{x}_0,R,y}),R)} - \frac{d(y)}{g(y,R)} \leq 0.$$

Equation (I.4.22) leads to

$$\frac{\partial A_y(J_1^{\hat{x}_0,R,y})}{\partial y} \times \frac{d(A_y(J_1^{\hat{x}_0,R,y}))}{g(A_y(J_1^{\hat{x}_0,R,y}),R)} - \frac{d(y)}{g(y,R)} = \frac{b_{\hat{x}_0}(y)}{g(y,R)} \times \frac{d(A_y(J_1^{\hat{x}_0,R,y}))}{b_{\hat{x}_0}(A_y(J_1^{\hat{x}_0,R,y}))} - \frac{d(y)}{g(y,R)}$$

This is equal to $\dfrac{\tilde{b}(y)}{g(y,R)}\left(\dfrac{d(A_y(J_1^{\hat{x}_0,R,y}))}{\tilde{b}(A_y(J_1^{\hat{x}_0,R,y}))} - \dfrac{d(y)}{\tilde{b}(y)}\right)$ if $y > \hat{x}_0$, and to $-\dfrac{d(y)}{g(y,R)}$ otherwise. In both cases, this quantity is non-positive because $d/\tilde{b}$ is non-increasing, $\tilde{b}(.)/g(.,R)$ and $d(.)/g(.,R)$ are non-negative, which ends the proof. □

Remark: The reader can check that the definition of the individual process ξ in the proof of Lemma I.4.14 and the one of Section I.4.1 have same distribution of sample paths, thanks to Lemma I.4.19 of Section I.4.8.

We can still define the number of offspring of an individual during its life as in (I.4.11):

$$N_{x_0,R,\xi_0} := \sup\{i \geq 0, J_i^{x_0,R,\xi_0} < T_{d,x_0,R,\xi_0}\},$$

and $m_{x_0,R}(\xi_0) := \mathbb{E}(N_{x_0,R,\xi_0})$. In the following, consistent with these notations, we write $m_{0,R}(\xi_0)$ for the expected number of birth jumps of an individual starting from ξ_0, but losing no energy when a birth occurs. Indeed, the previous coupling makes perfect sense for $x_0 = 0$ and we implicitly extend it to this remaining value in the following. We insist here on the fact that it makes sense to work with an individual process with $x_0 = 0$, but we cannot relate it to a population process anymore, because we do not want to see offspring with energy 0 appear in the population. We continue to use the expression "birth jump", but in the case $x_0 = 0$, the reader should be aware that this does not relate to any underlying branching process. Lemma I.4.14 still apply for $x_0 = 0$, and we use it to stochastically dominate $m_{\hat{x}_0,R}(.)$ by $m_{0,R}(.)$ for $\hat{x}_0 > 0$ in the upcoming lemma.

Lemma I.4.15. *We work under the general setting of Section I.1.1.1. We suppose Assumptions I.1.1 and I.1.2, and that $d/\tilde{b}$ is non-increasing. We fix $R \in \mathcal{R}_0$, and $(\hat{x}_0, \xi_0, \hat{\xi}_0)$, such that $\hat{x}_0 > 0$ and $\xi_0 \geq \max(\hat{x}_0, \hat{\xi}_0)$. Then, under the same coupling as in Lemma I.4.14, we have*

$$N_{0,R,\xi_0} \geq N_{\hat{x}_0,R,\hat{\xi}_0}.$$

Taking the expectation immediately leads to

$$m_{0,R}(\xi_0) \geq m_{\hat{x}_0,R}(\hat{\xi}_0).$$

Proof. The second point of Lemma I.4.14 allows us to assess that if an individual starting from $(\hat{x}_0, \hat{\xi}_0)$ has a child, then an individual starting from $(0, \xi_0)$ has one too. Moreover, the first point of Lemma I.4.14 shows that the new starting energies after the birth verify

$$\xi_{J_1^{\hat{x}_0,R,\hat{\xi}_0},\hat{x}_0,R,\hat{\xi}_0} + \hat{x}_0 \leq \xi_{J_1^{0,R,\xi_0},0,R,\xi_0}.$$

This implies that

$$\xi_{J_1^{0,R,\xi_0},0,R,\xi_0} \geq \max(\hat{x}_0, \xi_{J_1^{\hat{x}_0,R,\hat{\xi}_0},\hat{x}_0,R,\hat{\xi}_0}),$$

so the vector $(\hat{x}_0, \xi_{J_1^{0,R,\xi_0},0,R,\xi_0}, \xi_{J_1^{\hat{x}_0,R,\hat{\xi}_0},\hat{x}_0,R,\hat{\xi}_0})$ satisfies the assumptions of Lemma I.4.15. Thanks to Markov property, we can apply again the previous reasoning and obtain the same result for the second time of birth. This leads by induction to the fact that for every $i \geq 1$, if an individual starting from $(\hat{x}_0, \hat{\xi}_0)$ has i children, then an individual starting from $(0, \xi_0)$ has at least i children. This gives the wanted conclusion. □

Finally, we use this lemma to prove the following proposition.

Proposition I.4.16. *We work under the general setting of Section I.1.1.1, with Assumptions I.1.1, I.1.2 and I.1.4. Also, we assume that $d/\tilde{b}$ is non-increasing, and that $R \in \mathcal{R}_0$ satisfies that $d/g(.,R)$ has a divergent integral near $+\infty$. Then, we necessarily have*

$$\forall x > 0, \qquad \int_x^{+\infty} \left(\int_x^u \frac{\tilde{b}(s)}{g(s,R)} \mathrm{d}s \right) \frac{d(u)}{g(u,R)} \mathrm{d}u = +\infty. \tag{I.4.23}$$

Under the allometric setting of Section I.1.2, with Assumptions I.1.1, I.1.2 and I.1.4, Equation (I.4.23) implies that if $\delta \geq \alpha - 1$, then

$$\beta + \delta \geq 2(\alpha - 1).$$

Proof. We take $x > 0$ and $R \in \mathcal{R}_0$. Lemma I.4.15 gives in particular that

$$m_{x,R}(x) \leq m_{0,R}(x). \tag{I.4.24}$$

Without any loss of energy when a birth occurs, we can compute $m_{0,R}(x)$. The resource R has been chosen so that

$$\int_x^{+\infty} \frac{d(y)}{g(y,R)} \mathrm{d}y = +\infty,$$

so from the coupling of Lemma I.4.14 with $x_0 = 0$, we have $T_d < +\infty$ almost surely. We know the law of T_d when $x_0 = 0$, it is an inhomogeneous exponential law with rate depending on d and the flow $A_x(.)$. Conditionally to the time of death T_d, the number of jumps before T_d is then Poisson with parameter $\int_0^{T_d} \tilde{b}(A_x(w))\mathrm{d}w$. Hence:

$$m_{0,R}(x) = \int_0^{+\infty} \left(\int_0^t \tilde{b}(A_x(w))\mathrm{d}w \right) d(A_x(t)) e^{-\int_0^t d(A_x(\tau))\mathrm{d}\tau} \mathrm{d}u. \tag{I.4.25}$$

From the change of variables $s = A_x(w)$, then $u = A_x(t)$, we obtain

$$\begin{aligned} m_{0,R}(x) = \int_x^{+\infty} \left(\int_x^u \frac{\tilde{b}(s)}{g(s,R)} \mathrm{d}s \right) \frac{d(u)}{g(u,R)} e^{-\displaystyle\int_x^u \frac{d(\tau)}{g(\tau,R)}\mathrm{d}\tau} \mathrm{d}u \\ \leq \int_x^{+\infty} \left(\int_x^u \frac{\tilde{b}(s)}{g(s,R)} \mathrm{d}s \right) \frac{d(u)}{g(u,R)} \mathrm{d}u. \end{aligned} \tag{I.4.26}$$

Let us suppose by contradiction that the right-most integral above is finite for some $x > 0$, then

$$\int_y^{+\infty} \left(\int_x^u \frac{\tilde{b}(s)}{g(s,R)} \mathrm{d}s \right) \frac{d(u)}{g(u,R)} \mathrm{d}u \xrightarrow[y \to +\infty]{} 0, \tag{I.4.27}$$

as the remainder of a convergent integral. For $y > x$, as $\tilde{b}(.)/g(.,R)$ is positive for our choice of R, we also have

$$\int_y^{+\infty} \left(\int_y^u \frac{\tilde{b}(s)}{g(s,R)} \mathrm{d}s \right) \frac{d(u)}{g(u,R)} \mathrm{d}u \leq \int_y^{+\infty} \left(\int_x^u \frac{\tilde{b}(s)}{g(s,R)} \mathrm{d}s \right) \frac{d(u)}{g(u,R)} \mathrm{d}u. \tag{I.4.28}$$

From (I.4.24), (I.4.26), (I.4.28) and finally (I.4.27), we would get

$$m_{y,R}(y) \leq 1$$

for y high enough. One can check that we still get this result for this specific y, if we take a higher R because $g(y,.)$ is non-decreasing. This contradicts Assumption I.1.4 by Proposition I.1.7 and ends the proof.

Under the allometric setting of Section I.1.2 and Assumptions I.1.1, I.1.2 and I.1.4, the assumption $\delta \geq \alpha - 1$ ensures that the integral of $d/g(.,R)$ diverges near $+\infty$, because with point **1.** of Section I.4.2, we have $\gamma = \alpha$. Then, it follows from computations that if $\beta + \delta < 2(\alpha - 1)$, then (I.4.23) does not hold, considering that with Proposition I.4.12, we have $\beta \geq \delta$. □

Remark: This result gives a first insight on the possible regimes where $m_{x_0,R}(x_0) > 1$. First, we can be in the case $\beta = \delta = \alpha - 1$, and in that case $C_\beta > C_\delta$ according to Proposition I.4.12, so that the birth rate is higher than the death rate for every energy. Otherwise, still according to Proposition I.4.12, we have to be in the case $\beta > \delta$, so small individuals are more likely to die. However, there is a small probability that they reach high energies, and in that case, they could give birth to a lot of offspring, which could balance the high death rate of these newborns. The allometric coefficient on the birth rate should then be sufficiently high compared to the death rate, which is expressed in Proposition I.4.16.

I.4.7.2 Proof of 5. in Section I.4.2

Under the allometric setting of Section I.1.2 and if $\delta = \alpha - 1$, we can be more precise in our computations and extend the result of Proposition I.4.16. We begin with an intermediate result in the case $\beta = \delta = \alpha - 1$.

Lemma I.4.17. *Under the allometric setting of Section I.1.2 and Assumption I.1.8, if $\beta = \delta = \alpha - 1$, we have that $\mathcal{R}_0 \neq \varnothing$ and*

$$\forall R \in \mathcal{R}_0, \forall x > 0, \quad m_{0,R}(x) = C_\beta / C_\delta.$$

Proof. Thanks to Lemma I.4.13, we introduce R_0 such that $]R_0, +\infty[= \mathcal{R}_0$ and we work with $R > R_0$ in the following. Thanks to Proposition I.4.1, we have $\beta = \delta = \alpha - 1 = \gamma - 1$ and $C_\gamma > C_\alpha$. The result then follows from the equality in (I.4.26) and a straightforward computation. □

We are now ready to prove Proposition I.4.18, which entails point **5.** in Section I.4.2. Notice that under the allometric setting of Section I.1.2, if $\delta = \alpha - 1$, then Assumption I.1.1 is verified by point **2.** in Section I.4.2. Also, with Assumptions I.1.1, I.1.2 and I.1.4, $d/\tilde{b}$ is non-increasing by Proposition I.4.12, and $d/g(., R)$ has a divergent integral near $+\infty$ by Proposition I.4.1.

Proposition I.4.18. *Under the allometric setting of Section I.1.2 with $\delta = \alpha - 1$, under Assumptions I.1.2 and I.1.4, we have*

$$(\beta = \alpha - 1 \text{ and } C_\beta > C_\delta) \;\text{ or }\; \left(\beta \geq \alpha - 1 + \frac{C_\delta}{C_\gamma - C_\alpha}\right).$$

Proof. Thanks to Lemma I.1.9, we work under Assumption I.1.8. Thanks to Lemma I.4.13, we introduce R_0 such that $]R_0, +\infty[= \mathcal{R}_0$ and we work with $R > R_0$ and $x > 0$ in the following. First, thanks to Proposition I.4.12, we necessarily have $\beta \geq \alpha - 1$. If $\beta = \alpha - 1$, we obtain with (I.4.24) and Lemma I.4.17 that $m_{x,R}(x) \leq m_{0,R}(x) = C_\beta / C_\delta$, so $C_\beta > C_\delta$ is a necessary condition to verify Assumption I.1.4.
Now, let us suppose by contradiction that $\alpha - 1 < \beta < \alpha - 1 + \dfrac{C_\delta}{C_\gamma - C_\alpha}$. We recall that $C_R = \phi(R) C_\gamma - C_\alpha \leq C_\gamma - C_\alpha$ and we obtain from (I.4.26):

$$m_{0,R}(x) = \frac{C_\beta C_\delta}{C_R^2} \int_x^{+\infty} \left(\frac{u^{\beta-\alpha+1} - x^{\beta-\alpha+1}}{\beta - \alpha + 1} \right) \frac{1}{u} \left(\frac{x}{u} \right)^{C_\delta / C_R} \mathrm{d}u,$$

so according to (I.4.24), we have

$$m_{x,R}(x) \leq \frac{C_\beta C_\delta}{C_R^2 (\beta - \alpha + 1)} x^{C_\delta / C_R} \int_x^{+\infty} u^{\beta - \alpha - (C_\delta / C_R)} \mathrm{d}u.$$

From the definition of C_R, $\beta - \alpha - \dfrac{C_\delta}{C_R} \leq \beta - \alpha - \dfrac{C_\delta}{C_\gamma - C_\alpha} < -1$, so that the integral above converges and we finally get

$$m_{x,R}(x) \leq \frac{C_\beta C_\delta}{C_R^2 (\beta - \alpha + 1)(\alpha + \dfrac{C_\delta}{C_R} - \beta - 1)} x^{\beta - \alpha + 1}.$$

We can pick R_1 high enough so that $C_R \geq \dfrac{C_\gamma - C_\alpha}{2}$ for $R > R_1$, because $\phi(R) \underset{R\to+\infty}{\longrightarrow} 1$, and we obtain for $R > R_1$:

$$m_{x,R}(x) \leq \frac{4C_\beta C_\delta}{(C_\gamma - C_\alpha)^2(\beta - \alpha + 1)(\alpha + \dfrac{C_\delta}{C_\gamma - C_\alpha} - \beta - 1)} x^{\beta-\alpha+1}.$$

Finally, we realize that if x is close enough to 0, $m_{x,R}(x) \leq 1$ for any $R > R_1$. This contradicts Assumption I.1.4 by Proposition I.1.7 and concludes. □

I.4.8 Proof of 6. and 7. of Section I.4.2

In this section, we work under the allometric setting of Section I.1.2, with Assumptions I.1.1, I.1.2 and I.1.4. The result of point **7.** of Section I.4.2 in the case $\alpha - 1 < \beta \leq \alpha$ follows from Section I.4.7.2 (see the proof of Corollary I.4.24) and the fact that Assumption I.1.1 is verified if $\delta = \alpha - 1$ (point **2.** in Section I.4.2). Then, in this section, except in Corollary I.4.24, we work with $\beta > \alpha$ and prove points **6.** and **7.** of Section I.4.2. Recall that for $k \geq 1$, M^k is the event $\{N_{x_0,\xi_0} \geq k\}$ (*i.e.* an individual starting from ξ_0 with characteristic energy x_0 and resources R has at least k direct offspring during its life). Under Assumption I.1.5, we can write

$$m_{x_0,R}(\xi_0) = \mathbb{E}(N_{x_0,R,\xi_0}) = \sum_{k\geq 1} \mathbb{P}_{x_0,R,\xi_0}(M^k).$$

We will study the general term of this sum to obtain points **6.** and **7.** of Section I.4.2. In the following proof, we assume by contradiction that $\beta > \alpha$ and $\delta < \alpha - 1$. We will contradict Assumption I.1.4, *i.e.* obtain a subcritical or critical process. We highlight two energy regimes: the low-energy regime and the high-energy regime. The contribution to the expected number of offspring in the low-energy regime goes to 0 when x_0 goes to 0, thanks to the assumption $\delta < \alpha - 1$. For the high-energy regime, it is the assumption $\beta > \alpha$ that is important to obtain the same convergence. A biological interpretation is the following: in the low-energy regime, what prevents individuals to reproduce is that the death rate dominates the energy dynamics (which is expressed by $\delta < \alpha - 1$) so that individuals are way too likely to die; in the high-energy regime, what is problematic for reproduction is that the birth rate dominates the energy dynamics ($\beta > \alpha$), so that individuals lose energy too frequently and get back to the low-energy regime.

By Lemma I.1.9, we work under Assumption I.1.8, so Lemma I.4.13 holds true. We fix $R > R_0$ once and for all and also work with some $x_0 > 0$, which will eventually converge to 0. Thanks to Assumptions I.1.1 and I.1.2, by Theorem I.1.1, our process is almost surely biologically relevant. Recall that this means that ξ almost surely never reaches $\flat$ and dies in finite time.

In Section I.4.8.1, we will use results for the construction of Lemma I.4.14, with $\xi_0 > 0$. In Section I.4.8.2, we will use results for the construction of Section I.4.1. Thus, we prove now a technical result that allows us to use both constructions, started from the same (x_0, R, ξ_0). In the following, for $k \geq 1$, we write J_k for the k-th jump time associated to ξ^{aux} defined in the coupling of Lemma I.4.14, and $\hat{J}_k$ for the k-th jump time associated to ξ^{aux} defined in Section I.4.1, T_d and $\hat{T}_d$ for the associated death time, M^k or $\hat{M}^k$ for the associated events "the k-th first jumps are birth jumps", and N or $\hat{N}$ for the total number of birth jumps during the trajectory. Considering the auxiliary process ξ^{aux} in the coupling of Lemma I.4.14, we write S^k, for the maximal value reached by ξ^{aux} before J_k. In our setting, ξ^{aux} almost surely avoids $\flat$, and the energy is increasing outside birth jumps. S^k is then clearly the maximal energy reached at a jump time before J_k (*i.e.* $S^k := \sup_{t<J_k} \xi_t^{\mathrm{aux}} = \max_{1\leq i\leq k} \xi_{J_i-}^{\mathrm{aux}}$). In the same manner, considering the auxiliary process ξ^{aux} from the construction of Section I.4.1, we write $\hat{S}^k$, for the maximal value reached by ξ^{aux} before $\hat{J}_k$. We have to be more careful here, because ξ^{aux} can jump to ∂ in the construction of Section I.4.1 and some $\hat{J}_i$ can be equal to $+\infty$. Thus, we set $\xi_{+\infty}^{\mathrm{aux}} = \partial$ and $\partial < x$ for every $x > 0$. With these conventions, $\hat{S}^k$ is again the maximal energy reached at a jump time before $\hat{J}_k$ (*i.e.* $\hat{S}^k := \sup_{t<\hat{J}_k} \xi_t^{\mathrm{aux}} = \max_{1\leq i\leq k} \xi_{\hat{J}_i-}^{\mathrm{aux}}$).

Lemma I.4.19. *Under the general setting of Section I.1.1.1, under Assumptions I.1.1 and I.1.2, for every $k \geq 1$, J^k conditionally to M^k and $\hat{J}^k$ conditionally to $\hat{M}^k$ have the same law. It follows immediately from the definition of S^k and $\hat{S}^k$ that S^k conditionally to M^k and $\hat{S}^k$ conditionally to $\hat{M}^k$ have the same law.*

Proof. We prove the result by induction on k. Thanks to Theorem I.1.1, the process is almost surely biologically relevant, so the process avoid 0 and $+\infty$ and dies in finite time almost surely. In this setting, on the event $\hat{M}^1$, $\hat{J}_1 \neq +\infty$, and the law of $\hat{J}_1$ is an inhomogeneous exponential law given in (I.4.10), with rate depending on $b_{x_0} + d$. On the event M^1, $J_1 = J_1 \wedge T_d$, and from the construction of Lemma I.4.14, the law of J_1, respectively T_d, is an inhomogeneous exponential law with the same form as in (I.4.10), but replacing $b_{x_0} + d$ with b_{x_0}, respectively d. Furthermore, J_1 and T_d are independent. Hence, the law of $J_1 \wedge T_d$ is an inhomogeneous exponential law whose rate is the sum of those of J_1 and T_d. This entails that J_1 given M^1 and $\hat{J}_1$ given $\hat{M}^1$ have the same law.
Now, suppose that the result is valid for some $k \geq 1$. By Markov property, on the event M^{k+1}, respectively $\hat{M}^{k+1}$, we can apply again this reasoning starting from time J_k, respectively $\hat{J}_k$ (which is finite on the event $\hat{M}^k$), hence the result. □

For $j \geq 0$, we write $\heartsuit_j := x_0^{\frac{3}{\alpha-\beta}} + j$. We choose x_0 smaller than 1 and small enough to have $\heartsuit_0 \geq 2x_0$ (this is possible because $3/(\alpha - \beta) < 0$). Thanks to Corollary I.4.9, we work under Assumption I.1.5 and we have

$$\begin{aligned} m_{x_0,R}(x_0) &= \sum_{k\geq 1} \mathbb{P}_{x_0,R,x_0}(M^k) \\ &= \sum_{k\geq 1} \mathbb{P}_{x_0,R,x_0}(M^k \cap \{S^k \leq \heartsuit_0\}) + \sum_{k\geq 1}\sum_{j\geq 0} \mathbb{P}_{x_0,R,x_0}(M^k \cap \{\heartsuit_j < S^k \leq \heartsuit_{j+1}\}) \\ &= \sum_{k\geq 1} \mathbb{P}_{x_0,R,x_0}(M^k \cap \{S^k \leq \heartsuit_0\}) + \sum_{k\geq 1}\sum_{j\geq 0} \mathbb{P}_{x_0,R,x_0}(\hat{M}^k \cap \{\heartsuit_j < \hat{S}^k \leq \heartsuit_{j+1}\}). \end{aligned} \quad \text{(I.4.29)}$$

We have partitioned our events M^k depending on the maximal energy S^k and used Lemma I.4.19 for the third line. In the following, we seek for a contradiction with Assumption I.1.4. Our proof is divided in two parts, corresponding to the study of the two sums in the right-hand side above. The left-most simple sum accounts for the low-energy regime, where individuals are more likely to die fast than to give birth to many offspring. The double-sum represents the high-energy regime, and we will show that the probability to reach such a level of energy is decreasing to 0 when x_0 goes to 0, if $\beta > \alpha$. Thus, we want to show that the previously mentioned sums both converge to 0 when x_0 goes to 0. If this happens, thanks to the previous decomposition, we would have $m_{x_0,R}(x_0) < 1$

for x_0 small enough, which contradicts Assumption I.1.4 by Proposition I.1.7, and proves point **6.** of Section I.4.2. In the following, we refer to all the previously described setting as 'the setting of Section I.4.8'.

I.4.8.1 Low-energy regime

We study the sum

$$\sum_{k\geq 1} \mathbb{P}_{x_0,x_0}(M^k \cap \{S^k \leq \heartsuit_0\}).$$

We use the construction of Lemma I.4.14 and define another coupling between our initial individual process and a continuum of similar processes $(\zeta_t^x)_{t\geq 0}$ for $x > 0$. Precisely, for $x > 0$, we define $b_{x_0}^x : u \mapsto b_{x_0}(u)\mathbb{1}_{u\leq x} + b_{x_0}(x)\mathbb{1}_{u>x}$, and similarly $d^x : u \mapsto d(u)\mathbb{1}_{u\leq x} + d(x)\mathbb{1}_{u>x}$. These are simply the functions b_{x_0} and d frozen after x. We define the process $(\zeta_t^x)_{t\geq 0}$ in the exact same manner as $(\xi_t)_{t\geq 0}$, using the same random variables $(F_i)_{i\geq 0}$ and starting from ξ_0 at time 0, but simply replacing ξ, b_{x_0} and d by respectively ζ^x, $b_{x_0}^x$ and d^x in the procedure of Lemma I.4.14. This is indeed a coupling between ξ and ζ^x, because we use the same random variables to define the jump times. However, we insist on the fact that we possibly define different jump times $(J_i^x)_{i\geq 1}$ for ζ^x, even if we still use the same exponential variables as in the definition of the $(J_i)_{i\geq 1}$.

We naturally write N_{x_0,R,ξ_0}^x for the number of births occuring for the process ζ^x. In the same manner as in Section I.4.7, we also remark that it is possible to formally define ζ^x for $x_0 = 0$ and to compare it to the other processes. The previous coupling allows us to state the following lemma.

Lemma I.4.20. *Under the setting of Section I.4.8, for $R > R_0$, $x > 0$, $x_0 > 0$, $\xi_0 > 0$, $k \geq 1$,*

$$\{N_{x_0,R,\xi_0}\mathbb{1}_{S^k\leq x} \geq k\} \subseteq \{N_{x_0,R,\xi_0}^x \geq k\} \subseteq \{N_{0,R,\xi_0}^x \geq k\}.$$

Proof. On the event $\{S^k \leq x\}$, since b_{x_0} and d coincide respectively with $b_{x_0}^x$ and d^x before x, ξ and ζ^x also coincide until time J_k. Indeed, they both start from ξ_0 and as soon as the energy does not exceed x, the birth and death rates and the energy dynamics are the same. Hence, on the event $\{S^k \leq x\}$, if the k first jumps are births for ξ, they also are births for ζ^x. This justifies the left-most inclusion of the lemma.

Now, remark that in the same manner as in Section I.4.7, we can compare $\zeta_{x_0}^x$ and ζ_0^x and obtain the same conclusion for the N^x as for N in Lemma I.4.15. The only thing that differs between ξ and ζ^x is the shape of the birth and death rates, but they still verify the necessary condition highlighted in Lemma I.4.15, that is $d^x/b_{x_0}^x$ is non-increasing, since $\delta < \alpha < \beta$. One can check that the exact same reasoning applies and we obtain

$$N_{0,R,\xi_0}^x \geq N_{x_0,R,\xi_0}^x,$$

which concludes for the right-most inclusion. □

Applying Lemma I.4.20 leads to

$$\begin{aligned}\sum_{k\geq 1}\mathbb{P}_{x_0,R,x_0}(M^k\cap\{S^k\leq \heartsuit_0\}) &= \sum_{k\geq 1}\mathbb{P}(N_{x_0,R,x_0}\mathbb{1}_{S^k\leq\heartsuit_0}\geq k)\\ &\leq \sum_{k\geq 1}\mathbb{P}(N^{\heartsuit_0}_{0,R,x_0}\geq k)\\ &= \mathbb{E}(N^{\heartsuit_0}_{0,R,x_0}).\end{aligned}$$

Remark that we did not used any information about $\heartsuit_0$, so in fact we have

$$\forall x_0>0,\forall y>0,\quad \sum_{k\geq 1}\mathbb{P}_{x_0,R,x_0}(M^k\cap\{S^k\leq y\})\leq \mathbb{E}(N^{y}_{0,R,x_0}). \tag{I.4.30}$$

In the same manner as in the proof of Proposition I.4.16, we can compute this expectation because there is no energy loss anymore so it is easier to understand the trajectory of $\zeta^{\heartsuit_0}_{.,0,R,x_0}$. Since $x_0<\heartsuit_0$, there exists $t_\heartsuit>0$ such that $A_{x_0}(t_\heartsuit)=\heartsuit_0$. We obtain from Equation (I.4.25) applied to the rates $b^{\heartsuit_0}_{x_0}$ and $d^{\heartsuit_0}$:

$$\begin{aligned}\mathbb{E}(N^{\heartsuit_0}_{0,R,x_0}) &= \int_0^{+\infty}\left(\int_0^t b^{\heartsuit_0}_{x_0}(A_{x_0}(w))\mathrm{d}w\right)d^{\heartsuit_0}(A_{x_0}(t))e^{-\int_0^t d^{\heartsuit_0}(A_{x_0}(\tau))\mathrm{d}\tau}\mathrm{d}t\\ &= \int_0^{t_\heartsuit}\left(\int_0^t b_{x_0}(A_{x_0}(w))\mathrm{d}w\right)d(A_{x_0}(t))e^{-\int_0^t d(A_{x_0}(\tau))\mathrm{d}\tau}\mathrm{d}t\\ &+\int_{t_\heartsuit}^{+\infty}\left(\left(\int_0^{t_\heartsuit} b_{x_0}(A_{x_0}(w))\mathrm{d}w\right)+(t-t_\heartsuit)b_{x_0}(\heartsuit_0)\right)\\ &\qquad\qquad d(\heartsuit_0)e^{-\int_0^{t_\heartsuit} d(A_{x_0}(\tau))\mathrm{d}\tau}e^{-(t-t_\heartsuit)d(\heartsuit_0)}\mathrm{d}t.\end{aligned}$$

Now, we use the fact that $\int_{t_\heartsuit}^{+\infty}(t-t_\heartsuit)e^{-(t-t_\heartsuit)d(\heartsuit_0)}\mathrm{d}t=d(\heartsuit_0)^{-2}$. We also use the definition of $d^{\heartsuit_0}$ and $b^{\heartsuit_0}_{x_0}$, and the change of variables $u=A_{x_0}(t)$ to obtain

$$\begin{aligned}\mathbb{E}(N^{\heartsuit_0}_{0,R,x_0}) &= \int_{x_0}^{\heartsuit_0}\left(\int_{x_0}^u\frac{\tilde{b}}{g}(w)\mathrm{d}w\right)\frac{d}{g}(u)e^{-\int_{x_0}^u\frac{d}{g}(\tau)\mathrm{d}\tau}\mathrm{d}u\\ &+\left(\left(\int_{x_0}^{\heartsuit_0}\frac{\tilde{b}}{g}(w)\mathrm{d}w\right)+\frac{b_{x_0}}{d}(\heartsuit_0)\right)\exp\left(-\int_{x_0}^{\heartsuit_0}\frac{d}{g}(\tau)\mathrm{d}\tau\right),\end{aligned} \tag{I.4.31}$$

where we write g instead of $g(.,R)$ for the sake of simplicity. Also, from Proposition I.4.1, as soon as $\beta>\alpha$, $\tilde{b}/g$ is increasing. Hence, we compute the following upper bound:

$$\begin{aligned}\mathbb{E}(N^{\heartsuit_0}_{0,R,x_0}) &\leq x_0\frac{\tilde{b}}{g}(2x_0)\int_{x_0}^{2x_0}\frac{d}{g}(u)e^{-\int_{x_0}^u\frac{d}{g}(\tau)\mathrm{d}\tau}\mathrm{d}u+\heartsuit_0\frac{\tilde{b}}{g}(\heartsuit_0)\int_{2x_0}^{\heartsuit_0}\frac{d}{g}(u)e^{-\int_{x_0}^u\frac{d}{g}(\tau)\mathrm{d}\tau}\mathrm{d}u\\ &+\left(\heartsuit_0\frac{\tilde{b}}{g}(\heartsuit_0)+\frac{\tilde{b}}{d}(\heartsuit_0)\right)e^{-\int_{x_0}^{\heartsuit_0}\frac{d}{g}(\tau)\mathrm{d}\tau}\\ &\leq x_0\frac{\tilde{b}}{g}(2x_0)+\heartsuit_0\frac{\tilde{b}}{g}(\heartsuit_0)e^{-\int_{x_0}^{2x_0}\frac{d}{g}(\tau)\mathrm{d}\tau}+\left(\heartsuit_0\frac{\tilde{b}}{g}(\heartsuit_0)+\frac{\tilde{b}}{d}(\heartsuit_0)\right)e^{-\int_{x_0}^{\heartsuit_0}\frac{d}{g}(\tau)\mathrm{d}\tau}.\end{aligned}$$

At this point, one can check that all the terms in the upper bound converge to 0 when x_0 goes to 0, under the allometric setting of Section I.1.2 and with $\beta > \alpha > \delta+1$. Indeed, the exponential factors with general term d/g dominate all the other quantities near 0 (they both are of order $\exp(-cx_0^{\delta-\alpha+1})$ with some constant c, the other terms are powers of x_0), and $\frac{\tilde{b}}{g}(2x_0)$ converges to 0 when x_0 goes to 0.

I.4.8.2 High-energy regime

Now, we deal with the double sum:

$$\sum_{k\geq 1}\sum_{j\geq 0}\mathbb{P}_{x_0,R,x_0}(\hat{M}^k \cap \{\heartsuit_j < \hat{S}^k \leq \heartsuit_{j+1}\}).$$

We begin with another coupling argument. Recall that in this section, we use the construction of Section I.4.1. For $x > 0$, we define

$$\tilde{N}_x := \sup\{i \geq 1, \forall 1 \leq j \leq i, \quad U_j \leq r(x)\},$$

with the convention $\sup(\varnothing) := 0$ and $r := \frac{b_{x_0}}{b_{x_0}+d}$. Remark that $\tilde{N}_x + 1$ follows simply a geometric law with parameter $1 - r(x)$, but it is coupled to $\hat{N}_{x_0,R,\xi_0}$ because it depends on the same $(U_i)_{i\geq 1}$. This coupling via the only variables $(U_i)_{i\geq 1}$ allows us to assess two things.

Lemma I.4.21. *Under the setting of Section I.4.8, we have:*

1. *For $x, x_0, \xi_0 > 0$, $k \geq 1$, $\mathbb{P}_{x_0,R,\xi_0}(\hat{N}_{x_0,R,\xi_0}\mathbb{1}_{\hat{S}^k\leq x} \geq k) \leq \mathbb{P}_{x_0,R,\xi_0}(\tilde{N}_x \geq k)$.*
2. *$\tilde{N}_x$ is independent from $\hat{S}^k$.*

Proof. 1. On the event $\{\hat{S}^k \leq x\}$, since d/b_{x_0} is non-increasing, r is non-decreasing, so we can assess that for every $t \leq \hat{J}_k \wedge \hat{T}_d$, $1 - r(\xi_t^{\text{aux}}) \geq 1 - r(x)$. But on this event, if $\hat{N}_{x_0,R,\xi_0} \geq k$ (*i.e.* on the event $\hat{M}^k$), then $\hat{J}_k < \hat{T}_d$, and for every $i \leq k$, $U_i > 1 - r(\xi^{\text{aux}}_{\hat{J}_i-}) \geq 1 - r(x)$. Hence,

$$\begin{aligned}\mathbb{P}_{x_0,R,\xi_0}(\hat{N}_{x_0,R,\xi_0}\mathbb{1}_{\hat{S}^k\leq x} \geq k) &\leq \mathbb{P}_{x_0,R,\xi_0}(\forall 1 \leq i \leq k, U_i > 1 - r(x))\\ &= \mathbb{P}_{x_0,R,\xi_0}(\tilde{N}_x \geq k).\end{aligned}$$

2. $\tilde{N}_x$ is independent from $(E_i)_{i\geq 1}$, so independent from the $(\hat{J}_i)_{i\geq 1}$, hence independent from $\hat{S}^k$.

□

This last independence property is crucial for the third line of the following computation. We take $k \geq 1$ and $j \geq 0$, and by Lemma I.4.21:

$$\begin{aligned}\mathbb{P}_{x_0,R,x_0}(\hat{M}^k \cap \{\heartsuit_j < \hat{S}^k \leq \heartsuit_{j+1}\}) &= \mathbb{P}_{x_0,R,x_0}(\{\hat{N}_{x_0,R,x_0}\mathbb{1}_{\hat{S}^k\leq\heartsuit_{j+1}} \geq k\} \cap \{\heartsuit_j < \hat{S}^k\})\\ &\leq \mathbb{P}_{x_0,R,x_0}(\{\tilde{N}_{\heartsuit_{j+1}} \geq k\} \cap \{\heartsuit_j < \hat{S}^k\})\\ &= \mathbb{P}_{x_0,R,x_0}(\tilde{N}_{\heartsuit_{j+1}} \geq k)\mathbb{P}_{x_0,R,x_0}(\heartsuit_j < \hat{S}^k).\end{aligned}$$

Finally, interverting the two sums leads to

$$\sum_{k\geq 1}\sum_{j\geq 0}\mathbb{P}_{x_0,R,x_0}(\hat{M}^k\cap\{\heartsuit_j<\hat{S}^k\leq\heartsuit_{j+1}\})\leq\sum_{j\geq 0}\sum_{k\geq 1}\mathbb{P}_{x_0,R,x_0}(\tilde{N}_{\heartsuit_{j+1}}\geq k)\mathbb{P}_{x_0,R,x_0}(\heartsuit_j<\hat{S}^k). \tag{I.4.32}$$

Let $j\geq 0$, $k\geq 1$, we have

$$\begin{aligned}\mathbb{P}_{x_0,R,x_0}(\heartsuit_j<\hat{S}^k)&=\mathbb{P}_{x_0,R,x_0}(\max_{1\leq i\leq k}\xi^{\text{aux}}_{\hat{J}_i-}>\heartsuit_j)\\&=\sum_{i=1}^{k}\mathbb{P}_{x_0,R,x_0}(\{\xi^{\text{aux}}_{\hat{J}_i-}>\heartsuit_j\}\cap\{\forall n\in[\![1,i-1]\!],\xi^{\text{aux}}_{\hat{J}_n-}\leq\heartsuit_j\}).\end{aligned}$$

For $0<x\leq y$, $i\geq 1$ we define

$$q_i(x,y):=\mathbb{P}_{x_0,R,x}(\{\xi^{\text{aux}}_{\hat{J}_i-}>y\}\cap\{\forall n\in[\![1,i-1]\!],\xi^{\text{aux}}_{\hat{J}_n-}\leq y\})$$

so that

$$\mathbb{P}_{x_0,R,x_0}(\heartsuit_j<\hat{S}^k)=\sum_{i=1}^{k}q_i(x_0,\heartsuit_j).$$

Remark that with our convention $\partial<x$ for all $x>0$, on the event $\{\xi^{\text{aux}}_{\hat{J}_i-}>y\}$, the i first jumps are necessarily birth events, for any $i\geq 1$.

Lemma I.4.22. *Under the setting of Section I.4.8, we have*

$$\forall i\geq 1,\forall y\geq x_0,\forall x\in(0,y-x_0],\quad q_i(x,y)\leq\exp\left(-\int_{y-x_0}^{y}\frac{b_{x_0}+d}{g}(u)\mathrm{d}u\right).$$

Proof. We prove this lemma by induction on i. First, we know the law of $\hat{J}_1$, so for $y>x_0$ and $x\in(0,y-x_0]$:

$$\begin{aligned}q_1(x,y)&=\mathbb{P}_{x_0,R,x}(\{\xi^{\text{aux}}_{\hat{J}_1-}>y\})\\&=\int_y^{+\infty}\frac{b_{x_0}+d}{g}(u)\exp\left(-\int_x^u\frac{b_{x_0}+d}{g}(\tau)\mathrm{d}\tau\right)\mathrm{d}u\\&=\exp\left(-\int_x^y\frac{b_{x_0}+d}{g}(u)\mathrm{d}u\right)\\&\leq\exp\left(-\int_{y-x_0}^{y}\frac{b_{x_0}+d}{g}(u)\mathrm{d}u\right),\end{aligned}$$

because $\int_{\cdot}^{+\infty}(b_{x_0}/d)/g=+\infty$ as soon as $\beta>\alpha$ under the setting of Section I.4.8. Then, for some $i\geq 1$, we suppose that the property is true for q_i, and we use Markov property to assess

$$\begin{aligned}q_{i+1}(x,y)&=\mathbb{P}_{x_0,R,x}(\{\xi^{\text{aux}}_{\hat{J}_1-}\leq y\}\cap\{\xi^{\text{aux}}_{\hat{J}_{i+1}-}>y\}\cap\{\forall n\in[\![2,i]\!],\xi^{\text{aux}}_{\hat{J}_n-}\leq y\})\\&=\int_x^y\frac{b_{x_0}}{g}(u)\exp\left(-\int_x^u\frac{b_{x_0}+d}{g}(\tau)\mathrm{d}\tau\right)q_i(u-x_0,y)\mathrm{d}u\\&\leq\exp\left(-\int_{y-x_0}^{y}\frac{b_{x_0}+d}{g}(u)\mathrm{d}u\right)\int_x^y\frac{b_{x_0}+d}{g}(u)\exp\left(-\int_x^u\frac{b_{x_0}+d}{g}(\tau)\mathrm{d}\tau\right)\mathrm{d}u\\&\leq\exp\left(-\int_{y-x_0}^{y}\frac{b_{x_0}+d}{g}(u)\mathrm{d}u\right),\end{aligned}$$

where the third line comes from the induction hypothesis. □

This finally gives, because $2x_0 \le \heartsuit_j - x_0$,

$$\mathbb{P}_{x_0,R,x_0}(\heartsuit_j < \hat{S}^k) \le k \exp\left(-\int_{\heartsuit_j - x_0}^{\heartsuit_j} \frac{b_{x_0}+d}{g}(u)\mathrm{d}u\right).$$

Hence, as $\tilde{N}_{\heartsuit_{j+1}} + 1$ follows a geometric law with parameter $1 - r(\heartsuit_{j+1})$, we obtain from (I.4.32) that

$$\sum_{k\ge 1}\sum_{j\ge 0} \mathbb{P}_{x_0,R,x_0}(\hat{M}^k \cap \{\heartsuit_j < \hat{S}^k \le \heartsuit_{j+1}\})$$
$$\le \sum_{j\ge 0}\left(\sum_{k\ge 1} kr(\heartsuit_{j+1})^k\right)\exp\left(-\int_{\heartsuit_j - x_0}^{\heartsuit_j} \frac{b_{x_0}+d}{g}(u)\mathrm{d}u\right)$$
$$\le \sum_{j\ge 0}\frac{r(\heartsuit_{j+1})}{(1-r(\heartsuit_{j+1}))^2}\exp\left(-\int_{\heartsuit_j - x_0}^{\heartsuit_j} \frac{b_{x_0}}{g}(u)\mathrm{d}u\right)$$
$$= \sum_{j\ge 0}\frac{b_{x_0}^2 + b_{x_0}d}{d^2}(\heartsuit_{j+1})\exp\left(-\int_{\heartsuit_j - x_0}^{\heartsuit_j} \frac{b_{x_0}}{g}(u)\mathrm{d}u\right).$$

By assumption, $\beta > \alpha$, so that b_{x_0}/g is increasing, which leads to the first inequality in the following. Also, the reader can check that we can pick x_0 small enough so that for all $j \ge 0$, $(\heartsuit_j - x_0)^{\beta-\alpha} \ge \heartsuit_j^{\beta-\alpha}/2$, which entails the second line in the following.

$$\sum_{k\ge 1}\sum_{j\ge 0} \mathbb{P}_{x_0,R,x_0}(\hat{M}^k \cap \{\heartsuit_j < \hat{S}^k \le \heartsuit_{j+1}\})$$
$$\le \sum_{j\ge 0}\frac{b_{x_0}^2 + b_{x_0}d}{d^2}(\heartsuit_{j+1})\exp\left(-\frac{C_\beta}{C_R}x_0(\heartsuit_j - x_0)^{\beta-\alpha}\right)$$
$$\le \sum_{j\ge 0}\frac{b_{x_0}^2 + b_{x_0}d}{d^2}(\heartsuit_{j+1})\exp\left(-\frac{C_\beta}{2C_R}x_0\heartsuit_j^{\beta-\alpha}\right).$$

The term for $j = 0$, which we denote in the following as ς_0, converges to 0 when x_0 goes to 0, because it is a power of x_0 multiplied by an exponential term of order $\exp(-x_0^{-2})$. Now for $j \ge 1$, recall that $\heartsuit_j := x_0^{\frac{3}{\alpha-\beta}} + j$. Hence, because $\beta > \alpha$,

$$\begin{aligned} x_0\heartsuit_j^{\beta-\alpha} &= x_0(x_0^{\frac{3}{\alpha-\beta}} + j)^{\beta-\alpha} \\ &= x_0(x_0^{\frac{3}{\alpha-\beta}} + j)^{(\beta-\alpha)/3}(x_0^{\frac{3}{\alpha-\beta}} + j)^{(\beta-\alpha)/3}(x_0^{\frac{3}{\alpha-\beta}} + j)^{(\beta-\alpha)/3} \\ &\ge x_0\frac{1}{x_0}\frac{1}{x_0}j^{(\beta-\alpha)/3}. \end{aligned} \tag{I.4.33}$$

Thus, we have

$$\sum_{k\ge 1}\sum_{j\ge 0} \mathbb{P}_{x_0,R,x_0}(\hat{M}^k \cap \{\heartsuit_j < \hat{S}^k \le \heartsuit_{j+1}\})$$
$$\le \varsigma_0 + \sum_{j\ge 1}\frac{b_{x_0}^2 + b_{x_0}d}{d^2}(\heartsuit_{j+1})\exp\left(-\frac{C_\beta}{2C_R}\frac{1}{x_0}j^{\frac{\beta-\alpha}{3}}\right).$$

The general term of the sum in the right-hand side simply converges to 0 when x_0 goes to 0. Also, for $x_0 \leq 1$ and $j \geq 1$, we have

$$\exp\left(-\frac{C_\beta}{2C_R}\frac{1}{x_0}j^{\frac{\beta-\alpha}{3}}\right) = e^{-\frac{C_\beta}{4C_R}\frac{1}{x_0}j^{\frac{\beta-\alpha}{3}}} e^{-\frac{C_\beta}{4C_R}\frac{1}{x_0}j^{\frac{\beta-\alpha}{3}}} \leq e^{-\frac{C_\beta}{C_R}\frac{1}{4x_0}} e^{-\frac{C_\beta}{4C_R}j^{\frac{\beta-\alpha}{3}}}$$

We let the reader check that, uniformly on $x_0 \leq 1$, the term $\frac{b_{x_0}^2 + b_{x_0}d}{d^2}(\heartsuit_{j+1})e^{-\frac{C_\beta}{C_R}\frac{1}{4x_0}}$ is dominated by $P(j)$ with some polynomial function P that does not depend on x_0 (this comes essentially from the exponential term, that allows us to get rid of the dependence on x_0 by rough bounds).

This entails that our general term is dominated by $P(j)e^{-\frac{C_\beta}{4C_R}j^{\frac{\beta-\alpha}{3}}}$, which is the general term of a converging sum as $\beta > \alpha$. Hence, a dominated convergence argument allow us to conclude: the double sum in the left-hand side above goes to 0 when x_0 goes to 0.

I.4.8.3 Proof of 7. in Section I.4.2

In this section, we precise what happens for the previous decomposition if $\delta = \alpha - 1$. Precisely, we prove the following result.

Proposition I.4.23. *Under the allometric setting of Section I.1.2, Assumptions I.1.2 and I.1.4, we have*

$$(\beta > \alpha \text{ and } \delta = \alpha - 1) \Rightarrow (C_\delta \leq C_\gamma - C_\alpha).$$

Proof. We work with $R > R_0$ and $x_0 > 0$. Let us suppose by contradiction that $\beta > \alpha$, $\delta = \alpha - 1$ and $C_\delta > C_\gamma - C_\alpha$. We can still use the decomposition of (I.4.29). We begin by the low-energy regime. The upper bound of Equation (I.4.30) is still valid for $y = \heartsuit_0$, so our goal is to prove that $\mathbb{E}(N_{0,R,x_0}^{\heartsuit_0})$ goes to 0 when x_0 goes to 0. Equation (I.4.31) is still valid here, but we can be more subtle in the computations because $\delta = \alpha - 1$. Also, we use Proposition I.4.18 (Assumption I.1.1 is verified if $\delta = \alpha - 1$) to consider only the case $\beta \geq \alpha - 1 + C_\delta/(C_\gamma - C_\alpha)$. Recall also that $C_R = \phi(R)(C_\gamma - C_\alpha) \leq C_\gamma - C_\alpha$. First if $\beta > \alpha - 1 + C_\delta/(C_\gamma - C_\alpha)$, we get

$$\begin{aligned}
\mathbb{E}(N_{0,R,x_0}^{\heartsuit_0}) &= \int_{x_0}^{\heartsuit_0}\left(\int_{x_0}^{u}\frac{C_\beta}{C_R}w^{\beta-\alpha}\mathrm{d}w\right)\frac{C_\delta}{C_R}\frac{1}{u}\left(\frac{x_0}{u}\right)^{C_\delta/C_R}\mathrm{d}u \\
&+ \left(\left(\int_{x_0}^{\heartsuit_0}\frac{C_\beta}{C_R}w^{\beta-\alpha}\mathrm{d}w\right) + \frac{C_\beta}{C_\delta}\heartsuit_0^{\beta-\alpha+1}\right)\left(\frac{x_0}{\heartsuit_0}\right)^{C_\delta/C_R} \\
&\leq \mathfrak{C}x_0^{\frac{C_\delta}{C_R}}\heartsuit_0^{\beta-\alpha+1-\frac{C_\delta}{C_R}},
\end{aligned}$$

with some constant $\mathfrak{C}$ that does not depend on x_0. Finally, recall that $\heartsuit_0 := x_0^{\frac{3}{\alpha-\beta}}$. The reader can check that all of our previous reasonings still hold true if $\heartsuit_0$ is of the form $x_0^{\frac{\omega}{\alpha-\beta}}$ with $\omega > 1$, and we take such a $\heartsuit_0$ until the end of this proof. The previous upper bound then leads to

$$\mathbb{E}(N_{0,R,x_0}^{\heartsuit_0}) \leq \mathfrak{C}x_0^{\frac{C_\delta}{C_R}\left(1+\frac{\omega}{\beta-\alpha}\right)-\omega\left(1+\frac{1}{\beta-\alpha}\right)}.$$

We supposed that $C_\delta > C_\gamma - C_\alpha \geq C_R$. Hence, we can choose ω close enough to 1 to obtain that for all $R > R_0$, $\mathbb{E}(N_{0,R,x_0}^{\heartsuit_0})$ converges to 0 when x_0 goes to 0. If now $\beta = \alpha - 1 + C_\delta/(C_\gamma - C_\alpha)$, and $C_R < C_\gamma - C_\alpha$, the same reasoning applies. Otherwise, if

$C_R = C_\gamma - C_\alpha$, we obtain an upper bound of the form $\mathfrak{C} x_0^{C_\delta/C_R} \ln(x_0)$, which still converges to 0 when x_0 goes to 0.

The reasoning in the high-energy regime is exactly the same, since the crucial hypothesis in this case is $\beta > \alpha$ to use dominated convergence. However, we have to check that our new definition for $\heartsuit_0$ with $\omega > 1$ is still consistent with our reasoning. Precisely, we have to replace the lower bound obtained in (I.4.33) by

$$\begin{aligned} x_0 \heartsuit_j^{\beta-\alpha} &= x_0 (x_0^{\frac{\omega}{\alpha-\beta}} + j)^{\beta-\alpha} \\ &= x_0 (x_0^{\frac{\omega}{\alpha-\beta}} + j)^{(\beta-\alpha)/\omega} (x_0^{\frac{\omega}{\alpha-\beta}} + j)^{\frac{\beta-\alpha}{2}(1-\frac{1}{\omega})} (x_0^{\frac{\omega}{\alpha-\beta}} + j)^{\frac{\beta-\alpha}{2}(1-\frac{1}{\omega})} \\ &\geq x_0 \frac{1}{x_0} x_0^{\frac{1-\omega}{2}} j^{\frac{\beta-\alpha}{2}(1-\frac{1}{\omega})}, \end{aligned}$$

and the reader can check that all our other arguments hold true with these new exponents (what is important is that $1-\omega < 0$ and $1 - \frac{1}{\omega} > 0$). □

Finally, we highlight the following corollary, which completes the proof of point **7.** in Section I.4.2.

Corollary I.4.24. *Under the allometric setting of Section I.1.2, Assumptions I.1.1, I.1.2 and I.1.4, we have that if $\delta = \alpha - 1$,*

$$(\beta > \alpha - 1) \Rightarrow (C_\delta \leq C_\gamma - C_\alpha).$$

Proof. First, if $\beta \leq \alpha$ and $C_\delta > C_\gamma - C_\alpha$, then Proposition I.4.18 implies $\beta = \alpha - 1$, hence $\beta \in]\alpha-1, \alpha]$ implies $C_\delta \leq C_\gamma - C_\alpha$. Then if $\beta > \alpha$, we conclude with Proposition I.4.23. □

I.4.9 Proof of 8. in Section I.4.2

First, in Section I.4.9.1, we work under the general setting of Section I.1.1.1, under Assumptions I.1.1 and I.1.2, and we suppose in addition that $t \mapsto A_\xi(t)$ is well-defined on $\mathbb{R}^+$ for every $\xi > 0$ to obtain martingale properties for the individual process. Then, in Sections I.4.9.2, I.4.9.3 and I.4.9.4, we work under the allometric setting of Section I.1.2 with $\alpha \leq 1$, Assumptions I.1.2 and I.1.8 and $R \in \mathcal{R}_0$. For these last sections, Assumption I.1.8 ensures that Lemma I.4.13 holds true, so $\mathcal{R}_0 \neq \varnothing$. Remark that we do not need to suppose Assumption I.1.1 anymore, because if $R \in \mathcal{R}_0$, then $T_0 = +\infty$. Getting back to the notations of Section I.4.1, we obtain that, if it is not ∂ or $\flat$, $\xi_t^{\text{aux}} \leq A_{\xi_0}(t)$ for $t \geq 0$ (this is due to jumps for reproduction). We will show that when $\beta \leq \alpha$, there is a positive probability for ξ_t^{aux} to be asymptotically lower bounded by $(1-\varepsilon)A_{\xi_0}(t)$, for some $0 < \varepsilon < 1$. If in addition $\delta < \alpha - 1$, the death rate decreases too fast and there is a chance that the full individual process $\xi_t \notin \{\partial, \flat\}$ for every $t \geq 0$, so that an individual never dies. This would contradict Assumption I.1.2 thanks to Proposition I.4.10 and our choice of R.

First, we will give a third way to construct our process ξ thanks to a Poisson point process. Then, we will extract from this new definition some useful martingale properties. Finally, we were inspired by the concept of asymptotic pseudotrajectory developed by Benaïm and Hirsch [Ben99] to compare our trajectories to the integral curves associated to a deterministic flow. We fully exploit this concept in Section I.4.9.5, where we prove a stronger result in the case $\beta < \alpha < 1$, that is that there is a positive probability for ξ_t^{aux} to be asymptotically equivalent to $A_{\xi_0}(t)$.

I.4.9.1 Another construction of the individual process $(\xi_t)_{t\geq 0}$

We give here another way to construct $(\xi_t)_{t\geq 0}$, similar to the coupling of Lemma I.4.14, but using a Poisson point process for the construction of the jump events. For this construction, we fix $\xi_0 > 0$, $x_0 > 0$ and $R > 0$. We also suppose Assumptions I.1.1 and I.1.2, so almost surely $T_d < +\infty$ by Theorem I.1.1. In other terms, we work in this section with almost surely biologically relevant processes, so we ignore the value $\flat$ in the following construction, and the processes are well-defined for $t \geq 0$.

First, we define an auxiliary process $(X_t)_{t\geq 0}$ following all the wanted mechanisms except for deaths (so $(X_t)_{t\geq 0}$ takes its values in $\mathbb{R}_+^*$). Then, we kill this process to obtain ξ.

1. Let $X_0 = \xi_0$ be the initial energy.
2. Let $\mathcal{N}(\mathrm{d}s, \mathrm{d}h)$ be a Poisson point measure on $\mathbb{R}^+ \times \mathbb{R}^+$, with intensity $\mathrm{d}s \times \mathrm{d}h$. We write $\mathcal{N}_C(\mathrm{d}s, \mathrm{d}h) := \mathcal{N}(\mathrm{d}s, \mathrm{d}h) - \mathrm{d}s\mathrm{d}h$ for the associated compensated Poisson point measure. We define the process $(X_t)_{t\geq 0}$ by
$$X_t := A_{\xi_0}(t) + \int_0^t \int_{\mathbb{R}^+} \mathbb{1}_{\{h \leq b_{x_0}(X_{s-})\}} \Big(A_{X_{s-}-x_0}(t-s) - A_{X_{s-}}(t-s) \Big) \mathcal{N}(\mathrm{d}s, \mathrm{d}h). \tag{I.4.34}$$
3. We define E a random variable following an exponential law with parameter 1, independent from $\mathcal{N}$. We then stop the process X at the random time
$$T_d := \inf \left\{ t \geq 0, \int_0^t d(X_s)\mathrm{d}s = E \right\}, \tag{I.4.35}$$
and define $\xi_t := X_t \mathbb{1}_{t \leq T_d} + \partial \mathbb{1}_{t > T_d}$.

This construction using a Poisson point process is standard [FM04, CFM08]. Let $(J_n)_{n\geq 1}$ be the jump times for the process X, and $\mathcal{F}_t$ the canonical filtration associated to $\mathcal{N}$. Classical properties of Poisson point measures show that for $n \geq 1$ and given $\mathcal{F}_{J_n}$, the next jump time J_{n+1} is distributed as in the construction of Lemma I.4.14. This shows that X and ξ^{aux} of Lemma I.4.14 have same distribution of sample paths. Hence, this new definition of ξ is consistent with the constructions of Section I.4.1 and Lemma I.4.14.

We prove martingale properties for our process. For this purpose, we suppose in addition to Assumptions I.1.1 and I.1.2, that $t \mapsto A_\xi(t)$ is well-defined on $\mathbb{R}^+$ for every $\xi > 0$. In the following, if F is a $\mathcal{C}^{1,1}$ function from $\mathbb{R}^+ \times \mathbb{R}_+^*$ to $\mathbb{R}$, we write $\partial_1 F$ and $\partial_2 F$ the partial derivative according to the first and second variable, respectively. The quadratic variation of the process $(X_t)_{t\geq 0}$ given in the following proposition is a predictable quadratic variation (see Theorem 4.2. in [JS+87]).

Proposition I.4.25. *Let $F : (t, x) \mapsto F(t, x)$ be $\mathcal{C}^{1,1}$ and $\xi_0 > 0$. Under the general setting of Section I.1.1.1, under Assumptions I.1.1 and I.1.2, if $t \mapsto A_\xi(t)$ is well-defined on $\mathbb{R}^+$ for every $\xi > 0$ and $R \geq 0$, for all $t \geq 0$, we have:*
$$\begin{aligned} F(t, X_t) = F(0, \xi_0) &+ \int_0^t \Big(\partial_1 F(s, X_s) + \partial_2 F(s, X_s) g(X_s, R) \Big) \mathrm{d}s \\ &+ \int_0^t b_{x_0}(X_s) \Big(F(s, X_s - x_0) - F(s, X_s) \Big) \mathrm{d}s + M_{F,t}, \end{aligned}$$

where

$$M_{F,t} := \int_0^t \int_{\mathbb{R}^+} \mathbb{1}_{\{h \le b_{x_0}(X_{s-})\}} \Big(F(s, X_{s-} - x_0) - F(s, X_{s-}) \Big) \mathcal{N}_C(\mathrm{d}s, \mathrm{d}h)$$

is a martingale with predictable quadratic variation

$$\langle M_F \rangle_t := \int_0^t b_{x_0}(X_s) \Big(F(s, X_s - x_0) - F(s, X_s) \Big)^2 \mathrm{d}s.$$

The proof of Proposition I.4.25 can be found in Appendix A.2.1. The reader can deduce from this result that the process $(t, X_t)_{t\geq 0}$ is a solution to the martingale problem for the operator L, where for every $\mathcal{C}^{1,1}$ function F, $t \geq 0$, $x > 0$,

$$LF := (t, x) \mapsto \partial_1 F(t, x) + \partial_2 F(t, x) g(x, R) + b_{x_0}(x) \Big(F(x - x_0) - F(x) \Big).$$

We will design a relevant renormalization and time shift Z of X that verifies a similar martingale problem. Then, we will control the martingale part of Z. First, we want to compare X to the evolution of energy without any birth event, so we define

$$Y_t := \frac{X_t}{A_{\xi_0}(t)}.$$

The process $(Y_t)_{t\geq 0}$ takes values in $[0, 1]$.

Remark: The rescaling of X_t by $A_{\xi_0}(t)$ is not appropriate under the allometric setting of Section I.1.2 in the case $\alpha > 1$, precisely because $A_{\xi_0}(.)$ explodes in finite time, so Y_t would not be well-defined for all $t \geq 0$. This is one reason why we cannot adapt immediately the reasoning of this section for Theorem I.2.4.

Lemma I.4.26. *Let F be a $\mathcal{C}^1$ function $\mathbb{R}^+ \to \mathbb{R}$ and $\xi_0 > 0$. Under the general setting of Section I.1.1.1, under Assumptions I.1.1 and I.1.2, if $t \mapsto A_\xi(t)$ is well-defined on $\mathbb{R}^+$ for every $\xi > 0$ and $R \geq 0$, for all $t \geq 0$, we have:*

$$F(Y_t) = F(1) + \int_0^t \frac{F'(Y_s)}{A_{\xi_0}(s)} \Big(g(A_{\xi_0}(s) Y_s, R) - g(A_{\xi_0}(s), R) Y_s \Big) \mathrm{d}s$$
$$+ \int_0^t \int_{\mathbb{R}^+} \mathbb{1}_{\{h \le b_{x_0}(A_{\xi_0}(s) Y_{s-})\}} \left(F\left(Y_{s-} - \frac{x_0}{A_{\xi_0}(s)} \right) - F(Y_{s-}) \right) \mathcal{N}(\mathrm{d}s, \mathrm{d}h).$$

Proof. This simply comes from Proposition I.4.25 applied to $(t, x) \mapsto F\left(\frac{x}{A_{\xi_0}(t)}\right)$. □

I.4.9.2 Martingale properties under the allometric setting of Section I.1.2

In the following sections, we work under the allometric setting of Section I.1.2 with $\alpha \leq 1$, Assumptions I.1.2 and I.1.8, and $R \in \mathcal{R}_0$ ($\mathcal{R}_0 \neq \varnothing$ thanks to Lemma I.4.13). Remark that this implies that $t \mapsto A_\xi(t)$ is well-defined on $\mathbb{R}^+$ for every $\xi > 0$ by Proposition I.4.1 and (I.4.13). Also, $R \in \mathcal{R}_0$ so $R \notin \mathfrak{R}_0$, and we do not need to add Assumption I.1.1 for the process $(X_t)_{t\geq 0}$ of Section I.4.9.1 to be biologically relevant. Thus, for this special choice of R, we will freely use results from Section I.4.9.1, where Assumption I.1.1 was necessary only to ensure that $T_0 = +\infty$. We consider the following change of time scale:

$$Z_t := Y_{\pi(t)},$$

with $\pi'(t) = \dfrac{A_{\xi_0}(\pi(t))}{g(A_{\xi_0}(\pi(t)), R)} > 0$ and $\pi(0) = 0$ (π' is well-defined thanks to the choice of R). In our setting, $g(x, R) = C_R x^\alpha$ as in Equation (I.4.13), thanks to Proposition I.4.1. We also have the precise expression:

$$\forall(\xi_0, t) \in (\mathbb{R}^+)^2, \quad A_{\xi_0}(t) = \begin{cases} \left((1-\alpha)C_R t + \xi_0^{1-\alpha}\right)^{1/1-\alpha} & \text{if } \alpha < 1, \\ \xi_0 e^{C_R t} & \text{if } \alpha = 1. \end{cases} \tag{I.4.36}$$

This leads to $\pi'(t) = A_{\xi_0}(\pi(t))^{1-\alpha}/C_R$, so

$$\pi'(t) = \begin{cases} (1-\alpha)\pi(t) + \xi_0^{1-\alpha}/C_R & \text{if } \alpha < 1, \\ 1/C_R & \text{if } \alpha = 1. \end{cases} \tag{I.4.37}$$

Hence

$$\pi : t \mapsto \begin{cases} \dfrac{\xi_0^{1-\alpha}}{C_R(1-\alpha)}(e^{(1-\alpha)t} - 1) & \text{if } \alpha < 1, \\ t/C_R & \text{if } \alpha = 1. \end{cases} \tag{I.4.38}$$

In both cases, we obtain $A_{\xi_0}(\pi(u)) = \xi_0 e^u$. Also, π is a deterministic bijection from $\mathbb{R}^+$ to $\mathbb{R}^+$. To understand better the behavior of Z, we define another Poisson point measure adapted to our modified time scale. Let $(s_i, h_i)_{i\in\mathbb{N}}$ be an enumeration of the random points of $\mathcal{N}$ such that

$$\mathcal{N} = \sum_{i\in\mathbb{N}} \delta_{s_i, h_i},$$

we define

$$\widehat{\mathcal{N}} = \sum_{i\in\mathbb{N}} \delta_{\pi^{-1}(s_i), h_i\pi'(\pi^{-1}(s_i))}.$$

Lemma I.4.27. *$\widehat{\mathcal{N}}$ is a Poisson point measure, with intensity* $\mathrm{d}s\mathrm{d}h$. *Hence, $\mathcal{N}$ and $\widehat{\mathcal{N}}$ have the same law.*

Proof. From the definition of $\widehat{\mathcal{N}}$, for any A measurable set of $\mathbb{R}^+ \times \mathbb{R}^+$, we have $\widehat{\mathcal{N}}(A) = \mathcal{N}(\hat{A})$, where

$$\hat{A} := \{(s, h),\ (\pi^{-1}(s), h\pi'(\pi^{-1}(s))) \in A\}.$$

Thanks to the fact that π is a bijection, $\widehat{\mathcal{N}}$ is a Poisson point measure with some intensity measure λ on $\mathbb{R}^2_+$. If we take now $A = [s_1, s_2] \times [h_1, h_2]$, we obtain

$$\hat{A} = \left\{\{s\} \times \left[\frac{h_1}{\pi'(\pi^{-1}(s))}, \frac{h_2}{\pi'(\pi^{-1}(s))}\right],\ \pi(s_1) \le s \le \pi(s_2)\right\},$$

because π is positive increasing. Hence,

$$\lambda(A) = \mathrm{Leb}(\hat{A}) = \int_{\pi(s_1)}^{\pi(s_2)} \int_{h_1/\pi'(\pi^{-1}(s))}^{h_2/\pi'(\pi^{-1}(s))} \mathrm{d}h\mathrm{d}s = \int_{s_1}^{s_2}\int_{h_1}^{h_2} \mathrm{d}h\mathrm{d}u,$$

by the change of variables $u = \pi^{-1}(s)$, where Leb denotes the Lebesgue measure on $\mathbb{R}^+$. $\mathcal{N}$ and $\widehat{\mathcal{N}}$ thus have same intensity. □

In the following, we write $\widehat{\mathcal{N}}_C$ for the compensated Poisson point measure associated to $\widehat{\mathcal{N}}$.

Lemma I.4.28. *Let F be a $\mathcal{C}^1$ function $\mathbb{R}^+ \to \mathbb{R}$ and $\xi_0 > 0$. Under the general setting of Section I.1.1.1, under Assumptions I.1.1 and I.1.2, if $t \mapsto A_\xi(t)$ is well-defined on $\mathbb{R}^+$ for every $\xi > 0$ and $R \in \mathcal{R}_0$, for all $t \geq 0$, we have:*

$$F(Z_t) = F(1) + \int_0^t F'(Z_u)\left(\frac{g(A_{\xi_0}(\pi(u))Z_u, R)}{g(A_{\xi_0}(\pi(u)), R)} - Z_u\right)\mathrm{d}u + \int_0^t \pi'(u) b_{x_0}\left(A_{\xi_0}(\pi(u))Z_u\right)\left(F\left(Z_u - \frac{x_0}{A_{\xi_0}(\pi(u))}\right) - F(Z_u)\right)\mathrm{d}u + \mathcal{M}_{F,t},$$

where

$$\mathcal{M}_{F,t} := \int_0^t \int_{\mathbb{R}^+} \mathbb{1}_{\{h \leq \pi'(u) b_{x_0}\left(A_{\xi_0}(\pi(u))Z_{u-}\right)\}}\left(F\left(Z_{u-} - \frac{x_0}{A_{\xi_0}(\pi(u))}\right) - F(Z_{u-})\right)\widehat{\mathcal{N}}_C(\mathrm{d}u, \mathrm{d}h)$$

is a L^2-martingale with predictable quadratic variation

$$\langle \mathcal{M}_F \rangle_t := \int_0^t \pi'(u) b_{x_0}\left(A_{\xi_0}(\pi(u))Z_u\right)\left(F\left(Z_u - \frac{x_0}{A_{\xi_0}(\pi(u))}\right) - F(Z_u)\right)^2 \mathrm{d}u.$$

The proof of this lemma can be found in Appendix A.2.2. Under our setting, it translates into:

$$F(Z_t) = F(1) + \int_0^t F'(Z_u)\left(Z_u^\alpha - Z_u\right)\mathrm{d}u + \int_0^t \pi'(u) b_{x_0}\left(\xi_0 e^u Z_u\right)\left(F\left(Z_u - \frac{x_0}{\xi_0}e^{-u}\right) - F(Z_u)\right)\mathrm{d}u + \mathcal{M}_{F,t}.$$

I.4.9.3 Proof of point 8. in Section I.4.2 for $\beta < \alpha$

In this section, we study the case $\beta < \alpha$. We focus on the martingale part $\mathcal{M}_{\mathrm{Id}}$ of Z, where Id is the identity function on $\mathbb{R}^+$, and then deduce a useful control of this process. Lemma I.4.28 for $F = \mathrm{Id}$, along with Equations (I.4.37) and (I.4.38) gives

$$Z_t = 1 + \int_0^t \left(Z_u^\alpha - Z_u\right)\mathrm{d}u - \frac{x_0}{\xi_0}\int_0^t \frac{\xi_0^{1-\alpha}e^{(1-\alpha)u}}{C_R} b_{x_0}\left(\xi_0 e^u Z_u\right)e^{-u}\mathrm{d}u + \mathcal{M}_{\mathrm{Id},t}, \tag{I.4.39}$$

with

$$\mathcal{M}_{\mathrm{Id},t} = -\int_0^t \int_{\mathbb{R}^+} \mathbb{1}_{\left\{h \leq \frac{\xi_0^{1-\alpha}e^{(1-\alpha)u}}{C_R} b_{x_0}(\xi_0 e^u Z_{u-})\right\}} \frac{x_0}{\xi_0 e^u}\widehat{\mathcal{N}}_C(\mathrm{d}u, \mathrm{d}h)$$

and

$$\langle \mathcal{M}_{\mathrm{Id}} \rangle_t = \int_0^t \frac{\xi_0^{1-\alpha}e^{(1-\alpha)u}}{C_R} b_{x_0}\left(\xi_0 e^u Z_u\right)\left(\frac{x_0}{\xi_0 e^u}\right)^2 \mathrm{d}u. \tag{I.4.40}$$

Lemma I.4.29. *Let $\varepsilon > 0$, $x_0 > 0$, $R \in \mathcal{R}_0$. For all ξ_0 high enough, we have*

$$\mathbb{P}_{x_0,R,\xi_0}(\forall t,\ |\mathcal{M}_{\mathrm{Id},t}| < \varepsilon) > 0.$$

Proof. We take any (t, x_0, ξ_0). By definition of the predictable quadratic variation, $\mathcal{M}_{\mathrm{Id}}^2 - \langle \mathcal{M}_{\mathrm{Id}} \rangle$ is a local martingale. It is even a martingale, because of Lemma I.4.28, and $\mathbb{E}(\mathcal{M}_{\mathrm{Id},t}^2 - \langle \mathcal{M}_{\mathrm{Id}} \rangle_t) = \mathbb{E}(\mathcal{M}_{\mathrm{Id},0}^2 - \langle \mathcal{M}_{\mathrm{Id}} \rangle_0) = 0$. We use Doob's maximal inequality in its L^2 version (Corollary II.1.6 in [RY04]) to assess that

$$\mathbb{P}_{x_0,R,\xi_0}(\sup_{0 \le s \le t} |\mathcal{M}_{\mathrm{Id},s}| \ge \varepsilon) \le \frac{\mathbb{E}(\mathcal{M}_{\mathrm{Id},t}^2)}{\varepsilon^2} = \frac{\mathbb{E}(\langle \mathcal{M} \rangle_{\mathrm{Id},t})}{\varepsilon^2}. \tag{I.4.41}$$

Now, we consider (I.4.40). Remark that $0 \le Z_u \le 1$ (because this is the case for Y), and $b_{x_0}(x) = \mathbb{1}_{\{x > x_0\}} C_\beta x^\beta$. Hence, for $u \ge 0$, $b_{x_0}(\xi_0 e^u Z_u) \le C_\beta \xi_0^\beta e^{\beta u}$. Finally, we have a deterministic upper bound for $\langle \mathcal{M} \rangle_{\mathrm{Id},t}$, so from Equations (I.4.40) and (I.4.41), we get

$$\mathbb{P}_{x_0,R,\xi_0}(\sup_{0 \le s \le t} |\mathcal{M}_{\mathrm{Id},s}| \ge \varepsilon) \le \frac{C_\beta \xi_0^{\beta-\alpha-1} x_0^2}{C_R \varepsilon^2} \int_0^t e^{(\beta-\alpha-1)u} \mathrm{d}u.$$

Since $\beta < \alpha$, we finally obtain $\mathbb{P}_{x_0,R,\xi_0}(\sup_{0 \le s \le t} |\mathcal{M}_{\mathrm{Id},s}| \ge \varepsilon) \le \dfrac{C_\beta \xi_0^{\beta-\alpha-1} x_0^2}{C_R \varepsilon^2 (1 + \alpha - \beta)}$, which does not depend on t, and converges to 0 as ξ_0 goes to $+\infty$. Hence, for ξ_0 high enough,

$$\mathbb{P}_{x_0,R,\xi_0}(\exists t,\ |\mathcal{M}_{\mathrm{Id},t}| \ge \varepsilon) < 1.$$

□

This uniform control of the martingale part of Z allows us to prove the following proposition.

Proposition I.4.30. *Let $\varepsilon > 0$, $x_0 > 0$, $R \in \mathcal{R}_0$. For ξ_0 high enough, we have*

$$\mathbb{P}_{x_0,R,\xi_0}(\forall t,\ Z_t \ge 1 - \varepsilon) > 0.$$

Proof. We choose ξ_0 high enough so that the result of Lemma I.4.29 holds true for $\varepsilon/2$. Using (I.4.39), we obtain that on the event $\{\forall t,\ |\mathcal{M}_{\mathrm{Id},t}| < \varepsilon/2\}$,

$$\begin{aligned} Z_t &\ge 1 + \int_0^t (Z_u^\alpha - Z_u)\, \mathrm{d}u - \frac{C_\beta \xi_0^{\beta-\alpha} x_0}{C_R} \int_0^t e^{(\beta-\alpha)u} \mathrm{d}u + \mathcal{M}_{\mathrm{Id},t} \\ &\ge 1 - \varepsilon/2 - \frac{C_\beta \xi_0^{\beta-\alpha} x_0}{C_R(\alpha-\beta)} + \int_0^t (Z_u^\alpha - Z_u)\, \mathrm{d}u \\ &\ge 1 - \varepsilon/2 - \frac{C_\beta \xi_0^{\beta-\alpha} x_0}{C_R(\alpha-\beta)}, \end{aligned}$$

because $Z_u \le 1$ for all u. We simply choose ξ_0 higher if necessary to enforce $\dfrac{C_\beta \xi_0^{\beta-\alpha} x_0}{C_R(\alpha-\beta)} \le \varepsilon/2$. □

Now, we can try to evaluate what happens if we bring back death events. Let $\varepsilon > 0$, $R \in \mathcal{R}_0$ and $x_0 > 0$, thanks to Proposition I.4.30, we choose ξ_0 high enough so that the event $\{\forall t,\ Z_t \ge 1 - \varepsilon\}$ occurs with positive probability, and we work on this event, which can be rewritten, as π is a bijection, as

$$\begin{cases} \left\{\forall t,\ X_t \ge (1-\varepsilon)\left(\xi_0^{1-\alpha} + C_R(1-\alpha)t\right)^{1/(1-\alpha)}\right\} & \text{if } \alpha < 1, \\ \left\{\forall t,\ X_t \ge (1-\varepsilon)\xi_0 e^{C_R t}\right\} & \text{if } \alpha = 1. \end{cases} \tag{I.4.42}$$

In order to prove point **8.** of Section I.4.2 for $\beta < \alpha$, we suppose by contradiction that $\delta < \alpha - 1 \leq 0$. Then, d is decreasing. In the following, c is a constant depending on ξ_0 and ε, possibly varying from line to line. We first assume $\alpha < 1$. Observe that on the event $\{\forall t,\ Z_t \geq 1-\varepsilon\}$, we get for $t \geq 0$,

$$\int_0^t d(X_s)\mathrm{d}s \leq \int_0^t d\left(c(1+s)^{1/(1-\alpha)}\right)\mathrm{d}s \leq \int_0^t c(1+s)^{\delta/(1-\alpha)}\mathrm{d}s.$$

As we assumed that $\delta/(1-\alpha) < -1$, the above integral converges when $t \to +\infty$, and there exists a constant, still denoted as c, such that $\int_0^t d(X_s)\mathrm{d}s < c$ with positive probability. One can check that the same result holds if $\alpha = 1$. Finally, from the definition of T_d in (I.4.35), on the event $\{\forall t,\ Z_t \geq 1-\varepsilon\}$ (so on the event $\{\int_0^t d(X_s)\mathrm{d}s < c\}$ by the previous reasoning), we have $\{c < E\} \subseteq \{T_d = +\infty\}$, so we obtain

$$\begin{aligned}\mathbb{P}_{x_0,R,\xi_0}(T_d = +\infty) &\geq \mathbb{P}_{x_0,R,\xi_0}(\{T_d = +\infty\} \cap \{\forall t,\ Z_t \geq 1-\varepsilon\}) \\ &\geq \mathbb{P}_{x_0,R,\xi_0}\left(\{c < E\} \cap \{\forall t,\ Z_t \geq 1-\varepsilon\}\right) \\ &= \mathbb{P}_{x_0,R,\xi_0}(c < E)\mathbb{P}_{x_0,R,\xi_0}\left(\{\forall t,\ Z_t \geq 1-\varepsilon\}\right),\end{aligned}$$

because E is independent from X, so from Z. This lower bound is positive thanks to the choice of ξ_0. By Theorem I.1.1, this contradicts Assumption I.1.2 and ends the proof of point **8.** of Section I.4.2 for $\beta > \alpha$.

I.4.9.4 Proof of point 8. of Section I.4.2 for $\beta = \alpha$

Finally, in the case $\beta = \alpha$, we use another renormalization. Let $\kappa < 1$, we define

$$Y_{\kappa,t} := \frac{X_t}{A_{\xi_0}(t)^\kappa}.$$

Lemma I.4.31. *Let F be a $\mathcal{C}^1$ function $\mathbb{R}^+ \to \mathbb{R}$. Under the allometric setting of Section I.1.2 with $\alpha \leq 1$, Assumptions I.1.2 and I.1.8, for all $R \in \mathcal{R}_0$, for all $t \geq 0$, we have:*

$$\begin{aligned}F(Y_{\kappa,t}) = F(\xi_0^{1-\kappa}) &+ \int_0^t \frac{F'(Y_{\kappa,s})}{A_{\xi_0}(s)^\kappa}\left(g(A_{\xi_0}(s)^\kappa Y_{\kappa,s}, R) - \kappa A_{\xi_0}(s)^{\kappa-1}g(A_{\xi_0}(s),R)Y_{\kappa,s}\right)\mathrm{d}s \\ &+ \int_0^t\int_{\mathbb{R}^+} b_{x_0}(A_{\xi_0}(s)^\kappa Y_{\kappa,s})\left(F\left(Y_{\kappa,s} - \frac{x_0}{A_{\xi_0}(s)^\kappa}\right) - F(Y_{\kappa,s})\right)\mathrm{d}s \\ &+ \int_0^t\int_{\mathbb{R}^+} \mathbb{1}_{\{h \leq b_{x_0}(A_{\xi_0}(s)^\kappa Y_{\kappa,s-})\}}\left(F\left(Y_{\kappa,s-} - \frac{x_0}{A_{\xi_0}(s)^\kappa}\right) - F(Y_{\kappa,s-})\right)\mathcal{N}_C(\mathrm{d}s,\mathrm{d}h).\end{aligned}$$

Proof. This is Proposition I.4.25 applied to $(t,x) \mapsto F\left(\dfrac{x}{A_{\xi_0}^\kappa(t)}\right)$. □

Applying Lemma I.4.31 to $F := \mathrm{Id}$ leads to

$$\begin{aligned} Y_{\kappa,t} &= \xi_0^{1-\kappa} + \int_0^t \frac{1}{A_{\xi_0}(s)^\kappa}\Big(g(A_{\xi_0}(s)^\kappa Y_{\kappa,s}, R) - \kappa A_{\xi_0}(s)^{\kappa-1} g(A_{\xi_0}(s), R) Y_{\kappa,s}\Big)\mathrm{d}s \\ &\quad - \int_0^t \int_{\mathbb{R}+} b_{x_0}(A_{\xi_0}(s)^\kappa Y_{\kappa,s}) \frac{x_0}{A_{\xi_0}(s)^\kappa}\mathrm{d}s \\ &\quad - \int_0^t \int_{\mathbb{R}+} \mathbb{1}_{\{h \le b_{x_0}(A_{\xi_0}(s)^\kappa Y_{\kappa,s-})\}} \frac{x_0}{A_{\xi_0}(s)^\kappa}\mathcal{N}_C(\mathrm{d}s, \mathrm{d}h) \\ &\ge \xi_0^{1-\kappa} + \int_0^t \Big((C_R - x_0 C_\beta) A_{\xi_0}(s)^{\kappa(\alpha-1)} Y_{\kappa,s}^\alpha - C_R \kappa A_{\xi_0}(s)^{\alpha-1} Y_{\kappa,s}\Big)\mathrm{d}s \\ &\quad - \int_0^t \int_{\mathbb{R}+} \mathbb{1}_{\{h \le b_{x_0}(A_{\xi_0}(s)^\kappa Y_{\kappa,s-})\}} \frac{x_0}{A_{\xi_0}(s)^\kappa}\mathcal{N}_C(\mathrm{d}s, \mathrm{d}h). \end{aligned}$$

We choose x_0 small enough such that $C_R - x_0 C_\beta \ge C_R \kappa$, and we obtain, for all $t > 0$:

$$Y_{\kappa,t} \ge \xi_0^{1-\kappa} + \int_0^t C_R \kappa A_{\xi_0}(s)^{\alpha-1}\Big(A_{\xi_0}(s)^{(\kappa-1)(\alpha-1)} Y_{\kappa,s}^\alpha - Y_{\kappa,s}\Big)\mathrm{d}s - \tilde{\mathcal{M}}_t,$$

with $\tilde{\mathcal{M}}_t := \int_0^t \int_{\mathbb{R}+} \mathbb{1}_{\{h \le b_{x_0}(A_{\xi_0}(s)^\kappa Y_{\kappa,s-})\}} \frac{x_0}{A_{\xi_0}(s)^\kappa}\mathcal{N}_C(\mathrm{d}s, \mathrm{d}h)$. Also, we know that $X_t \le A_{\xi_0}(t)$, so $Y_{\kappa,t} \le A_{\xi_0}(t)^{1-\kappa}$ for $t \ge 0$. We verify that for every $x > 0$, the function $y \mapsto x^{1-\alpha} y^\alpha - y$ is non-negative on $]0, x]$, so for every $s \ge 0$:

$$A_{\xi_0}(s)^{(\kappa-1)(\alpha-1)} Y_{\kappa,s}^\alpha - Y_{\kappa,s} \ge 0.$$

This entails that for $t \ge 0$ and x_0 small enough,

$$Y_{\kappa,t} \ge \xi_0^{1-\kappa} - \tilde{\mathcal{M}}_t. \tag{I.4.43}$$

Lemma I.4.32. *Let $\varepsilon > 0$, $x_0 > 0$, $R \in \mathcal{R}_0$ and $\kappa > \frac{1}{2}$. Then we can take ξ_0 high enough to have*

$$\mathbb{P}_{x_0,R,\xi_0}(\forall t,\ |\tilde{\mathcal{M}}_t| < \varepsilon) > 0.$$

Proof. We take $t > 0$ and $\alpha < 1$. As in the proof of Lemma I.4.29, we use Doob's maximal inequality in its L^2 version to obtain

$$\begin{aligned} \mathbb{P}_{x_0,R,\xi_0}(\sup_{0\le s\le t} |\tilde{\mathcal{M}}_s| \ge \varepsilon) &\le \frac{\mathbb{E}(\langle \tilde{\mathcal{M}}\rangle_t)}{\varepsilon^2} \\ &\le \frac{x_0^2}{\varepsilon^2}\int_0^t \frac{b_{x_0}(A_{\xi_0}(s)^\kappa Y_{\kappa,s})}{A_{\xi_0}(s)^{2\kappa}}\mathrm{d}s \\ &\le \frac{C_\beta x_0^2}{\varepsilon^2}\int_0^t A_{\xi_0}(s)^{\alpha-2\kappa}\mathrm{d}s, \end{aligned}$$

as $b_{x_0}(x) \le C_\beta x^\beta$, $Y_{\kappa,t} \le A_{\xi_0}(t)^{1-\kappa}$ and $\beta = \alpha$. Also, we choose $\kappa > 1/2$, which implies, using Equation (I.4.36),

$$\mathbb{P}_{x_0,\xi_0}(\sup_{0\le s\le t} |\tilde{\mathcal{M}}_s| \ge \varepsilon) \le \frac{C_\beta x_0^2 \xi_0^{1-2\kappa}}{\varepsilon^2 C_R (2\kappa - 1)},$$

which does not depend on t, and converges to 0 as ξ_0 goes to $+\infty$. Hence, for ξ_0 high enough,

$$\mathbb{P}_{x_0,\xi_0}(\exists t,\ |\tilde{\mathcal{M}}_t| \geq \varepsilon) < 1.$$

One can check that in the case $\alpha = 1$, we obtain the same result. □

Corollary I.4.33. *Let $\varepsilon > 0$, $R \in \mathcal{R}_0$ and $\kappa > 1/2$. For x_0, ξ_0 respectively small and high enough, we have*

$$\mathbb{P}_{x_0,\xi_0}(\forall t,\ Y_{\kappa,t} \geq \xi_0^{1-\kappa} - \varepsilon) > 0.$$

Proof. Let x_0 be small enough such that (I.4.43) holds true, and let ξ_0 be high enough so that the result of Lemma I.4.32 holds true for such x_0. Then, on the event $\{\forall t,\ |\tilde{\mathcal{M}}_t| < \varepsilon\}$, we have thanks to (I.4.43), for all $t \geq 0$,

$$Y_{\kappa,t} \geq \xi_0^{1-\kappa} - \varepsilon.$$

□

In the same way as the previous section, we evaluate what happens if we bring back death events. Let $\varepsilon > 0$ and $\kappa > 1/2$. Thanks to Corollary I.4.33, we choose x_0 small enough and ξ_0 high enough (and even higher if necessary to have $\xi_0^{1-\kappa} \geq 1$), so that the event $\{\forall t,\ Y_{\kappa,t} \geq 1 - \varepsilon\}$ occurs with positive probability, and we work on this event, which can be rewritten from Equation (I.4.36) as

$$\begin{cases} \left\{\forall t,\ X_t \geq (1-\varepsilon)\left(\xi_0^{1-\alpha} + C_R(1-\alpha)t\right)^{\kappa/(1-\alpha)}\right\} & \text{if } \alpha < 1, \\ \left\{\forall t,\ X_t \geq (1-\varepsilon)\xi_0^{\kappa} e^{\kappa C_R t}\right\} & \text{if } \alpha = 1. \end{cases} \tag{I.4.44}$$

To prove point **8.** of Section I.4.2 when $\beta = \alpha$, we suppose by contradiction that $\delta < \alpha - 1 \leq 0$, so d is decreasing. Once this is done, we also choose $\kappa \in (1/2, 1)$ such that $\delta < \frac{\alpha-1}{\kappa}$. In the following, c is a constant depending on ξ_0 and ε, possibly varying from line to line. We first take $\alpha < 1$. On the event (I.4.44), we get for $t \geq 0$,

$$\int_0^t d(X_s)\mathrm{d}s \leq \int_0^t d(c(1+s)^{\kappa/(1-\alpha)})\mathrm{d}s \leq \int_0^t c(1+s)^{\delta\kappa/(1-\alpha)}\mathrm{d}s,$$

with $\delta\kappa/(1-\alpha) < -1$. Hence, the above integral converges when $t \to +\infty$. One can check that the same result holds if $\alpha = 1$. We conclude with the same arguments as in the end of Section I.4.9.3.

I.4.9.5 An asymptotic pseudotrajectory result for $\beta < \alpha < 1$

In this section, the goal is to show Proposition I.4.36, which precises the asymptotic behavior of $(Z_t)_{t\geq 0}$ (see Corollary I.4.37), and is stronger than Proposition I.4.30, in the specific case where $\beta < \alpha < 1$. We begin with some definitions. A *flow* Φ on $\mathbb{R}^+\times]0,1]$ is a continuous map

$$\begin{aligned} \Phi: \quad \mathbb{R}^+ \times \mathbb{R}^+\times]0,1] &\to \quad \mathbb{R}^+\times]0,1] \\ (u,t,x) &\mapsto \quad \Phi_u(t,x) := (\Phi_u^1(t,x), \Phi_u^2(t,x)) \end{aligned}$$

with $\Phi_0 : (t,x) \mapsto (t,x)$ and $\Phi_{u+v}(t,x) = \Phi_u(\Phi_v(t,x))$ for all (u,v,t,x).

Definition I.4.34 (**Asymptotic pseudotrajectory**). *Let Φ be a flow on $\mathbb{R}^+\times]0,1]$, then a function $Z:\mathbb{R}^+\to\mathbb{R}^+\times]0,1]$ is an asymptotic pseudotrajectory, or simply APT in the following, for Φ if for any $T>0$,*

$$\lim_{t\to+\infty}\sup_{h\in[0,T]} d(Z(t+h),\Phi_h(Z(t)))=0,$$

where d is the distance on $\mathbb{R}^+\times]0,1]$ defined by $d((x,y),(x',y'))=\max(|x-x'|,|y-y'|)$ for every (x,y,x',y').

Being an APT for Φ means that, given $T>0$, the trajectory of $h\in[0,T]\mapsto Z(t+h)$ is arbitrarily close to the trajectory of Φ starting from $Z(t)$ for t high enough. We can think of Φ as the flow associated to an ODE that is inhomogeneous in time, and Z a perturbation of this flow. This concept was introduced by Benaïm and Hirsch to study the asymptotic behavior of stochastic approximation algorithms [Ben99]. Let us define

$$\varphi:(t,x)\in\mathbb{R}^+\times]0,1]\mapsto\left(1,x^\alpha-x-\frac{C_\beta x_0}{C_R}\xi_0^{\beta-\alpha}e^{(\beta-\alpha)t}x^\beta\right)=:(\varphi_1(t,x),\varphi_2(t,x)) \tag{I.4.45}$$

and denote as $\Phi:(u,t,x)\in\mathbb{R}^+\times\mathbb{R}^+\times]0,1]\mapsto\Phi_u(t,x)$ the flow induced by $\Phi_0(t,x):=(t,x)$ and

$$\frac{\partial\Phi_u(t,x)}{\partial u}:=\varphi(\Phi_u(t,x)). \tag{I.4.46}$$

We first state a useful lemma.

Lemma I.4.35. *Let $x_0>0$, $R>R_0$, $\xi_0>0$ and $\beta<\alpha<1$, and Φ the flow defined in (I.4.46). For every $(u,t,x)\in\mathbb{R}^+\times\mathbb{R}^+\times]0,1]$, we have*

$$\Phi_u^1(t,x)=t+u.$$

Moreover, there exists $c>0$ and $t_0>0$ (depending on x_0, R, α and β) such that

$$\forall t\geq t_0,\forall\xi_0\geq 1,\quad \varphi_2(t,c)\geq 0.$$

Finally, there exists a constant $K>0$ (depending on c) such that

$$\forall t\geq 0,\forall(x,y)\in[c,1],\quad |\varphi_2(t,x)-\varphi_2(t,y)|\leq K|x-y|.$$

Proof. The first assertion of the lemma is straightforward from (I.4.45). Then, there exists $c\in]0,1]$ such that $c^\alpha-c>0$ because $\alpha<1$. As $e^{(\beta-\alpha)t}\underset{t\to+\infty}{\longrightarrow}0$, from Equation (I.4.45) again, we can choose t_0 high enough such that $\varphi_2(t_0,c)\geq 0$, uniformly on $\xi_0\geq 1$. For every $x\in]0,1]$, $\varphi_2(.,z)$ is non-decreasing, because $\beta<\alpha$, which concludes for the second part of the lemma. Finally, for $t\geq 0$, $e^{(\beta-\alpha)t}\leq 1$, and $z\mapsto z^\alpha-z$ and $z\mapsto z^\beta$ are Lipschitz-continuous on $[c,1]$, because $\beta<\alpha<1$. This ends the proof. □

We are ready to state the main result of this section. The essential ideas of the proof are adapted from the work of Michel Benaïm (Section 4.3 in [Ben99]).

Proposition I.4.36. *Let $x_0>0$, $R>R_0$ and $\beta<\alpha<1$. We consider the process $(Z_t)_{t\geq 0}$ defined in Section I.4.9.2. Then, for ξ_0 high enough, $(t,Z_t)_{t\geq 0}$ is an APT for Φ defined in (I.4.46), with positive probability.*

Proof. We fix $x_0 > 0$ and $R > R_0$. We choose $c > 0$ and $t_0 > 0$ for which the results of Lemma I.4.35 hold true. We apply Proposition I.4.30 with $\varepsilon = c$, *i.e.* we choose ξ_0 high enough so that

$$\mathbb{P}_{x_0,R,\xi_0}(\forall t,\ Z_t \geq c) > 0.$$

If necessary, we choose $\xi_0 \geq 1$ so that we can still apply the first part of Lemma I.4.35. In the following, we work on the event $\{\forall t,\ Z_t \geq c\}$, and we denote by $\mathbb{P}$ the probability measure $\mathbb{P}_{x_0,R,\xi_0}(.|\forall t,\ Z_t \geq c)$. We fix $T > 0$, and for $t, u \geq 0$, we write

$$\Delta(t,u) := \sup_{0\leq h\leq u} |Z_{t+h} - \Phi_h^2(t, Z_t)|.$$

From Lemma I.4.35, we have for every (t,h),

$$|t + h - \Phi_h^1(t, Z_t)| = 0$$

Thus, to prove our result, it suffices to show that

$$\mathbb{P}\left(\lim_{t\to+\infty} \Delta(t,T) = 0\right) = 1.$$

First step: Lipschitz and Grönwall tricks.

We choose $t \geq t_0$, $h \in [0,T]$, then by definition of Φ in (I.4.46) and Lemma I.4.35:

$$\begin{aligned}
\left|Z_{t+h} - \Phi_h^2(t,Z_t)\right| &= \left|Z_{t+h} - Z_t - \int_0^h \varphi_2(t+u, \Phi_u^2(t,Z_t))\mathrm{d}u\right| \\
&\leq \left|Z_{t+h} - Z_t - \int_0^h \varphi_2(t+u, Z_{t+u})\mathrm{d}u\right| \\
&\qquad + \int_0^h \left|\varphi_2(t+u, Z_{t+u}) - \varphi_2(t+u, \Phi_u^2(t,Z_t))\right| \mathrm{d}u \\
&\leq \left|Z_{t+h} - Z_t - \int_t^{t+h} \varphi_2(s, Z_s)\mathrm{d}s\right| + K\int_0^h \left|Z_{t+u} - \Phi_u^2(t,Z_t)\right| \mathrm{d}u.
\end{aligned}$$

The last inequality follows from Lemma I.4.35 and the fact that we work on the event $\{\forall t,\ Z_t \geq c\}$. We just have to verify that for $u \in [0,h]$, $\Phi_u^2(t,Z_t) \geq c$. This comes again from Lemma I.4.35 and the definition of Φ, we observe that $\Phi_u^2(t,Z_t)$ cannot go under c over time, as soon as it is true for $u = 0$, and its derivative is positive at this precise value.

Now, we take the supremum over $[0,u]$ for the $\left|Z_{t+u} - \Phi_u^2(t,Z_t)\right|$-term in the last integral and extend it from $[0,h]$ to $[0,T]$, which keeps the inequality true. Then we take the supremum over $[0,T]$ in the first term of the right-hand side, and eventually in the left-hand side to obtain

$$\Delta(t,T) \leq \tilde{\Delta}(t,T) + K\int_0^T \Delta(t,u)\mathrm{d}u,$$

with $\tilde{\Delta}(t,u) := \sup_{0\leq h\leq u}\left|Z_{t+h} - Z_t - \int_t^{t+h} \varphi_2(s,Z_s)\mathrm{d}s\right|$ for $u \geq 0$. Remark that $u \mapsto \tilde{\Delta}(t,u)$ is non-decreasing. We apply Grönwall's lemma and get

$$\Delta(t,T) \leq \tilde{\Delta}(t,T)e^{KT}. \tag{I.4.47}$$

Second step: martingale properties.

We fix $\varepsilon > 0$ and observe that

$$\mathbb{P}\left(\tilde{\Delta}(t,T) \geq \varepsilon\right) \leq \mathbb{P}\left(\sup_{0\leq h\leq T}\left(Z_{t+h} - Z_t - \int_t^{t+h}\varphi_2(s,Z_s)\mathrm{d}s\right) \geq \varepsilon\right) + \mathbb{P}\left(\sup_{0\leq h\leq T}\left(-\left(Z_{t+h} - Z_t - \int_t^{t+h}\varphi_2(s,Z_s)\mathrm{d}s\right)\right) \geq \varepsilon\right)$$

From now on, we take $\theta \in \{-1,1\}$, and we try to estimate

$$\mathcal{P} := \mathbb{P}\left(\sup_{0\leq h\leq T}\left(\theta\left(Z_{t+h} - Z_t - \int_t^{t+h}\varphi_2(s,Z_s)\mathrm{d}s\right)\right) \geq \varepsilon\right).$$

which is equal to

$$\mathbb{P}\left(\sup_{0\leq h\leq T}\exp\left(r\theta\left(Z_{t+h} - Z_t - \int_t^{t+h}\varphi_2(s,Z_s)\mathrm{d}s\right)\right) \geq e^{r\varepsilon}\right)$$

for any $r \geq 0$.

We now use Lemma I.4.28 with $F : x \mapsto e^{r\theta x}$ to obtain that $F(Z_t) - \int_0^t L_sF(Z_s)\mathrm{d}s$ is a martingale, with

$$L_sF(x) := F'(x)\left(x^\alpha - x\right) + \frac{\xi_0^{1-\alpha}e^{(1-\alpha)s}}{C_R}b_{x_0}\left(\xi_0 e^s x\right)\left(F\left(x - \frac{x_0}{\xi_0}e^{-s}\right) - F(x)\right).$$

It implies that $N_t := F(Z_t)\exp\left(-\int_0^t \frac{L_sF(Z_s)}{F(Z_s)}\mathrm{d}s\right)$ is a martingale (Corollary 3.3 p.66 in [EK86]). This gives naturally rise to another martingale for any fixed $t \geq 0$: $(N_{t+h}/N_t)_{h\geq 0}$. We get that

$$\left(\exp\left(r\theta\Big(Z_{t+h} - Z_t\Big) - \int_t^{t+h}\frac{L_sF(Z_s)}{F(Z_s)}\mathrm{d}s\right)\right)_{h\geq 0}$$

is a martingale for any $t \geq 0$. Now, we take t_0 higher if necessary to have $\xi_0 e^{t_0} Z_s > x_0$ (recall that we work on the event $\{\forall t,\ Z_t \geq c\}$), and we obtain for $s \geq t_0$,

$$\begin{aligned}\frac{L_sF(Z_s)}{F(Z_s)} &= r\theta\left(Z_s^\alpha - Z_s\right) + \frac{\xi_0^{1-\alpha}e^{(1-\alpha)s}}{C_R}b_{x_0}\left(\xi_0 e^s Z_s\right)\left(\exp\left(-r\theta\frac{x_0}{\xi_0}e^{-s}\right) - 1\right)\\ &= r\theta\varphi^2(s,Z_s) + \frac{C_\beta\xi_0^{\beta-\alpha+1}e^{(\beta-\alpha+1)s}}{C_R}Z_s^\beta\left(\exp\left(-r\theta\frac{x_0}{\xi_0}e^{-s}\right) + r\theta\frac{x_0}{\xi_0}e^{-s} - 1\right).\end{aligned}$$

We know that $e^u - u - 1 \leq u^2$ for u close enough to 0, so we can choose t_0 higher if necessary so that for $s \geq t_0$,

$$\exp\left(-r\theta\frac{x_0}{\xi_0}e^{-s}\right) + r\theta\frac{x_0}{\xi_0}e^{-s} - 1 \leq \frac{x_0^2}{\xi_0^2}r^2e^{-2s}.$$

Thus, for $s \geq t_0$, we obtain

$$\frac{L_sF(Z_s)}{F(Z_s)} - r\theta\varphi_2(s,Z_s) \leq Cr^2e^{(\beta-\alpha-1)s},$$

with $C := \dfrac{C_\beta x_0^2 \xi_0^{\beta-\alpha-1}}{C_R} \max(1, c^\beta)$. We used here the fact that $c \leq Z_s \leq 1$ over time. We choose ξ_0 higher if necessary to have $C \leq 1$. Finally, we know that

$$\exp\left(r\theta\Big(Z_{t+h} - Z_t - \int_t^{t+h} \varphi_2(s, Z_s)\mathrm{d}s\Big) - \int_t^{t+h} \left[\frac{L_s F(Z_s)}{F(Z_s)} - r\theta\varphi_2(s, Z_s)\right]\mathrm{d}s\Big)\right)$$

is a martingale (simply adding and substracting a term in the expression of $(N_{t+h}/N_t)_{h\geq 0}$), so the process

$$\widetilde{N}_h := \exp\left(r\theta\Big(Z_{t+h} - Z_t - \int_t^{t+h} \varphi_2(s, Z_s)\mathrm{d}s\Big) - \int_t^{t+h} Cr^2 e^{(\beta-\alpha-1)s}\mathrm{d}s\right)$$

is a supermartingale for $t \geq t_0$.

Third step: Doob's inequality.

In the following, we take $t \geq t_0$. First, we assess that

$$\mathcal{P} \leq \mathbb{P}\left(\sup_{0\leq h\leq T} \widetilde{N}_h \geq \exp\left(r\varepsilon - \int_t^{t+T} Cr^2 e^{(\beta-\alpha-1)s}\mathrm{d}s\right)\right).$$

All this work has been done to apply Doob's inequality for supermartingales and obtain

$$\begin{aligned}\mathcal{P} &\leq \mathbb{E}(\widetilde{N}_0)\exp\left(-r\varepsilon + \int_t^{t+T} Cr^2 e^{(\beta-\alpha-1)s}\mathrm{d}s\right) \\ &= \exp\left(-r\varepsilon + C' r^2 e^{(\beta-\alpha-1)t}\right),\end{aligned}$$

with $C' := \dfrac{Ce^{(\beta-\alpha-1)T} - 1}{\beta - \alpha - 1}$. The time has come to make a particular choice for r. We fix

$$r := \frac{\varepsilon}{2C'} e^{(\alpha+1-\beta)t}$$

which is positive because $C \leq 1$ and $\beta < \alpha$, and we obtain

$$\mathcal{P} \leq \exp\left(-\frac{\varepsilon^2}{4C'} e^{(\alpha+1-\beta)t}\right).$$

Finally,

$$\mathbb{P}(\tilde{\Delta}(t,T) \geq \varepsilon) \leq 2\exp\left(-\frac{\varepsilon^2}{4C'} e^{(\alpha+1-\beta)t}\right). \tag{I.4.48}$$

Last step: Borel-Cantelli to conclude

By (I.4.47) and (I.4.48), there exists $\mathfrak{C} > 0$ such that for every $k \in \mathbb{N}^*$ with $kT > t_0$,

$$\mathbb{P}(\Delta(kT,T) \geq \varepsilon) \leq 2\exp\left(-\mathfrak{C}\varepsilon^2 e^{(\alpha+1-\beta)k^2T^2}\right),$$

which is the general term of a convergent sum in k. This is valid for every $\varepsilon > 0$, so by Borel-Cantelli lemma, we obtain that

$$\mathbb{P}(\Delta(kT,T) \underset{k\to+\infty}{\longrightarrow} 0) = 1.$$

For $t \geq t_0$, if we take $k \in \mathbb{N}$ such that $kT \leq t < (k+1)T$, we also assess that

$$\Delta(t,T) \leq 2\Delta(kT,T) + \Delta((k+1)T,T)$$

which ends the proof. □

Remark: We not only obtained that Z is almost surely an APT for Φ, but also have a strong exponential control on the possible deviations. Namely, there exists constants $\mathfrak{K}$ and $\mathfrak{C}'$ such that

$$\mathbb{P}(\Delta(t,T) \geq \varepsilon) \leq \mathfrak{K} \exp\left(-\mathfrak{C}'\varepsilon^2 e^{2\alpha t/1-\alpha}\right).$$

Nevertheless, the constants $\mathfrak{C}'$ and $\mathfrak{K}$ are unknown, and the previous 'fast convergence' result can be worsened by the precise values of these constants. One could wonder about the interest of such a long technical proof to get only an asymptotic APT result. All the power of this concept of APT is highlighted in [Ben99], and entails the following corollary.

Corollary I.4.37. *Under the assumptions of Proposition I.4.36, with positive probability,*

$$Z_t \xrightarrow[t \to +\infty]{} 1,$$

which immediately entails that with positive probability,

$$\frac{X_t}{A_{\xi_0}(t)} \xrightarrow[t \to +\infty]{} 1.$$

Proof. Without loss of generality, we can start our process from ξ_0 high enough so that $(t, Z_t)_{t\geq 0}$ is an APT for Φ defined in (I.4.46), with positive probability, thanks to Proposition I.4.36 (from any initial condition $\xi_0' > 0$, there exists a positive probability for $(X_t)_{t\geq 0}$ to reach ξ_0 in finite time and by Markov property, we can always start from this stopping time). We define $L(Z)$ the (random) *limit set* of Z as

$$L(Z) := \bigcap_{t\geq 0} \overline{Z([t, +\infty[)}.$$

This is the set of limits of convergent sequences $(Z_{t_k})_{k\geq 0}$, $t_k \to +\infty$. Note that $(Z_t)_{t\geq 0}$ lies in the compact set $[0, 1]$, so $L(Z) \neq \emptyset$. As $(t, Z_t)_{t\geq 0}$ is an APT for Φ, and by the Limit Set Theorem (Theorem 5.7 in [Ben99]) we assess that $L(Z)$ is internally chain transitive for Φ. It means that every point of $L(Z)$ should be a chain recurrent point for $\Phi_{|L(Z)}$ (see subsection 5.1 of [Ben99]). The only possible chain recurrent point for Φ is 1. Indeed, $\{1\}$ is a global attractor for Φ: the solution to Equation (I.4.46) starting from any $0 < z \leq 1$ converges to 1, because this is the equilibrium of $z' = C_R(z^\alpha - z)$, and the right-most term goes to 0 over time. We also know that $L(Z)$ is non-empty, hence

$$L(Z) = \{1\}.$$

We end the proof by a compactness argument: $(Z_t)_{t\geq 0}$ lies in a compact and its limit set is a singleton, so it should converge to this unique value. □

Remark: A simple monotony argument led us to $X_t \leq A_{\xi_0}(t)$. The previous corollary highlights that in fact, without any death events and with positive probability, an individual will 'extract' itself from the low-energy regime where births are likely to occur. It will then grow up indefinitely with fewer and fewer birth events, mimicking the behavior of $A_{\xi_0}(t)$.

I.4.10 Converse implication in Theorem I.2.1

In this section, we work under the allometric setting of Section I.1.2. Also, we work under (I.2.6) or (I.2.8), and they both imply Assumption I.1.8 by Proposition I.4.1. Hence, we pick R_0 verifying the result of Lemma I.4.13. We begin by two preliminary results on the operator $K_{x_0,R}\mathbf{1}$ for $x_0 > 0$ and $R > R_0$.

Lemma I.4.38. *Let $x_0 > 0$, $k \geq 0$. Under the allometric setting of Section I.1.2 and (I.2.6), we have for $R > R_0$,*

$$\forall \xi \geq k x_0, \quad K^k_{x_0,R}\mathbf{1}(\xi) = \left(\frac{C_\beta}{C_\beta + C_\delta}\right)^k.$$

Proof. It is straightforward to prove this result by induction on k, using (I.1.2), where we notice that under (I.2.6), for $R > R_0$, we have $\int_0^{t_{\max}(\xi_0,R)} (b_{x_0} + d)(A_{\xi_0,R}(u))\mathrm{d}u = +\infty$. □

Lemma I.4.39. *Let $x_0 > 0$, $k \geq 1$. Under the allometric setting of Section I.1.2 and (I.2.6), we have for $R > R_0$,*

$$K^k_{x_0,R}\mathbf{1}(x_0) \geq \left(\frac{C_\beta}{C_\beta + C_\delta}\right)^k k^{-\frac{C_\beta + C_\delta}{C_R}}.$$

Proof. We compute for $k \geq 1$,

$$\begin{aligned} K^k_{x_0,R}\mathbf{1}(x_0) &:= \int_{x_0}^{+\infty} \frac{b_{x_0}(u)}{g(u,R)} e^{-\int_{x_0}^{u} \frac{b_{x_0}(\tau) + d(\tau)}{g(\tau,R)}\mathrm{d}\tau} K^{k-1}_{x_0,R}\mathbf{1}(u - x_0)\mathrm{d}u \\ &\geq \int_{kx_0}^{+\infty} \frac{b_{x_0}(u)}{g(u,R)} e^{-\int_{x_0}^{u} \frac{b_{x_0}(\tau) + d(\tau)}{g(\tau,R)}\mathrm{d}\tau} \left(\frac{C_\beta}{C_\beta + C_\delta}\right)^{k-1} \mathrm{d}u \end{aligned}$$

by Lemma I.4.38. Under the allometric setting of Section I.1.2 and (I.2.6), a straightforward computation leads to the result. □

We are now ready to prove the converse implication in Theorem I.2.1.

Proposition I.4.40. *Under the allometric setting of Section I.1.2, under Assumptions I.1.1 and I.1.2, we have*

$$\text{(I.2.8)} \Rightarrow \textit{Assumption I.1.4}.$$

Proof. We pick $x_0 > 0$ and $R > R_0$. Under Assumptions I.1.1 and I.1.2, by Lemma I.4.8 and Lemma I.4.39, we can write

$$\begin{aligned} m_{x_0,R}(x_0) := \mathbb{E}(N_{x_0,R,x_0}) = \sum_{k\geq 1} \mathbb{P}_{x_0,R,x_0}(M^k) &= \sum_{k\geq 1} K^k_{x_0,R}\mathbf{1}(x_0) \\ &\geq \sum_{k \geq 1} \left(\frac{C_\beta}{C_\beta + C_\delta}\right)^k k^{-\frac{C_\beta + C_\delta}{C_R}}. \end{aligned}$$

Now, if (I.2.8), we have $C_\beta + C_\delta < C_\gamma - C_\alpha$ and we know that $C_R \underset{R\to+\infty}{\longrightarrow} C_\gamma - C_\alpha$, so we can choose $R > R_0$ high enough such that $\frac{C_\beta+C_\delta}{C_R} < 1$, and the previous lower bound, with the fact that $C_\beta > (e-1)C_\delta$, gives

$$m_{x_0,R}(x_0) > \sum_{k\geq 1}\left(\frac{C_\beta}{C_\beta + C_\delta}\right)^k k^{-1} > \sum_{k\geq 1}\frac{(1-e^{-1})^k}{k} = 1,$$

which concludes since this is valid for every $x_0 > 0$. □

Finally, we prove that (I.2.6) is not a sufficient condition to obtain Assumption I.1.4. We begin again by a preliminary lemma.

Lemma I.4.41. *Under the allometric setting of Section I.1.2, if (I.2.6), then for $x_0 > 0$ and $R > R_0$, we have*

$$\forall k \geq 1, \forall \xi > 0, \quad K^k_{x_0,R}\mathbf{1}(\xi) \leq \left(\frac{C_\beta}{C_\beta + C_\delta}\right)^k, \tag{I.4.49}$$

and

$$\exists \mathfrak{C}(C_\beta, C_\delta) < 1, \quad K^2_{x_0,R}\mathbf{1}(x_0) \leq \mathfrak{C}(C_\beta, C_\delta)\left(\frac{C_\beta}{C_\beta + C_\delta}\right)^2. \tag{I.4.50}$$

Proof. It is straightforward to prove (I.4.49) by induction on k. Then, remark that

$$K_{x_0,R}\mathbf{1}(\xi)\begin{cases} = \frac{C_\beta}{C_\beta+C_\delta} & \text{if } \xi \geq x_0, \\ \leq \frac{C_\beta}{C_\beta+C_\delta}\left(\frac{\xi}{x_0}\right)^{\frac{C_\delta}{C_\gamma - C_\alpha}} & \text{else.}\end{cases}$$

We use this fact and (I.4.49) to obtain

$$\begin{aligned} K^2_{x_0,R}\mathbf{1}(x_0) \leq \frac{C_\beta}{C_\beta + C_\delta}\Bigg(&\int_{x_0}^{\frac{3}{2}x_0}\frac{b_{x_0}(u)}{g(u,R)}e^{-\int_{x_0}^{u}\frac{b_{x_0}(\tau)+d(\tau)}{g(\tau,R)}\mathrm{d}\tau}\left(\frac{1}{2}\right)^{\frac{C_\delta}{C_\gamma - C_\alpha}}\mathrm{d}u \\ &+ \int_{3/2x_0}^{+\infty}\frac{b_{x_0}(u)}{g(u,R)}e^{-\int_{x_0}^{u}\frac{b_{x_0}(\tau)+d(\tau)}{g(\tau,R)}\mathrm{d}\tau}\mathrm{d}u\Bigg). \end{aligned}$$

Under the allometric setting of Section I.1.2 and (I.2.6), this gives

$$K^2_{x_0,R}\mathbf{1}(x_0) \leq \left(\frac{C_\beta}{C_\beta + C_\delta}\right)^2 \mathfrak{C}(C_\beta, C_\delta), \tag{I.4.51}$$

with $\mathfrak{C}(C_\beta, C_\delta) := \left(\left(\frac{1}{2}\right)^{\frac{C_\delta}{C_\gamma - C_\alpha}}\left(1 - \left(\frac{2}{3}\right)^{\frac{C_\beta + C_\delta}{C_\gamma - C_\alpha}}\right) + \left(\frac{2}{3}\right)^{\frac{C_\beta + C_\delta}{C_\gamma - C_\alpha}}\right) < 1$, which concludes. □

Proposition I.4.42. *Under the allometric setting of Section I.1.2, under Assumptions I.1.1, I.1.2 and I.1.4, if $\beta = \delta = \alpha - 1$, there exists $\Xi : \mathbb{R}^+ \to]1, +\infty[$ such that*

$$\frac{C_\beta}{C_\delta} > \Xi(C_\delta) > 1.$$

If in addition, $C_\delta \geq C_\gamma - C_\alpha$, then

$$\Xi(C_\delta) > 1.07.$$

In particular, (I.2.6) *is not a sufficient condition to verify Assumption I.1.4 under Assumptions I.1.1 and I.1.2.*

Proof. We pick $x_0 > 0$ and $R > R_0$. Under our assumptions, we necessarily have $C_\beta > C_\delta$ by point **4.** of Section I.4.2. Under Assumptions I.1.1 and I.1.2, by Lemma I.4.8 and Lemma I.4.41, we can write

$$\begin{aligned} m_{x_0,R}(x_0) = \sum_{k\geq 1} \mathbb{P}_{x_0,R,x_0}(M^k) &= \sum_{k\geq 1} K^k_{x_0,R}\mathbf{1}(x_0) \\ &\leq \sum_{k\geq 1} \left(\frac{C_\beta}{C_\beta + C_\delta}\right)^k - (1 - \mathfrak{C}(C_\beta, C_\delta))\left(\frac{C_\beta}{C_\beta + C_\delta}\right)^2 \\ &= \frac{C_\beta}{C_\delta} - (1 - \mathfrak{C}(C_\beta, C_\delta))\left(\frac{C_\beta}{C_\beta + C_\delta}\right)^2. \end{aligned}$$

For a given value of C_δ, this upper bound goes to $1 - \frac{L}{4}$, with some $L > 0$, when C_β converges to C_δ (see the explicit expression of $\mathfrak{C}(C_\beta, C_\delta)$ in the proof of Lemma I.4.41). Hence, for a given value of C_δ, if C_β is too close from C_δ, $m_{x_0,R}(x_0) < 1$ so Assumption I.1.4 is not verified. This is valid for any C_δ, hence the result.

If in addition, we assume that $C_\delta \geq C_\gamma - C_\alpha$, we can go further in (I.4.51) and obtain

$$K^2_{x_0,R}\mathbf{1}(x_0) \leq \left(\frac{C_\beta}{C_\beta + C_\delta}\right)^2 \left(\frac{1}{2}\left(1 - \left(\frac{2}{3}\right)^{\frac{C_\beta}{C_\delta}+1}\right) + \left(\frac{2}{3}\right)^{\frac{C_\beta}{C_\delta}+1}\right) \leq \frac{13}{18}\left(\frac{C_\beta}{C_\beta + C_\delta}\right)^2.$$

Hence, we can pick $\mathfrak{C}(C_\beta, C_\delta) \equiv \frac{13}{18}$ in the previous reasoning, to obtain that

$$m_{x_0,R}(x_0) \leq \frac{C_\beta}{C_\delta} - \frac{5}{18}\left(\frac{C_\beta}{C_\beta + C_\delta}\right)^2,$$

which is less than or equal to 1 if $C_\beta \leq 1,07 C_\delta$. □

I.5 Simulations

In this section, we work under the allometric setting of Section I.1.2 and $\alpha \leq 1$. We present simulations of individual trajectories $(\xi_{t,x_0,R,\xi_0})_{t\geq 0}$, and are mainly interested in the behavior of the average number of offspring $m_{x_0,R}(x_0)$, for different values of parameters. In particular, our main aim is to determine when we have $m_{x_0,R}(x_0) > 1$, leading then to the supercriticality of the population process (see Proposition I.1.7). Precisely, according to Theorem I.2.1, we first fix the following parameters (unless specified otherwise):

Parameter	Value
α	0.75
γ	α
δ	$\alpha - 1$
$\phi(R)$	2/3
C_γ	2
C_α	1

We picked the usual value $\alpha = 0.75$ highlighted by the Metabolic Theory of Ecology [Pet86, BGA+04, SDF08]. In this section, we will independently make vary the following parameters: x_0, β, C_β and C_δ. We estimate $m_{x_0,R}(x_0)$ by a Monte-Carlo method and use `Python` for the simulations. We present our numerical results, that lead to Conjecture I.2.2 on a necessary and sufficient condition to verify Assumption I.1.4 under Assumptions I.1.1 and I.1.2, and Conjecture I.2.3 about the behavior of $m_{x_0,R}(\xi_0)$ in the I_2 case.

Recall that the allometric coefficients $\beta < \alpha - 1$ are non-admissible because they contradict Assumption I.1.2 (see Section I.4.5), that is they allow individuals with $T_d = +\infty$. These are obviously trajectories that we cannot observe with numerical simulations, so we focus in the following on the case $\beta \geq \alpha - 1$.

I.5.1 Typical look of individual trajectories

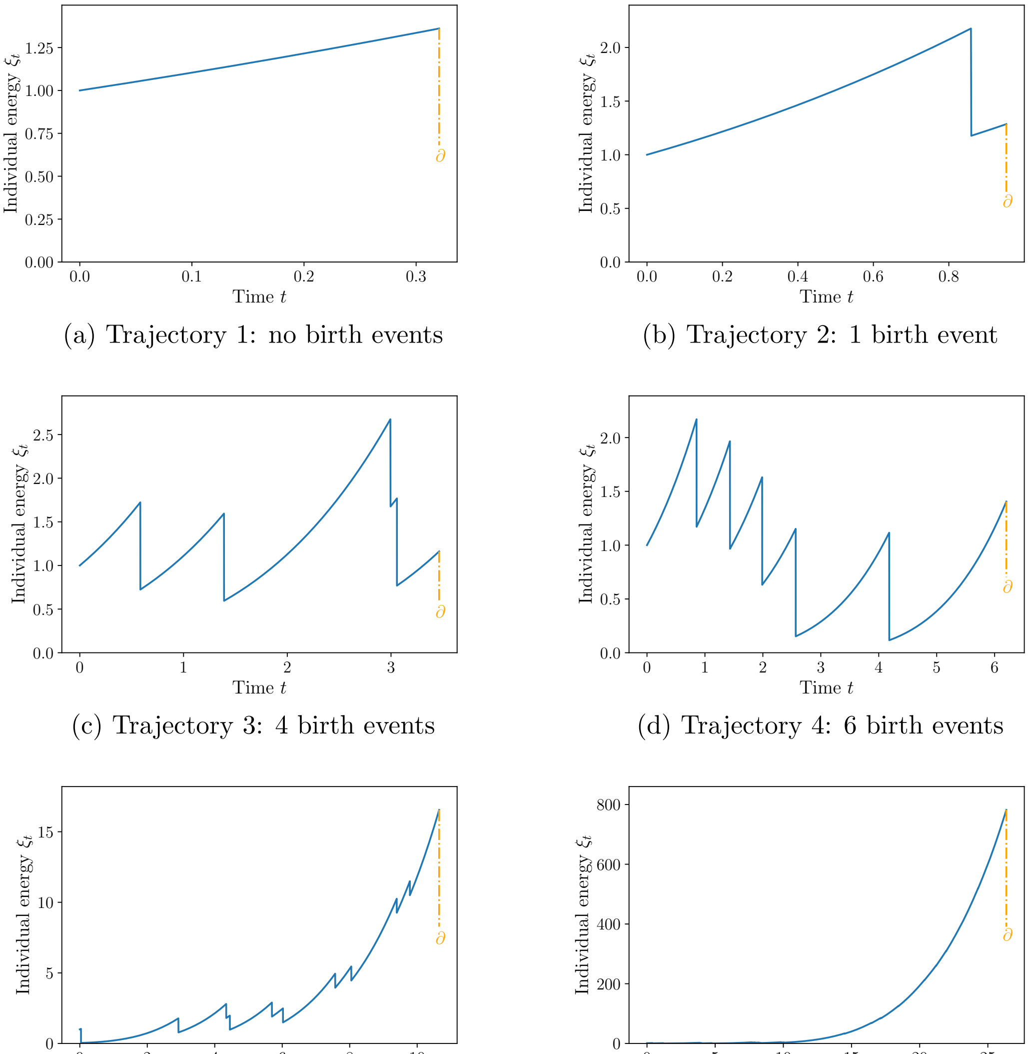
We run independent simulations of the individual process with a set of parameters satisfying $(\gamma, \delta, \beta) \in I_2$ (in particular $\beta > \alpha - 1$). On Figure I.3, we present six individual trajectories, illustrating the three distinct shapes that we observed in our simulations. The most common trajectories are shown on Figures I.3a and I.3b, where individuals die fast and have 0 or 1 offspring. Rarely, lifetime and energy increase and there is a huge amount of birth events like on Figures I.3e and I.3f (on Figure I.3f, the energy increases so fast that we barely see the negative jumps due to birth events anymore). Finally, we have intermediate trajectories, as shown on Figures I.3c and I.3d, where individual energy stays close to the initial energy x_0. They occur more often than I.3e and I.3f, but less often than I.3a and I.3b. Modifying the parameter $\beta > \alpha - 1$ leads to similar observations. Now if $\beta = \alpha - 1$, we still encounter the shapes described earlier, but the frequencies of observation of these three different shapes become relatively similar.

I.5.2 Subcriticality for $\beta > \alpha - 1$, not in the I_2 case

We illustrate that condition (I.2.7) is necessary to obtain Assumption I.1.4 if $\beta > \alpha - 1$. To do so, we choose values of β outside the I_2 case, and we observe that there exists some $x_0 > 0$ with $m_{x_0,R}(x_0) \leq 1$. On Figure I.4, we represent $m_{x_0,R}(x_0)$ depending on x_0, for values of β outside the I_2 case. We verify that $m_{x_0,R}(x_0) \leq 1$ for small values of x_0. We also observe that we need to look at smaller and smaller values for x_0 when β goes to $\alpha - 1$ to obtain a subcritical regime. Moreover, it seems that over a given value for x_0, depending on β, the regime is supercritical, but the mean number of offspring goes to 1 when x_0 goes to $+\infty$. There is a value for x_0 that maximizes the mean number of offspring and it does not seem to depend on β.

I.5.3 Explosion of $m_{x_0,R}(x_0)$ in the I_2 case

When we continue to increase the value of β and look now at the I_2 case, we obtain an unstable behavior of the Monte Carlo estimate, which leads to Conjecture I.2.3. Let us write n for the number of individual trajectories simulated to estimate $m_{x_0,R}(x_0)$. On Figure I.5, we represent $m_{x_0,R}(x_0)$ depending on n, for various values of β in I_2. Notice that now, x_0 is fixed equal to 1. The Monte-Carlo estimation of the mean number of offspring behaves badly. We see higher and higher peaks occuring for random values of n when β increases. It becomes harder and harder to estimate correctly this expectation with Monte Carlo estimation, because very rare trajectories like on Figure I.3f have a very high contribution. This is a typical heavy-tailed situation, where the mean number of



(a) Trajectory 1: no birth events

(b) Trajectory 2: 1 birth event

(c) Trajectory 3: 4 birth events

(d) Trajectory 4: 6 birth events

(e) Trajectory 5: 10 birth events

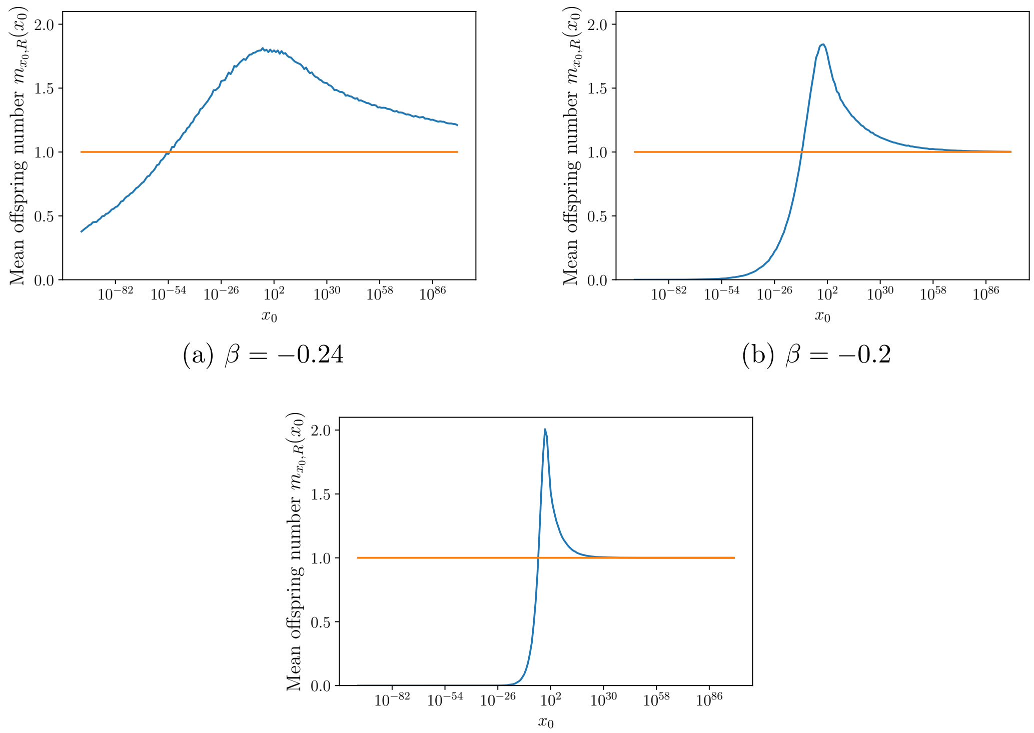
(f) Trajectory 6: 25 birth events

Figure I.3: Different shapes for individual trajectories, with parameters $x_0 = \xi_0 = 1$, $\beta = -0.2$, $C_\beta = 2$, $C_\delta = 0.5$. Most of the time, we obtain the shapes of I.3a and I.3b, and the rarest trajectory is I.3f. The dashed vertical orange line represents the time of death.

offspring can be very high due to very rare events with a huge amount of births, which highlights a classical weakness of Monte-Carlo estimates. Simulations for others value of x_0 lead to a similar behavior, so we conjecture that we observe this unstability because $m_{x_0,R}(x_0) = +\infty$ for every x_0 for these values of β.

I.5.4 Sufficient and non sufficient conditions to have supercriticality in the I_1 case ($\beta = \alpha - 1$)

We illustrate that condition (I.2.8) is sufficient to obtain Assumption I.1.4, and that (I.2.6) is not sufficient (see Section I.4.10 for theoretical proofs). On Figure I.6, we represent the average number of offspring $m_{x_0,R}(x_0)$ depending on x_0. We illustrate that $m_{x_0,R}(x_0) < C_\beta/C_\delta$ (the blue line is always below the orange one), which is proven in Section I.4.10



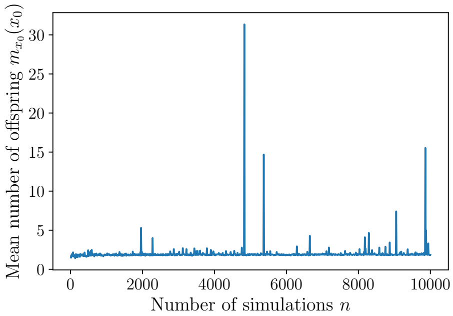
(a) $\beta = -0.24$

(b) $\beta = -0.2$

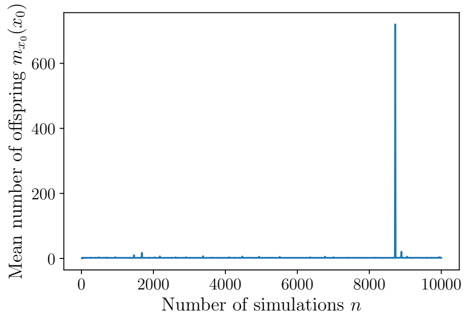
(c) $\beta = -0.1$

Figure I.4: Monte Carlo estimation of the average number of offspring $m_{x_0,R}(x_0)$ for $x_0 \in [10^{-100}, 10^{100}]$, plotted on a log-scale, with parameters $C_\beta = 2$, $C_\delta = 0.5$, and different values of $\alpha - 1 = -0.25 < \beta < -0.05 = \alpha - 1 + \frac{C_\delta}{C_\gamma - C_\alpha}$. For every value of x_0, we simulated $n = 50000$ Monte-Carlo samples to obtain an estimation of $m_{x_0,R}(x_0)$. If at some x_0, the blue line is above the orange line, the numerical simulation suggests that the population process is supercritical for the corresponding value of x_0, and if it is below, the population process is subcritical.

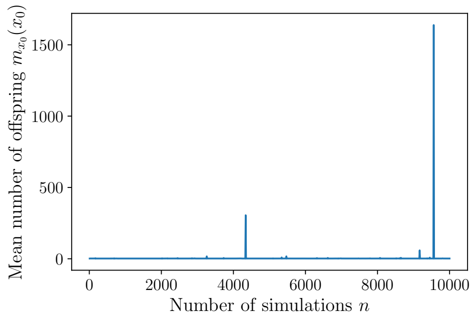
(see (I.4.49)). This is a major difference between the I_1 and the I_2 case: in the I_1 case, we prove that $m_{x_0,R}(x_0)$ is always finite, and we conjecture that in the I_2 case, it is always infinite. The scale of the x-axis is extremely large comparatively to the scale of the y-axis. Hence, we can consider that the variations of the blue curve are essentially due to the intrinsic randomness of the simulations, and that $m_{x_0,R}(x_0)$ is independent from x_0, which is really different from the situation depicted on Figure I.4. This illustrates the independence of our model dynamics along the energy at birth x_0 when $\beta = \delta = \alpha - 1$. It was possible to foresee this result, because in this particular case, the ratio d/b_{x_0} is constant on $]x_0, +\infty[$. A simple time change shows that the birth and death dynamics of the processes $(\xi_{x_0,R,x_0})_{x_0>0}$ are the same (up to a time scaling), no matter the value of x_0. On Figure I.6a, we choose parameters that verify (I.2.6) (and not (I.2.8)), but Assumption I.1.4 is not verified as $m_{x_0,R}(x_0) < 1$, so that (I.2.6) is not sufficient for Assumption I.1.4. On Figure I.6d, we choose parameters that verify (I.2.8) and Assumption I.1.4 is verified. Notice that on Figures I.6b and I.6c, Assumption I.1.4 is verified but not (I.2.8), so that this condition is not necessary.



(a) $\beta = 0.26$



(b) $\beta = 1$



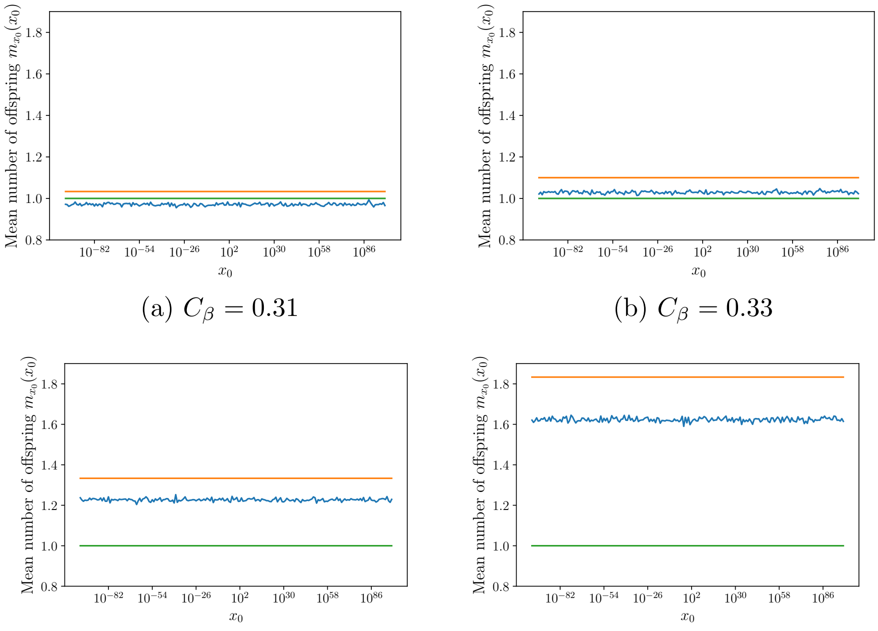
(c) $\beta = 3$

Figure I.5: Monte-Carlo estimation of the mean number of offspring $m_{x_0}(x_0)$ for $x_0 = 1$, $C_\beta = 2$, $C_\delta = 0.5$ and different values of β. The x-axis represents the number n of independent individual trajectories simulated to estimate $m_{x_0}(x_0)$.

I.5.5 Necessary and sufficient condition for supercriticality in the I_1 case

We have verified numerically in Section I.5.4 (and we prove it in Section I.4.10) that for the case $\beta = \delta = \alpha - 1$, the necessary condition (I.2.6) and the sufficient condition (I.2.8), respectively highlighted in Theorem I.2.1 and Proposition I.2.2, are not sharp. This is why, for a given value of $C_\gamma - C_\alpha$, we numerically search for a necessary and sufficient condition on the parameters to have a real equivalence in Theorem I.2.1 (this gives rise to Conjecture I.2.2). Precisely, we numerically find a function $\Xi : C_\delta/(C_\gamma - C_\alpha) \in]0,1[\mapsto \Xi(C_\delta/(C_\gamma - C_\alpha)) \in]1,+\infty[$ such that Assumptions I.1.1, I.1.2 and I.1.4 are verified, if and only if $C_\delta < C_\gamma - C_\alpha$ and $\frac{C_\beta}{C_\delta} > \Xi(C_\delta/(C_\gamma - C_\alpha))$.

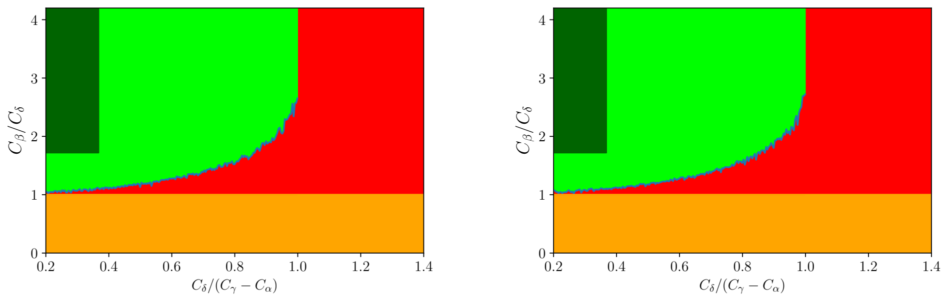
On Figure I.7, we represent in green the region of couples $(C_\beta/C_\delta, C_\delta/(C_\gamma - C_\alpha))$ for which the regime is supercritical. We observe that if $C_\delta/(C_\gamma - C_\alpha) \geq 1$ then the population process is subcritical. Moreover, the boundary between subcritical and supercritical regimes (see the blue curve) draws a convex increasing curve, leading to the function Ξ in our Conjecture I.2.2, defined on $]0, 1[$. Finally, Figures I.7a and I.7b are strikingly similar (and it is a general observation for other values of $C_\gamma - C_\alpha$), which is why we formulate Conjecture I.2.2 with Ξ depending on the quotient $C_\delta/(C_\gamma - C_\alpha)$, rather than on the couple $(C_\delta, C_\gamma - C_\alpha)$.



(a) $C_\beta = 0.31$ (b) $C_\beta = 0.33$

(c) $C_\beta = 0.4$ (d) $C_\beta = 0.55$

Figure I.6: Monte Carlo estimation of the average number of offspring $m_{x_0,R}(x_0)$ for $x_0 \in [10^{-100}, 10^{100}]$, plotted on a log-scale, and different values of C_β, with parameters $\beta = \alpha - 1$, $C_\delta = 0.3$. For every value of x_0, we simulated $n := 50000$ Monte-Carlo samples to obtain an estimation of $m_{x_0,R}(x_0)$, represented in blue. The orange line is constant equal to C_β/C_δ, and the green line constant equal to 1. If at some x_0, the blue line is above the green line, the population process is supercritical for the corresponding value of x_0, and if it is below, the population process is subcritical.



(a) $C_\gamma - C_\alpha = 0.1$ (b) $C_\gamma - C_\alpha = 1$

Figure I.7: Supercriticality of the population process with respect to C_β/C_δ and $C_\delta/(C_\gamma - C_\alpha)$, for $x_0 = 1$, $\beta = \alpha - 1$ and two values of $C_\gamma - C_\alpha$. The dark and light green areas correspond to parameters leading to a supercritical regime, whereas the orange and red areas lead to critical or subcritical regimes. The set of parameters for which the necessary condition for supercriticality (I.2.6) is not verified is the orange area, and the set of parameters verifying the sufficient condition for supercriticality (I.2.8) are in the dark green area. For $0.2 \leq C_\delta/(C_\gamma - C_\alpha) < 1$, the blue curve describes the value of transition from (sub)critical to supercritical regime. For $C_\delta/(C_\gamma - C_\alpha) \geq 1$, our simulations led only to (sub)critical regime.

Chapter II– An individual-based stochastic model with variable resources: a tightness result with unbounded growth, birth and death rates

In Section I.1.1 of Chapter I, we described a model for a population consuming a constant resource over time, and thus ignored competition between individuals through resource consumption. This gave rise to a Galton-Watson process (see Proposition I.1.6) and the associated branching property was at the core of most of our reasoning. We investigated under which conditions on allometric coefficients our model ensures survival of the population with positive probability (*i.e.* satisfies Assumption I.1.4). In particular, Proposition I.1.7 reduced the study of the whole population process into the study of one single individual trajectory starting from energy x_0. In Chapter II, the situation becomes very different, because we allow the amount of resources to vary over time, and add a competition term between individuals *via* resource consumption. As in Section I.1.1.3, we gather all individual trajectories into a measure-valued population process. To distinguish between individuals, we define the set of indices as

$$\mathcal{U} := \bigcup_{n \in \mathbb{N}} (\mathbb{N}^*)^{n+1}.$$

Over time, every individual in the population will have an index of the form $u := u_1...u_{n+1}$ with some $n \geq 0$, and some positive integers $u_1, ..., u_n$. At time $t \geq 0$, an alive individual u in the population is characterized by its energy $\xi_t^u \in \mathbb{R}_+^*$, compete for a fluctuating stock of resources $R_t \in \mathbb{R}^+$, can die and reach the cemetery state ∂, and can reproduce several times during its life. When a birth occurs, we add an individual to the population, with a new index in $\mathcal{U}$. In the population, at time $t \geq 0$, we denote as V_t the set of alive individuals at this time. Every alive individual $u \in V_t$ is represented by a Dirac mass at ξ_t^u. Thus, we define the population process μ_t as a point measure given at time t by

$$\mu_t := \sum_{u \in V_t} \delta_{\xi_t^u}.$$

The population represents a specific species, which is characterized by a trait $x_0 > 0$, the energy parents transmit to their offspring. In all the rest of this chapter, we fix the parameter x_0 once and for all. The construction of our processes, and thus the main results of this chapter, will implicitly depend on this parameter, but we will not make it vary in our reasonings. Note that this is an important difference with Chapter I, where the main goal was to construct a model with allometric parameters valid for every value of x_0. In Chapter I, we thus made x_0 vary to obtain necessary conditions on the allometric coefficients for our model to verify Assumptions I.1.1, I.1.2 and I.1.4. Also, the definition (2.2.1) of the measure-valued process $(\mu_t)_t$ is the same as the one of Section I.1.1.3, but importantly in this chapter, as resources are meant to vary over time, we cannot construct individual trajectories independently from the evolution of $(R_t)_t$ over time. This also entails that contrary to the general setting described in Section I.1.1, we will not benefit from any branching property anymore. Our construction in Section II.1.2 thus focuses directly on the measure-valued population process, instead of constructing and studying only individual trajectories as in Section I.4.1. Whereas Chapter I was dedicated to the study of individual trajectories constructed as $\mathbb{R}_+^*$-valued Piecewise Deterministic Markov Processes, Chapter II focuses on the large population asymptotic study of a measure-valued process with martingales properties, constructed with Poisson point measures.

The first aim of Chapter II is to provide in Section II.1 a rigorous construction of the process $(\mu_t, R_t)_t$, and to give assumptions for this process to be almost surely well-defined for every $t \geq 0$. Then, we introduce in Section II.2 a scaling parameter K, representing the initial population size. We present our main result in Theorem II.3.1 of Section II.3. For any $T \geq 0$, we establish the tightness of a sequence of renormalized processes $(\mu_t^K, R_t^K)_{t\in[0,T]}$ following the previous dynamics, and characterize any accumulation point $(\mu_t^*, R_t^*)_{t\in[0,T]}$ with a deterministic system of integro-differential equations. The proof of Theorem II.3.1 is developed in Section II.4, and we provide extensions of this result in Section II.5. In particular, if there exists a unique solution to the previously mentioned system of equations, then our tightness result becomes a convergence result (see Corollary II.5.2). Finally, in Section II.6, we introduce a new setting with allometric functional parameters that has not been studied yet, and we draw links between our model and specific frameworks existing in the literature. In Appendix B.1, we discuss the assumptions made in Section II.1. In Appendix B.2, we develop the topological background of this chapter. In Appendix B.3, we provide the proofs of intermediate theorems used in Section II.4.

This work is in line with a rich literature about similar tightness results for individual-based models, originating with [FM04], followed by [CFM08], adding an age structure in [Tra08], an interaction with resources in [CF15], and a diffusion term in [Tch24]. One common feature of these papers is that birth and death rates are bounded, and any accumulation point μ^* integrates bounded functions. Our work goes beyond these previous results for the following reasons. First, our birth and death rates are not necessarily bounded, which adds potential problems of explosion of individual energy or population size in finite time. Thus, our tightness result in Theorem II.3.1 occurs in a weighted space of measures integrating a weight ω, adapted to these unbounded rates. Then, any accumulation point μ^* integrates a broader set of functions, not necessarily bounded (see Corollary II.5.3). Because of our biological interests, we illustrate our results with the example of allometric functional parameters.

II.1 Definitions and assumptions

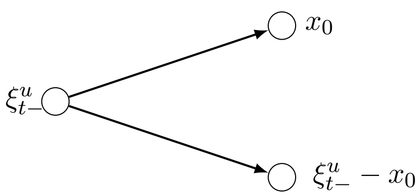
First, in Section II.1.1, we define individual dynamics, *i.e.* deterministic metabolism and resource consumption, and random birth or death events. Then, in Section II.1.2, we introduce notations to gather all the individual flows and resource dynamics into one deterministic flow between random jumps. Then, we provide an algorithmic construction of the process $(\mu_t, R_t)_t$ with Poisson point measures, valid up to a stopping time accounting for two possible problematic events. The first one is the possibility for individual trajectories to reach 0 or $+\infty$ in finite time between random jumps, and the second one is the possible accumulation of jump times at the population level. In Section II.1.3, we address the first problematic event, and make assumptions for individual trajectories to almost surely not reach 0 or $+\infty$. Then, in Section II.1.4, we resolve the second problem, by giving assumptions under which the population process is almost surely well-defined for $t \in \mathbb{R}^+$ (*i.e.* there is no accumulation of jump times). Once this is done, in Section II.1.5, we show that $(\mu_t, R_t)_{t\geq 0}$ is a Jumping Markov Process (JMP). This particular type of Feller process was initially introduced by Davis [Dav84] for $\mathbb{R}^n$-valued processes, and called Piecewise Deterministic Markov Processes (PDMP). For a general definition and systematic study of this kind of processes, adapted to our measure-valued setting, we refer to the work of Jacod and Skorokhod [JS96]. We further characterize the Feller process $(\mu_t, R_t)_{t\geq 0}$ *via* its extended generator (see p.45 in [JS96]).

II.1.1 Individual and resource dynamics

In this section, we introduce the functional parameters and main mechanisms of our model, but note that the formal construction of the process is done in Section II.1.2. An individual trajectory will be deterministic between some random jump times, corresponding to birth or death events. We give first a general framework under which, for any $t \geq 0$, $u \in \mathcal{U}$, an individual energy ξ^u_t is in $\mathbb{R}^*_+ \cup \{\partial\}$, where ∂ is a cemetery state. *A priori*, without further assumptions, an individual energy could reach 0 or $+\infty$ in finite time, and we precise our convention in that case in the following. In Section II.1.3, we will give assumptions to ensure that almost surely, the previous situation does not happen.

Births

For $t \geq 0$, an individual indexed by $u \in V_{t-}$ with energy $\xi^u_{t-} \in \mathbb{R}^*_+$ gives birth to a single offspring at rate $b(\xi^u_{t-})$. This individual transfers a constant amount of energy x_0 to the offspring. The energy of the individual goes from ξ^u_{t-} to $\xi^u_t := \xi^u_{t-} - x_0$. The point measure then jumps to $\mu_t := \mu_{t-} - \delta_{\xi^u_{t-}} + \delta_{\xi^u_{t-}-x_0} + \delta_{x_0}$. If the parent has the label $u := u_1...u_n$ and this is the k-th birth jump for this parent for $k \geq 1$, the index of the offspring is $uk := u_1...u_nk$. Then, we set $V_t := V_{t-} \cup \{uk\}$.



We assume that the birth rate b is equal to 0 for every $x \leq x_0$, so that no individual with negative energy appears during a birth event. Also, we assume that b is non-negative and continuous for $x > x_0$.

Deaths

For $t \geq 0$, an individual indexed by $u \in V_{t-}$ with energy $\xi^u_{t-} \in \mathbb{R}^*_+$ dies at positive and continuous rate $d(\xi^u_{t-})$. Then, the individual process jumps to $\xi^u_t := \partial$ and we set $\xi^u_s := \partial$

for every $s \geq t$. The point measure jumps to $\mu_t := \mu_{t-} - \delta_{\xi^u_{t-}}$, and we set $V_t := V_{t-} \setminus \{u\}$.

Energy loss and resource consumption

1. For $t \geq 0$, an individual $u \in V_t$ with energy $\xi^u_t \in \mathbb{R}^*_+$ loses energy over time, at non-negative and $\mathcal{C}^1(\mathbb{R}^*_+)$ rate $\ell(\xi^u_t)$.

2. In order to balance this energy loss, this individual consumes the resource at rate $f(\xi^u_t, R_t)$. Importantly, we suppose that f is of the form $f(\xi, R) := \phi(R)\psi(\xi)$, where ψ is a $\mathcal{C}^1(\mathbb{R}^*_+)$ positive function. Also, we assume that ϕ is a $\mathcal{C}^1(\mathbb{R}^+)$ non-decreasing function on $\mathbb{R}^+$, verifying $\phi(0) = 0$ and $\lim_{x\to+\infty} \phi(x) = 1$. Finally, we suppose that ϕ is Lipschitz continuous on $\mathbb{R}^+$, meaning that

$$\exists k > 0, \forall R_1, R_2 \geq 0, \forall x > 0, \quad |f(x, R_1) - f(x, R_2)| \leq k|R_1 - R_2|\psi(x). \tag{II.1.1}$$

Thus, between two random jump times (due to birth or death events), the energy evolves according to the following equation:

$$\frac{\mathrm{d}\xi^u_t}{\mathrm{d}t} = f(\xi^u_t, R_t) - \ell(\xi^u_t) =: g(\xi^u_t, R_t). \tag{II.1.2}$$

Remark that $g : (x, R) \mapsto g(x, R)$ is $\mathcal{C}^{1,1}(\mathbb{R}^*_+ \times \mathbb{R}^+)$, meaning that it is differentiable with continuous derivatives in both its variables. Also, one can replace f by g in (II.1.1). It is possible that at some time $t \geq 0$, ξ^u_t reaches either 0 or $+\infty$ (vanishing energy or explosion of (II.1.2) in finite time before a jump event). We will make assumptions in Section II.1.3 to avoid this situation almost surely. For now, if this happens at some time t, we adopt the convention $\xi^w_s := \partial$ for every $s \geq t$ and $w \in V_{t-}$. Also, we set the point measure to $\mu_s := 0$ for all $s \geq t$, and $V_s := \varnothing$ for $s \geq t$. Finally, for any $t \geq 0$ and $u \in \mathcal{U}$, if $u \notin V_t$, we set $\xi^u_t := \partial$ (in particular, individuals that are not born yet at time t have energy ∂).

Resource dynamics

Let $\chi > 1$, $R_{\max} > 0$ and $\varsigma \in \mathcal{C}^1(\mathbb{R}^+)$, such that $\varsigma(0) \geq 0$ and $\varsigma(R) < 0$ if $R \geq R_{\max}$. Between random jumps, the quantity of resource $R_t \in \mathbb{R}^+$ satisfies the following equation

$$\frac{\mathrm{d}R_t}{\mathrm{d}t} = \rho(\mu_t, R_t) := \varsigma(R_t) - \chi \int_{\mathbb{R}^*_+} f(x, R_t)\mu_t(\mathrm{d}x). \tag{II.1.3}$$

The function ρ is well-defined on $\mathcal{M}_P(\mathbb{R}^*_+) \times \mathbb{R}^+$, where $\mathcal{M}_P(\mathbb{R}^*_+)$ is the space of finite point measures on $\mathbb{R}^*_+$. Equation (II.1.3) means that in the absence of individuals in the population, the amount of resource eventually stabilizes at some $R_{\mathrm{eq}} \in [0, R_{\max}]$. For the renewal function ς, one can think for example of a logistic growth $R \geq 0 \mapsto R(R_{\mathrm{eq}} - R)$ with $R_{\mathrm{eq}} \leq R_{\max}$. This is a classical assumption in literature for biotic resources [BLLD11, YKR18, FBC21]. Another example is the case of a chemostat, where we can take $\varsigma : R \geq 0 \mapsto D(R_{\mathrm{in}} - R)$, with a dilution rate $D > 0$, and the constant $R_{\mathrm{in}} \leq R_{\max}$ can be interpreted as an abiotic nutrient flow in the chemostat [CF15]. The coefficient χ can be interpreted as the inverse of the conversion efficiency. The ratio $1/\chi < 1$ represents the proportion of resource consumed by individuals effectively converted to energy. In this chapter, we suppose that χ is a constant, which is a usual assumption in literature [LL05], even if in [FBC21], the authors make χ depend on individual energy over time. The integral quantity represents the speed at which the whole population consumes the resource at time t. This non-linear term is an indirect source of competition between individuals.

If we choose an initial condition $R_0 \in [0, R_{\max}]$, Equation (II.1.3) enforces that $R_t \in [0, R_{\max}]$ for every $t \geq 0$, as long as $\int_{\mathbb{R}_+^*} \psi(x)\mu_t(\mathrm{d}x) < +\infty$ (in particular, this is the case if $\mu_t \in \mathcal{M}_P(\mathbb{R}_+^*)$). Moreover, if $R_0 > R_{\max}$, even with no individuals in the population, the resource will decrease to $R_{\max}$. Hence, without loss of generality, we assume that $R_0 \in [0, R_{\max}]$. Also, we make the following assumption.

Assumption II.1.1. $\forall x > 0 \quad g(x, R_{\max}) > 0.$

If Assumption II.1.1 is not verified, as ϕ is non-decreasing, we obtain

$$\exists M > 0, \forall x \leq M, \forall R \in [0, R_{\max}] \quad g(x, R) \leq 0.$$

Considering (II.1.2), this means that if the initial energy of an individual is in $(0, M]$, then it will remain in this compact set over time. In other terms, if Assumption II.1.1 is not verified, we impose an *a priori* upper bound on the maximal energy in our model. Similar mass-structured models where the maximal mass M of an individual is deterministically bounded are already developed in previous works [CF15, CCF16]. Although observed living species obviously have bounded masses, we want to design a model where this bound is not artificially imposed by the model, but results from interaction with a limiting resource. This is why in our setting, we allow individual energies to increase indefinitely, at least if there are sufficient resources (*i.e.* with $R_{\max}$ resources are available), which is expressed in Assumption II.1.1.

Lemma II.1.2. *Assumption II.1.1 implies that*

$$\phi(R_{\max}) > 0 \quad \text{and} \quad \forall x > 0,\ \psi(x) > \ell(x).$$

Proof. This is straightforward from our assumptions on ψ, ϕ and ℓ. □

Remark: We recover a condition expressed in Assumption I.1.8. It was a natural necessary condition in the allometric case to obtain a supercritical branching process, in the case where there was no competition for resources. In all the rest of this chapter, we implicitly work under Assumption II.1.1.

II.1.2 Algorithmic construction of the process with Poisson point measures

II.1.2.1 Deterministic flow between random jumps

We begin with the definition of the deterministic flow associated to individual energies and the amount of resources between random jumps. We provide $\mathcal{U}$ with the lexicographical order, denoted as $\prec$, and consider a finite subset $V \subseteq \mathcal{U}$ of cardinality $|V| \in \mathbb{N}$. It means that there exists $u_1 \prec ... \prec u_{|V|}$ elements of $\mathcal{U}$, such that $V = \{u_1, ..., u_{|V|}\}$, with $V = \varnothing$ if $|V| = 0$. Let us fix an initial condition $R_0 \in [0, R_{\max}]$ and $(\xi_0^{u_j})_{1 \leq j \leq |V|} \in \left(\mathbb{R}_+^*\right)^{|V|}$ individual energies indexed by V. In the following, we will lighten this notation into $\Xi_0 := (\xi_0^u)_{u \in V}$. We write $\left((X_t^u(\Xi_0, R_0))_{u \in V}, X_t^{\Re}(\Xi_0, R_0)\right)$ for a solution to the system of $|V| + 1$ coupled equations

$$\frac{\mathrm{d}X_t^{\Re}(\Xi_0, R_0)}{\mathrm{d}t} = \rho\left(\sum_{u \in V} \delta_{X_t^u(\Xi_0, R_0)}, X_t^{\Re}(\Xi_0, R_0)\right), \tag{II.1.4}$$

$$\frac{\mathrm{d}X_t^u(\Xi_0, R_0)}{\mathrm{d}t} = g(X_t^u(\Xi_0, R_0), X_t^{\Re}(\Xi_0, R_0)) \quad \text{for } u \in V, \tag{II.1.5}$$

with initial condition at time 0

$$\begin{array}{rcl} X_0^{\Re}(\Xi_0, R_0) & = & R_0, \\ X_0^u(\Xi_0, R_0) & = & \xi_0^u \quad \text{for } u \in V. \end{array}$$

Proposition II.1.3. *Let $V \subseteq \mathcal{U}$ be finite and $((\xi_0^u)_{u\in V}, R_0) \in (\mathbb{R}_+^*)^{|V|} \times [0, R_{\max}]$. Then, there exists a neighborhood $O \subseteq (\mathbb{R}_+^*)^{|V|} \times [0, R_{\max}]$ of $((\xi_0^u)_{u\in V}, R_0)$, and a neighborhood $J \subseteq \mathbb{R}^+$ of 0, such that*

1. *For every $((\xi^u)_{u\in V}, R) \in O$, there exists a unique local solution with values in $(\mathbb{R}_+^*)^{|V|} \times [0, R_{\max}]$ to the system of coupled equations (II.1.4)-(II.1.5), starting from $((\xi^u)_{u\in V}, R)$ at time 0. This solution is at least defined on J, and denoted as $\big((X_t^u(\Xi, R))_{u\in V}, X_t^{\Re}(\Xi, R)\big)$, with $\Xi := (\xi^u)_{u\in V}$.*

2. *The function $(t, (\xi^u)_{u\in V}, R) \in J \times O \mapsto \big((X_t^u(\Xi, R))_{u\in V}, X_t^{\Re}(\Xi, R)\big)$ is $\mathcal{C}^{1,1}(J \times O)$, and $\mathcal{C}^2$ in the variable t.*

Proof. As $\varsigma \in \mathcal{C}^1(\mathbb{R}^+)$, and f, g are $\mathcal{C}^{1,1}(\mathbb{R}_+^* \times \mathbb{R}^+)$, classical arguments entails the result (see Corollaire II.2. and Théorème II.10. in Chapter X of [ZQ96]). □

Remark: The previous objects do not depend on the set of indices V. We use these notations to be able to identify any individual by an index and keep track of its energy over time, in the upcoming construction of our population process.

We introduce $t_{\exp}(\Xi_0, R_0) \in (0, +\infty]$ the maximal time of existence of the solution to (II.1.4)-(II.1.5) starting from (Ξ_0, R_0) at time 0 highlighted in Proposition II.1.3. The deterministic time $t_{\exp}(\Xi_0, R_0)$ is finite, if and only if one of the $X_t^u(\Xi_0, R_0)$ reaches 0 or $+\infty$ in finite time. Finally, we define a flow X with a measure-valued first component, to be able to use it in the upcoming definition of the stochastic measure-valued process $(\mu_t, R_t)_t$. Suppose that $\mu_0 \in \mathcal{M}_P(\mathbb{R}_+^*)$ is such that

$$\mu_0 = \sum_{u\in V} \delta_{\xi_0^u},$$

with $\mu_0 = 0$ if $V = \varnothing$. Then, in the following, we write $t_{\exp}(\mu_0, R_0) := t_{\exp}(\Xi_0, R_0)$, and this does not depend on the set of indices V. Also, for $t \in [0, t_{\exp}(\mu_0, R_0))$, we define

$$X_t(\mu_0, R_0) := \left(\sum_{u\in V} \delta_{X_t^u(\Xi_0, R_0)}, X_t^{\Re}(\Xi_0, R_0)\right),$$

and this again does not depend on the indexing by V. Finally, we adopt the convention $X_t(\mu_0, R_0) := (0, 0)$ for every $t \geq t_{\exp}(\mu_0, R_0)$. We thus have defined the deterministic flow with measure-valued first component

$$\begin{array}{rcccl} X & : & \mathcal{M}_P(\mathbb{R}_+^*) \times \mathbb{R}^+ \times \mathbb{R}^+ & \to & \mathcal{M}_P(\mathbb{R}_+^*) \times \mathbb{R}^+ \\ & & (\mu, R, t) & \mapsto & X_t(\mu, R). \end{array}$$

In the following, for μ measure on $\mathbb{R}_+^*$ and f measurable from $\mathbb{R}_+^*$ to $\mathbb{R}$, we write

$$\langle \mu, f \rangle := \int_{\mathbb{R}_+^*} f \mathrm{d}\mu.$$

II.1.2.2 Algorithmic construction of the process

First, we define individual energies $((\xi_t^u)_{u\in\mathcal{U}})_t$, the set of alive individuals $(V_t)_t$ and the amount of resources $(R_t)_t$ inductively, by constructing a sequence of successive random jump times $(J_n)_{n\geq 0}$, between which the dynamics are deterministic (note immediately that our construction will then be valid only up to time $\sup_{n\in\mathbb{N}} J_n$). Then, we gather individual processes into a measure-valued process $(\mu_t)_t$. Between two jump times, the process $(\mu_t, R_t)_t$ will be deterministic and will follow the flow with measure-valued first component X defined in Section II.1.2.1. We consider $\mathcal{N}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h)$ and $\mathcal{N}'(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h)$ two independent Poisson point measures on $\mathbb{R}^+\times\mathcal{U}\times\mathbb{R}_+^*$, with intensity $\mathrm{d}s\times n(\mathrm{d}u)\times\mathrm{d}h$, with $n(\mathrm{d}u) := \sum_{w\in\mathcal{U}}\delta_w(\mathrm{d}u)$. The support of $\mathcal{N}$, respectively $\mathcal{N}'$, on $\mathbb{R}^+\times\mathcal{U}\times\mathbb{R}_+^*$ is a countable random set, denoted as $\mathrm{supp}(\mathcal{N})$, respectively $\mathrm{supp}(\mathcal{N}')$. This is a random variable verifying $\mathcal{N}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h) = \sum_{(s,u,h)\in\mathrm{supp}(\mathcal{N})}\delta_{(s,u,h)}$, respectively the same equation with $\mathcal{N}'$.

For the initial condition, let (μ_0, R_0) be a random variable taking values in $\mathcal{M}_P(\mathbb{R}_+^*)\times[0, R_{\max}]$. We define $N_0 := \langle\mu_0, 1\rangle$ the initial number of individuals, $V_0 := \{1, ..., N_0\}\subseteq\mathcal{U}$, and $\Xi_0 := (\xi_0^u)_{u\in V_0}$ the initial individual energies. Thus, we index alive individuals at time 0 so that $\mu_0 := \sum_{u\in V_0}\delta_{\xi_0^u}$. Also, for $u\notin V_0$, we set $\xi_0^u := \partial$. The Poisson point measures $\mathcal{N}$ and $\mathcal{N}'$ are independent from (μ_0, R_0). The canonical filtration associated to (μ_0, R_0), $\mathcal{N}$ and $\mathcal{N}'$ is $(\mathcal{F}_t)_{t\geq 0}$.

We now define the sequence $(J_n)_{n\in\mathbb{N}}$ of successive jump times of the population process. First, we set $J_0 := 0$, and then suppose that our process is described until some time $J_n < +\infty$ with $n \geq 0$. At time J_n, there exists a finite $V_{J_n}\subseteq\mathcal{U}$, associated to individual energies $\Xi_{J_n} := (\xi_{J_n}^u)_{u\in V_{J_n}}$. With the convention $\inf(\varnothing) = +\infty$, we define

$$\begin{aligned}J_{n+1}^b := \inf\{t \in (J_n, J_n + t_{\exp}(\Xi_{J_n}, R_{J_n})),\ (t, u, h) \in \mathrm{supp}(\mathcal{N}),\\ u \in V_{J_n},\ h \leq b\left(X_{t-J_n}^u(\Xi_{J_n}, R_{J_n})\right)\},\\ J_{n+1}^d := \inf\{t \in (J_n, J_n + t_{\exp}(\Xi_{J_n}, R_{J_n})),\ (t, u, h) \in \mathrm{supp}(\mathcal{N}'),\\ u \in V_{J_n},\ h \leq d\left(X_{t-J_n}^u(\Xi_{J_n}, R_{J_n})\right)\}.\end{aligned}$$

First if $J_n + t_{\exp}(\Xi_{J_n}, R_{J_n}) = J_{n+1}^b \wedge J_{n+1}^d = +\infty$, it means that there are no jumps anymore, and no explosion of the solution to (II.1.4)-(II.1.5) starting from (Ξ_{J_n}, R_{J_n}). Then, we set $J_{n+1} = +\infty$, and for $t \geq J_n$, $V_t = V_{J_n}$. Concerning individual energies, for $t \geq J_n$, if $u \notin V_{J_n}$, then $\xi_t^u = \partial$, and

$$\left((\xi_t^u)_{u\in V_{J_n}}, R_t\right) = \left((X_{t-J_n}^u(\Xi_{J_n}, R_{J_n}))_{u\in V_{J_n}}, X_{t-J_n}^{\Re}(\Xi_{J_n}, R_{J_n})\right).$$

Else if $J_n + t_{\exp}(\Xi_{J_n}, R_{J_n}) < J_{n+1}^b \wedge J_{n+1}^d = +\infty$, it means that one or several individual energies reach 0 or $+\infty$ at time $J_n + t_{\exp}(\Xi_{J_n}, R_{J_n})$ (which is an event that we will avoid almost surely in Section II.1.3). Then, we set $J_{n+1} = J_n + t_{\exp}(\Xi_{J_n}, R_{J_n})$, and for $t \in (J_n, J_{n+1})$, we set $V_t = V_{J_n}$ and if $u \notin V_{J_n}$, then $\xi_t^u = \partial$. In addition, for $t \in (J_n, J_{n+1})$, we set

$$\left((\xi_t^u)_{u\in V_{J_n}}, R_t\right) = \left((X_{t-J_n}^u(\Xi_{J_n}, R_{J_n}))_{u\in V_{J_n}}, X_{t-J_n}^{\Re}(\Xi_{J_n}, R_{J_n})\right).$$

Next, we set $V_{J_{n+1}} = \varnothing$, $R_{J_{n+1}} = R_{J_{n+1}-}$ and for $u \in \mathcal{U}$, $\xi_{J_{n+1}}^u = \partial$. Remark that with these conventions, we necessarily get back to the first case described above for the definition of J_{n+2} (so $J_{n+2} = +\infty$), and eventually obtain that for all $t \geq J_{n+1}$, $V_t = \varnothing$ and for $u \in \mathcal{U}$, $\xi_t^u = \partial$.

Finally, if $J^b_{n+1} \wedge J^d_{n+1} < J_n + t_{\exp}(\Xi_{J_n}, R_{J_n})$, it means that one birth or death event occurs. By property of Poisson point measures, we almost surely have $J^b_{n+1} \neq J^d_{n+1}$, and the infimum in the definition of J^b_{n+1} or J^d_{n+1} is reached at a single element $(t, u, h) \in \mathrm{supp}(\mathcal{N})$ or $\mathrm{supp}(\mathcal{N}')$. We distinguish again between two cases.

- First if $J^b_{n+1} < J^d_{n+1}$, it means that one birth event occurs. This event concerns an individual indexed by some $w \in V_{J_n}$. Then, we set $J_{n+1} = J^b_{n+1}$, and for all $t \in [J_n, J_{n+1})$, $V_t = V_{J_n}$. Concerning individual energies, if $u \notin V_{J_n}$, we set $\xi^u_t = \partial$ and
$$\left((\xi^u_t)_{u \in V_{J_n}}, R_t\right) = \Big((X^u_{t-J_n}(\Xi_{J_n}, R_{J_n}))_{u \in V_{J_n}}, X^{\Re}_{t-J_n}(\Xi_{J_n}, R_{J_n})\Big).$$
Then, at time J_{n+1}, a new individual appears in the population. We set $V_{J_{n+1}} = V_{J_n} \cup \{wk\}$, where $k-1$ is the number of offspring individual w already produced (*i.e.* the cardinality of the set $\{(t, u, h) \in \mathrm{supp}(\mathcal{N}),\ u = w,\ \exists\, 1 \le m \le n,\ J^b_m = t\}$). We also set $\xi^{wk}_{J_{n+1}} = x_0$, $\xi^w_{J_{n+1}} = \xi^w_{J_{n+1}-} - x_0$ and $\xi^u_{J_{n+1}} = \xi^u_{J_{n+1}-}$ for $u \in \mathcal{U} \setminus \{w, wk\}$. Finally, $R_{J_{n+1}} = R_{J_{n+1}-}$.

- Else if $J^d_{n+1} < J^b_{n+1}$, it means that a death event occurs. This event concerns an individual indexed by some $w \in V_{J_n}$. Then, we set $J_{n+1} = J^d_{n+1}$, and for all $t \in [J_n, J_{n+1})$, $V_t = V_{J_n}$. Concerning individual energies, if $u \notin V_{J_n}$, we set $\xi^u_t = \partial$ and
$$\left((\xi^u_t)_{u \in V_{J_n}}, R_t\right) = \Big((X^u_{t-J_n}(\Xi_{J_n}, R_{J_n}))_{u \in V_{J_n}}, X^{\Re}_{t-J_n}(\Xi_{J_n}, R_{J_n})\Big).$$
Then, at time J_{n+1}, individual w disappears from the population. We set $V_{J_{n+1}} = V_{J_n} \setminus \{w\}$. We also set $\xi^w_{J_{n+1}} = \partial$ and $\xi^u_{J_{n+1}} = \xi^u_{J_{n+1}-}$ for $u \in \mathcal{U} \setminus \{w\}$. Finally, $R_{J_{n+1}} = R_{J_{n+1}-}$.

By convention, for all $n \in \mathbb{N}$, $J^b_{n+1} = J^d_{n+1} = J_{n+1} = +\infty$ if $J_n = +\infty$. The sequence $(J_n)_{n \in \mathbb{N}^*}$ is non-decreasing, so we can define $J_\infty := \lim_{n \to +\infty} J_n$. Eventually, for every $t \in [0, J_\infty)$, we define μ_t as in (2.2.1), with $\mu_t = 0$ if $V_t = \varnothing$. We verify immediately that, if no individual energy reaches 0 or explodes, for all $n \in \mathbb{N}$, for all $t \in [J_n, J_{n+1})$, μ_t coincide with the deterministic flow $X_{t-J_n}(\mu_{J_n}, R_{J_n})$, and is modified at any jump time according to the rules given in Section II.1.1. Note that μ_t (as well as the $(\xi^u_t)_{u \in \mathcal{U}}$, V_t and R_t) is well-defined only for $t \in [0, J_\infty)$, with J_∞ possibly finite or infinite. For all $t \in [0, J_\infty)$, V_t is the set containing the indices of alive individuals at time t. The previously described update rules of the set V_t at each jump event make it adapted with respect to the filtration $(\mathcal{F}_t)_t$. Remark that for every $t \in [0, J_\infty)$, if $u \notin V_t$, then $\xi^u_t = \partial$; and if $u \in V_t$, then $\xi^u_t \in \mathbb{R}^*_+$. In the following, we want to avoid almost surely the following situations:

Situation 1 One of the individual energy vanishes/explodes.

Situation 2 There is an accumulation of jump times.

We define
$$\tau_{\exp} := \inf\{J_n,\ J_{n+1} = J_n + t_{\exp}(\Xi_{J_n}, R_{J_n}) < +\infty\},$$
with the convention $\inf(\varnothing) = +\infty$. To avoid Situation 1, respectively Situation 2, we need to ensure that almost surely, $\tau_{\exp} = +\infty$, respectively $J_\infty = +\infty$. In Section II.1.3, we give assumptions under which $\tau_{\exp} = +\infty$ holds true almost surely. In Section II.1.4, we work under the assumptions of Section II.1.3, and we give further assumptions, under which $J_\infty = +\infty$ holds true almost surely. To end this section, we also provide in Section II.1.5 martingale properties for our process when $J_\infty \wedge \tau_{\exp} = +\infty$ almost surely.

II.1.2.3 Classical writing of the process on $[0, J_\infty \wedge \tau_{\exp})$

Before time $\tau_{\exp}$, individual energies never vanish or explode, hence the global jump rate of the population is finite at any time in $[0, J_\infty \wedge \tau_{\exp})$. Thus, for every $J_n < \tau_{\exp}$, the next jump time J_{n+1} is almost surely either $+\infty$ or a birth/death jump in the inductive construction of Section II.1.2. Then, for every $t \in [0, J_\infty \wedge \tau_{\exp})$, we can almost surely write, with the convention on the flow X in mind,

$$
\begin{aligned}
(\mu_t, R_t) = X_t(\mu_0, R_0)& \\
&+ \int_0^t \int_{\mathcal{U}\times\mathbb{R}_+^*} \mathbb{1}_{\{u\in V_{s-}\}}\mathbb{1}_{\{h\leq b(\xi^u_{s-})\}}[X_{t-s}(\mu_{s-} + \delta_{x_0} + \delta_{\xi^u_{s-}-x_0} - \delta_{\xi^u_{s-}}, R_s) \\
&\qquad\qquad - X_{t-s}(\mu_{s-}, R_s)]\mathcal{N}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h) \\
&+ \int_0^t \int_{\mathcal{U}\times\mathbb{R}_+^*} \mathbb{1}_{\{u\in V_{s-}\}}\mathbb{1}_{\{h\leq d(\xi^u_{s-})\}}[X_{t-s}(\mu_{s-} - \delta_{\xi^u_{s-}}, R_s) \\
&\qquad\qquad - X_{t-s}(\mu_{s-}, R_s)]\mathcal{N}'(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h).
\end{aligned}
$$

This formal writing is classical in the literature (Definition 2.4. in [Tra08], Section 4.1 in [CF15]), and should be understood as a telescopic sum. First, individual energies and resources evolve deterministically, following the flow $X_t(\mu_0, R_0)$. Then, at each birth or death event, we erase the current flow and replace it with a new flow, modified according to our birth and death rules.

II.1.3 Assumptions for non vanishing/exploding individual energies

In this section, we introduce assumptions for our stochastic process to verify $\tau_{\exp} = +\infty$ almost surely (see Corollary II.1.7). We adopt the convention that for any $x > 0$, if $\ell(x) = 0$, then $\frac{d}{\ell}(x) = +\infty$. Also, recall that under Assumption II.1.1, $g(., R_{\max})$ is a positive function on $\mathbb{R}_+^*$.

Assumption II.1.4 (Individual energy avoids 0). *For all $x > 0$,*

$$\int_0^x \frac{d}{\ell}(y)\mathrm{d}y = +\infty.$$

Assumption II.1.5 (Individual energy avoids $+\infty$). *For all $x > 0$,*

$$\int_x^{+\infty} \frac{b(y) + d(y)}{g(y, R_{\max})}\mathrm{d}y = +\infty.$$

Proposition II.1.6. *Under Assumptions II.1.4 and II.1.5, we have almost surely*

$$(J_1 = +\infty) \quad \text{or} \quad (J_1 < t_{\exp}(\mu_0, R_0)).$$

Proof. In all the following proof, we suppose by contradiction that $\mathbb{P}(t_{\exp}(\mu_0, R_0) \leq J_1 < +\infty) > 0$, and work under this event. Importantly, this implies that until time $t_{\exp}(\mu_0, R_0)$, there are no random birth or death jumps in the population, so the process $(\mu_t)_t$ is well-defined and deterministic on $[0, t_{\exp}(\mu_0, R_0))$. Thus, under the event

$\{t_{\exp}(\mu_0, R_0) \le J_1 < +\infty\}$, we have $V_0 \neq \varnothing$ (otherwise $t_{\exp}(\mu_0, R_0) = +\infty$) and for every $u \in V_0$ and $s < t_{\exp}(\mu_0, R_0)$, we have $\xi_s^u = X_s^u(\Xi_0, R_0)$ with the notations of Section II.1.2. We assess that one of the two following situations occurs:

(i) $\exists u \in V_0, \quad \xi_s^u \xrightarrow[s \to t_{\exp}(\mu_0, R_0)]{} 0,$

(ii) $\exists u \in V_0, \quad \xi_s^u \xrightarrow[s \to t_{\exp}(\mu_0, R_0)]{} +\infty.$

Indeed, if (i) and (ii) are not verified, then by definition, we would necessarily have $t_{\exp}(\mu_0, R_0) = +\infty$. By construction of our process with Poisson point measures, under the event $\{t_{\exp}(\mu_0, R_0) \le J_1 < +\infty\}$, we have in particular

$$\int_0^{t_{\exp}(\mu_0, R_0)} \left(\sum_{u \in V_0} (b+d)(\xi_s^u) \right) \mathrm{d}s < +\infty. \tag{II.1.6}$$

Indeed, before time $t_{\exp}(\mu_0, R_0)$, individual energies ξ_s^u follow the deterministic flows $X_\cdot^u(\Xi_0, R_0)$, so the previous integral is a deterministic quantity. If the previous integral was infinite, the first jump time J_1 given by the Poisson point measures $\mathcal{N}$ and $\mathcal{N}'$ would follow an inhomogeneous exponential law, and we would have

$$\mathbb{P}(J_1 < t_{\exp}(\mu_0, R_0)) = 1 - \exp\left(\int_0^{t_{\exp}(\mu_0, R_0)} \left(\sum_{u \in V_0} (b+d)(\xi_s^u) \right) \mathrm{d}s \right) = 1.$$

First, suppose that $u \in V_0$ is such that (i) is verified. We have

$$\int_0^{t_{\exp}(\mu_0, R_0)} d(\xi_s^u) \mathrm{d}s \ge \int_0^{t_{\exp}(\mu_0, R_0)} d(\xi_s^u) \mathbb{1}_{\{g(\xi_s^u, R_s) < 0\}} \mathrm{d}s.$$

By (i) and considering (II.1.2), there exists a sequence $(O_n)_{n \in \mathbb{N}}$ of disjoint open intervals such that

$$\{s \in (0, t_{\exp}(\mu_0, R_0)),\ g(\xi_s^u, R_s) < 0\} = \bigsqcup_{n \in \mathbb{N}} O_n \quad \text{and} \quad (0, \xi_0^u) \subseteq \{\xi_s^u,\ s \in \bigsqcup_{n \in \mathbb{N}} O_n\}. \tag{II.1.7}$$

For every $n \in \mathbb{N}$, from (II.1.2), $s \mapsto \xi_s^u$ is a decreasing bijection from O_n to $\xi^u(O_n)$, and we write $\tilde{\xi}^u$ for its inverse function. We perform the change of variables $w = \xi_s^u$ on each interval O_n to obtain

$$\int_0^{t_{\exp}(\mu_0, R_0)} d(\xi_s^u) \mathrm{d}s \ge \sum_{n \in \mathbb{N}} \int_{O_n} d(\xi_s^u) \mathrm{d}s = \sum_{n \in \mathbb{N}} \int_{\xi^u(O_n)} \frac{d(w)}{-g(w, R_{\tilde{\xi}_w^u})} \mathrm{d}w \ge \int_0^{\xi_0^u} \frac{d}{\ell}(w) \mathrm{d}w,$$

because of (II.1.7), and $R \mapsto g(x, R)$ is non-decreasing for every $x > 0$. By (II.1.6), the left-most integral is finite, which contradicts Assumption II.1.4.

Else if there exists $u \in V_0$ such that (ii) is verified, we write in the same manner

$$\int_0^{t_{\exp}(\mu_0, R_0)} (b+d)(\xi_s^u) \mathrm{d}s \ge \int_0^{t_{\exp}(\mu_0, R_0)} (b+d)(\xi_s^u) \mathbb{1}_{\{g(\xi_s^u, R_s) > 0\}} \mathrm{d}s.$$

By (ii) and considering (II.1.2), there exists a sequence $(O'_n)_{n\in\mathbb{N}}$ of disjoint open intervals such that

$$\{s \in (0, t_{\exp}(\mu_0, R_0)),\ g(\xi^u_s, R_s) > 0\} = \bigsqcup_{n\in\mathbb{N}} O'_n \quad \text{and} \quad (\xi^u_0, +\infty) \subseteq \{\xi^u_s,\ s \in \bigsqcup_{n\in\mathbb{N}} O'_n\}. \tag{II.1.8}$$

For every $n \in \mathbb{N}$, from (II.1.2), $s \mapsto \xi^u_s$ is an increasing bijection from O'_n to $\xi^u(O'_n)$, and we continue to write $\tilde{\xi}^u$ for its inverse function. We perform the change of variables $w = \xi^u_s$ on each interval O'_n to obtain

$$\begin{aligned}\int_0^{t_{\exp}(\mu_0,R_0)} (b+d)(\xi^u_s)\mathrm{d}s &\geq \sum_{n\in\mathbb{N}} \int_{O'_n} (b+d)(\xi^u_s)\mathrm{d}s \\ &= \sum_{n\in\mathbb{N}} \int_{\xi^u(O_n)} \frac{(b+d)(w)}{g(w, R_{\tilde{\xi}^u_w})}\mathrm{d}w \geq \int_{\xi^u_0}^{+\infty} \frac{b(w)+d(w)}{g(w, R_{\max})}\mathrm{d}w,\end{aligned}$$

because of (II.1.8), and $R \mapsto g(x, R)$ is non-decreasing for every $x > 0$. By (II.1.6), the left-most integral is finite, which contradicts Assumption II.1.5 and this ends the proof. $\square$

Corollary II.1.7. *Under Assumptions II.1.4 and II.1.5, we have $\tau_{\exp} = +\infty$ almost surely.*

Proof. By definition of $\tau_{\exp}$ and construction of the process, it suffices to show that for every $n \in \mathbb{N}$, we almost surely have

$$(J_{n+1} = +\infty) \quad \text{or} \quad (J_{n+1} < J_n + t_{\exp}(\mu_{J_n}, R_{J_n}) < +\infty).$$

Let us fix $n \in \mathbb{N}$ and work in the following conditionnally to the event $\{J_n + t_{\exp}(\mu_{J_n}, R_{J_n}) < +\infty\} \cap \{J_{n+1} < +\infty\}$. We aim to show that $J_{n+1} < J_n + t_{\exp}(\mu_{J_n}, R_{J_n})$ almost surely. We define $(\tilde{\mu}_0, \tilde{R}_0)$ a random variable with same law as (μ_{J_n}, R_{J_n}), and $(\tilde{\mu}_t, \tilde{R}_t)_t$ a process starting from the random initial condition $(\tilde{\mu}_0, \tilde{R}_0)$ and constructed with the algorithmic procedure described above. The jump times with indices 0 and 1 associated to $(\tilde{\mu}_t, \tilde{R}_t)_t$ are naturally written $\tilde{J}_0$ and $\tilde{J}_1$, and note that by construction, $\tilde{J}_0 := 0$. Under the event $\{J_{n+1} < +\infty\}$, from the strong Markov property for Poisson point processes (see Example 10.4(a) in [DVJ07]), the law of $J_n + t_{\exp}(\mu_{J_n}, R_{J_n}) - J_{n+1}$ conditionnally to $\{J_n + t_{\exp}(\mu_{J_n}, R_{J_n}) < +\infty\}$ and $\mathcal{F}_{J_n}$ is equal to the law of $\tilde{J}_0 + t_{\exp}(\tilde{\mu}_0, \tilde{R}_0) - \tilde{J}_1 = t_{\exp}(\tilde{\mu}_0, \tilde{R}_0) - \tilde{J}_1$, which concludes thanks to Proposition II.1.6. $\square$

II.1.4 Assumptions for a well-defined population process $(\mu_t)_t$ for every $t \geq 0$

At this step, thanks to Assumptions II.1.4 and II.1.5, we have $\tau_{\exp} = +\infty$ almost surely (it is Corollary II.1.7), and we work under this event. The process $(\mu_t, R_t)_t$ is still well-defined only on $[0, J_\infty)$, with J_∞ possibly finite, *i.e.* there is a possible accumulation of jump times. We begin with intermediate results in Section II.1.4.1. Then, in Section II.1.4.2, we define an appropriate weight function ω. Finally, we give in Section II.1.4.3 a setting under which almost surely, $J_\infty = +\infty$. In fact, under this setting, we will even obtain in Proposition II.1.16 a stronger result, which implies in particular that the expectation of the population size $|V_t|$ is finite for every $t \geq 0$, and this entails the non-accumulation of jump times. To obtain this stronger proposition, that will also be important in the proof of Theorem II.3.1, we use the weight function ω.

II.1.4.1 Preliminary lemmas

Lemma II.1.8. *Let Θ be a measurable function from $\mathcal{M}_P(\mathbb{R}_+^*) \times [0, R_{\max}]$ to $\mathbb{R}$. Under Assumptions II.1.4 and II.1.5, we have almost surely, for all $t \in [0, J_\infty)$,*

$$\begin{aligned}\Theta(\mu_t, R_t) &= \Theta(X_t(\mu_0, R_0)) \\ &\quad + \int_0^t \int_{\mathcal{U}\times\mathbb{R}_+^*} \mathbb{1}_{\{u\in V_{s-}\}}\mathbb{1}_{\{h\le b(\xi_{s-}^u)\}}\Big(\Theta(X_{t-s}(\mu_{s-} + \delta_{x_0} + \delta_{\xi_{s-}^u - x_0} - \delta_{\xi_{s-}^u}, R_s)) \\ &\qquad\qquad - \Theta(X_{t-s}(\mu_{s-}, R_s))\Big)\mathcal{N}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h) \\ &\quad + \int_0^t \int_{\mathcal{U}\times\mathbb{R}_+^*} \mathbb{1}_{\{u\in V_{s-}\}}\mathbb{1}_{\{h\le d(\xi_{s-}^u)\}}\Big(\Theta(X_{t-s}(\mu_{s-} - \delta_{\xi_{s-}^u}, R_s)) \\ &\qquad\qquad - \Theta(X_{t-s}(\mu_{s-}, R_s))\Big)\mathcal{N}'(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h).\end{aligned}$$

Proof. Let $t \in [0, J_\infty)$, then there exists $n \geq 1$ such that $t \leq J_n$. Thus, there is a finite number of jumps on the interval $[0, t]$, and we obtain (as for (8) in Section 3 of [CF15])

$$\Theta(\mu_t, R_t) = \Theta(X_t(\mu_0, R_0)) + \sum_{s\le t}\left[\Theta(X_{t-s}(\mu_s, R_s)) - \Theta(X_{t-s}(\mu_{s-}, R_{s-}))\right], \qquad \text{(II.1.9)}$$

which is a valid notation with the convention on the flow X. From the algorithmic construction of the process $(\mu_t, R_t)_t$ in Section II.1.2, times of jump (*i.e.* times when $(\mu_s, R_s) \neq (\mu_{s-}, R_{s-})$) are given by the Poisson point measures $\mathcal{N}$ and $\mathcal{N}'$, so are almost surely distinct. Furthermore, under Assumptions II.1.4 and II.1.5, these jumps are not associated to vanishing/exploding individual energies, but only to birth or death events, hence the result (integrals against Poisson point measures are only a formal rewriting of the telescopic sum in (II.1.9)). □

Remark: We need Assumptions II.1.4 and II.1.5 to ensure, thanks to Corollary II.1.7, that almost surely individual energies do not vanish/explode before J_∞. This is why the decomposition in Lemma II.1.8 is not well-defined pathwisely, but only almost surely.

Let $\varphi : (t, x) \mapsto \varphi_t(x)$ be a $\mathcal{C}^{1,1}(\mathbb{R}^+ \times \mathbb{R}_+^*)$ function. We write $\partial_t\varphi$ and $\partial_x\varphi$ for the associated partial derivatives, and also for every $t \geq 0$ we define

$$\Phi_t : (R, x) \in \mathbb{R}^+ \times \mathbb{R}_+^* \mapsto \partial_t\varphi(t, x) + g(x, R)\partial_x\varphi(t, x). \qquad \text{(II.1.10)}$$

Note that Φ depends on φ, but to lighten the notations, we choose to write it this way in all the rest of this chapter. The notation Φ will always be related to the definition in (II.1.10) with a function φ we work with without ambiguity.

Lemma II.1.9. *Let $\varphi \in \mathcal{C}^{1,1}(\mathbb{R}^+ \times \mathbb{R}_+^*)$. Under Assumptions II.1.4 and II.1.5, we have almost surely, for all $t \in [0, J_\infty)$,*

$$\begin{aligned}\langle \mu_t, \varphi_t\rangle &= \langle \mu_0, \varphi_0\rangle \\ &\quad + \int_0^t \langle \mu_s, \Phi_s(R_s, .)\rangle \mathrm{d}s \\ &\quad + \int_0^t \int_{\mathcal{U}\times\mathbb{R}_+^*} \mathbb{1}_{\{u\in V_{s-}\}}\mathbb{1}_{\{h\le b(\xi_{s-}^u)\}}\Big(\varphi_s(x_0) + \varphi_s(\xi_{s-}^u - x_0) - \varphi_s(\xi_{s-}^u)\Big)\mathcal{N}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h) \\ &\quad - \int_0^t \int_{\mathcal{U}\times\mathbb{R}_+^*} \mathbb{1}_{\{u\in V_{s-}\}}\mathbb{1}_{\{h\le d(\xi_{s-}^u)\}}\varphi_s(\xi_{s-}^u)\mathcal{N}'(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h),\end{aligned}$$

with Φ associated to φ as in (II.1.10).

Proof. Let $t \in [0, J_\infty)$ and $\varphi \in \mathcal{C}^{1,1}(\mathbb{R}^+ \times \mathbb{R}_+^*)$. We apply Lemma II.1.8 to $\Theta : (\mu, R) \mapsto \langle \mu, \varphi_t \rangle$ and with (2.2.1) in mind, we obtain

$$\begin{aligned}\langle \mu_t, \varphi_t \rangle = &\sum_{w \in V_0} \varphi_t(X_t^w(\mu_0, R_0)) \\ &+ \int_0^t \int_{\mathcal{U}\times\mathbb{R}_+^*} \mathbb{1}_{\{u \in V_{s-}\}} \mathbb{1}_{\{h \le b(\xi_{s-}^u)\}} \Bigg(\sum_{w \in V_s} \varphi_t(X_{t-s}^w(\mu_{s-} + \delta_{x_0} + \delta_{\xi_{s-}^u - x_0} - \delta_{\xi_{s-}^u}, R_s)) \\ &\qquad - \sum_{w \in V_{s-}} \varphi_t(X_{t-s}^w(\mu_{s-}, R_s)) \Bigg) \mathcal{N}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h) \\ &+ \int_0^t \int_{\mathcal{U}\times\mathbb{R}_+^*} \mathbb{1}_{\{u \in V_{s-}\}} \mathbb{1}_{\{h \le d(\xi_{s-}^u)\}} \Bigg(\sum_{w \in V_s} \varphi_t(X_{t-s}^w(\mu_{s-} - \delta_{\xi_{s-}^u}, R_s)) \\ &\qquad - \sum_{w \in V_{s-}} \varphi_t(X_{t-s}^w(\mu_{s-}, R_s)) \Bigg) \mathcal{N}'(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h).\end{aligned}$$

Let $0 \le s \le t$, $r \ge 0$ and $\mu \in \mathcal{M}_P(\mathbb{R}_+^*)$, written of the form $\mu := \sum_{w \in V} \delta_{\xi^w}$ with $V \subseteq \mathcal{U}$ and $(\xi_w)_{w \in V}$ in $\mathbb{R}_+^*$. The chain rule gives that for $w \in V$, $\varphi_t(X_{t-s}^w(\mu, r)) = \varphi_s(\xi^w) + \int_s^t \Phi_\tau(X_{\tau-s}^{\mathfrak{R}}(\mu, r), X_{\tau-s}^w(\mu, r))\mathrm{d}\tau$, and we obtain

$$\begin{aligned}\langle \mu_t, \varphi_t \rangle = &\langle \mu_0, \varphi_0 \rangle + T_0 + T_1 + T_2 \\ &+ \int_0^t \int_{\mathcal{U}\times\mathbb{R}_+^*} \mathbb{1}_{\{u \in V_{s-}\}} \mathbb{1}_{\{h \le b(\xi_{s-}^u)\}} \Big(\varphi_s(x_0) + \varphi_s(\xi_{s-}^u - x_0) - \varphi_s(\xi_{s-}^u) \Big) \mathcal{N}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h) \\ &- \int_0^t \int_{\mathcal{U}\times\mathbb{R}_+^*} \mathbb{1}_{\{u \in V_{s-}\}} \mathbb{1}_{\{h \le d(\xi_{s-}^u)\}} \varphi_s(\xi_{s-}^u) \mathcal{N}'(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h),\end{aligned}$$

with

$$\begin{aligned}T_0 &= \sum_{w \in V_0} \int_0^t \Phi_\tau(X_\tau^{\mathfrak{R}}(\mu_0, R_0), X_\tau^w(\mu_0, R_0))\mathrm{d}\tau, \\ T_1 &= \int_0^t \int_{\mathcal{U}\times\mathbb{R}_+^*} \mathbb{1}_{\{u \in V_{s-}\}} \mathbb{1}_{\{h \le b(\xi_{s-}^u)\}} \int_s^t \Bigg(\sum_{w \in V_s} \Phi_\tau\Big(X_{\tau-s}^{\mathfrak{R}}(\mu_{s-} + \delta_{x_0} + \delta_{\xi_{s-}^u - x_0} - \delta_{\xi_{s-}^u}, R_s), \\ &\qquad X_{\tau-s}^w(\mu_{s-} + \delta_{x_0} + \delta_{\xi_{s-}^u - x_0} - \delta_{\xi_{s-}^u}, R_s) \Big) \\ &\qquad - \sum_{w \in V_{s-}} \Phi_\tau(X_{\tau-s}^w(\mu_{s-}, R_s)) \Bigg) \mathrm{d}\tau \mathcal{N}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h), \\ T_2 &= \int_0^t \int_{\mathcal{U}\times\mathbb{R}_+^*} \mathbb{1}_{\{u \in V_{s-}\}} \mathbb{1}_{\{h \le d(\xi_{s-}^u)\}} \\ &\qquad \int_s^t \Bigg(\sum_{w \in V_s} \Phi_\tau\Big(X_{\tau-s}^{\mathfrak{R}}(\mu_{s-} - \delta_{\xi_{s-}^u}, R_s), X_{\tau-s}^w(\mu_{s-} - \delta_{\xi_{s-}^u}, R_s) \Big) \\ &\qquad - \sum_{w \in V_{s-}} \Phi_\tau(X_{\tau-s}^w(\mu_{s-}, R_s)) \Bigg) \mathrm{d}\tau \mathcal{N}'(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h).\end{aligned}$$

As $t < J_\infty$, there exists $n \in \mathbb{N}^*$ such that $t \leq J_n$, so that integrals against Poisson point measures are a formal pathwise writing of finite sums and we can use Fubini theorem almost surely (under the almost sure event $\{\tau_{\exp} = +\infty\}$) for T_0, T_1 and T_2 to obtain, for T_2 for example:

$$T_2 = \int_0^t \int_0^\tau \int_{\mathcal{U}\times\mathbb{R}_+^*} \mathbb{1}_{\{u\in V_{s-}\}}\mathbb{1}_{\{h\leq d(\xi^u_{s-})\}} \left(\sum_{w\in V_s} \Phi_\tau\Big(X^{\mathfrak{R}}_{\tau-s}(\mu_{s-} - \delta_{\xi^u_{s-}}, R_s), X^w_{\tau-s}(\mu_{s-} - \delta_{\xi^u_{s-}}, R_s)\Big) - \sum_{w\in V_{s-}} \Phi_\tau(X^w_{\tau-s}(\mu_{s-}, R_s))\right) \mathcal{N}'(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h)\mathrm{d}\tau.$$

Using Lemma II.1.8 again, we realize that

$$T_0 + T_1 + T_2 = \int_0^t \langle \mu_s, \Phi_s(R_s, .)\rangle \mathrm{d}s,$$

which concludes. □

For every $F : (r, x) \mapsto F(r, x)$ in $\mathcal{C}^{1,1}([0, R_{\max}] \times \mathbb{R})$, we write $\partial_r F$ and $\partial_x F$ for the associated partial derivatives.

Lemma II.1.10. *Let $\varphi \in \mathcal{C}^{1,1}(\mathbb{R}^+ \times \mathbb{R}_+^*)$, and $F \in \mathcal{C}^{1,1}([0, R_{\max}] \times \mathbb{R})$. Under Assumptions II.1.4 and II.1.5, we have almost surely, for all $t \in [0, J_\infty)$,*

$$\begin{aligned} F(R_t, \langle \mu_t, \varphi_t\rangle) = F(R_0, \langle \mu_0, \varphi_0\rangle) &+ \int_0^t \Big[\rho(\mu_s, R_s)\partial_r F(R_s, \langle \mu_s, \varphi_s\rangle) + \langle \mu_s, \Phi_s(R_s, .)\rangle \partial_x F(R_s, \langle \mu_s, \varphi_s\rangle)\Big]\mathrm{d}s \\ &+ \int_0^t \int_{\mathcal{U}\times\mathbb{R}_+^*} \mathbb{1}_{\{u\in V_{s-}\}}\mathbb{1}_{\{h\leq b(\xi^u_{s-})\}} \Big(F(R_s, \langle \mu_{s-} + \delta_{x_0} + \delta_{\xi^u_{s-}-x_0} - \delta_{\xi^u_{s-}}, \varphi_s\rangle) \\ &\qquad - F(R_s, \langle \mu_{s-}, \varphi_s\rangle)\Big)\mathcal{N}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h) \\ &+ \int_0^t \int_{\mathcal{U}\times\mathbb{R}_+^*} \mathbb{1}_{\{u\in V_{s-}\}}\mathbb{1}_{\{h\leq d(\xi^u_{s-})\}} \Big(F(R_s, \langle \mu_{s-} - \delta_{\xi^u_{s-}}, \varphi_s\rangle) \\ &\qquad - F(R_s, \langle \mu_{s-}, \varphi_s\rangle)\Big)\mathcal{N}'(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h), \end{aligned}$$

with Φ associated to φ as in (II.1.10).

Proof. Let $t \in [0, J_\infty)$, $\varphi \in \mathcal{C}^{1,1}(\mathbb{R}^+ \times \mathbb{R}_+^*)$ and $F \in \mathcal{C}^{1,1}([0, R_{\max}] \times \mathbb{R})$. We can apply Lemma II.1.9 to obtain a decomposition of $\langle \mu_t, \varphi_t\rangle$, and (II.1.3) gives

$$R_t = R_0 + \int_0^t \rho(\mu_s, R_s)\mathrm{d}s.$$

As $t < J_\infty$, there exists $n \in \mathbb{N}^*$ such that $t \leq J_n$, so that $\langle \mu_t, \varphi_t\rangle$ coincides with the stopped process $\langle \mu_{t\wedge J_n}, \varphi_{t\wedge J_n}\rangle$ at time t. Hence, to obtain the desired expression, it suffices to show that we can use Itô's formula for finite variation processes with right-continuous sample paths (Theorem 31 page 78 in [Pro05]) with $F : (r, x) \mapsto F(r, x)$ applied to the process $(\langle \mu_{t\wedge J_n}, \varphi_{t\wedge J_n}\rangle, R_{t\wedge J_n})_{t\geq 0}$ for any $n \in \mathbb{N}^*$. By assumption, $F \in \mathcal{C}^{1,1}([0, R_{\max}] \times \mathbb{R})$ and by construction, the stopped process $(\langle \mu_{t\wedge J_n}, \varphi_{t\wedge J_n}\rangle, R_{t\wedge J_n})_{t\geq 0}$ has right-continuous sample paths, so we only need to show that it is a finite variation process (see Definition page 39 in [Pro05]). This is immediately the case for the component $(R_{t\wedge J_n})_{t\geq 0}$, because its derivative is bounded before time J_n. Almost surely, any individual trajectory of the stopped process $(\mu_{t\wedge J_n}, R_{t\wedge J_n})_{t\geq 0}$ has bounded jumps and is deterministic with a bounded derivative between jumps, thanks to Corollary II.1.7. Hence, $(\langle \mu_{t\wedge J_n}, \varphi_{t\wedge J_n}\rangle)_{t\geq 0}$ has also bounded variations, which ends the proof. □

Remark: Let $\varphi \in \mathcal{C}^1(\mathbb{R}_+^*)$, $F \in \mathcal{C}^{1,1}([0, R_{\max}] \times \mathbb{R})$ and $f : (\mu, R) \in \mathcal{M}_P(\mathbb{R}_+^*) \times [0, R_{\max}] \mapsto F(R, \langle \mu, \varphi\rangle)$. If $J_\infty = +\infty$ almost surely, then Lemma II.1.10 allows us to express

$$\lim_{t\to 0} \frac{\mathbb{E}\left(f(\mu_t, R_t)\right) - f(\mu_0, R_0)}{t},$$

for any $(\mu_0, R_0) \in \mathcal{M}_P(\mathbb{R}_+^*) \times [0, R_{\max}]$. Hence, this result is a stepping stone in the definition of the infinitesimal generator of the process $(\mu_t, R_t)_t$ (see Section 4.6. in [JS96]). In fact, we will provide the extended generator of this process in Proposition II.1.20.

In the following, for $t \in [0, J_\infty)$, we define $E_t := \langle \mu_t, \mathrm{Id}\rangle$ the total energy of the population at time t. In the dynamics described in (II.1.3), as ς is $\mathcal{C}^1$, the speed of renewal of R_t is upper bounded by $||\varsigma||_{\infty,[0,R_{\max}]} < +\infty$, for every $t \geq 0$. This resource renewal is the only income of biomass into the system described by $(\mu_t, R_t)_t$, so this shall give us a control on the total biomass of the system over time, *i.e.* for every $t \geq 0$, we can upper bound $R_t + E_t$.

Proposition II.1.11. *Under Assumptions II.1.4 and II.1.5, we have almost surely*

$$\forall t \in [0, J_\infty), \quad R_t + E_t \leq R_0 + E_0 + t||\varsigma||_{\infty,[0,R_{\max}]} < +\infty. \tag{II.1.11}$$

Proof. Let $t \in [0, J_\infty)$, by Lemma II.1.10 applied to $F : (r, x) \mapsto r + x$ and $\varphi : (t, x) \mapsto x$, we have

$$\begin{aligned} R_t + E_t &= R_0 + E_0 + \int_0^t \Big[\rho(\mu_s, R_s) + \langle \mu_s, \Phi_s(R_s, .)\rangle\Big] \mathrm{d}s \\ &\quad - \int_0^t \int_{\mathcal{U}\times\mathbb{R}_+^*} \mathbb{1}_{\{u\in V_{s-}\}} \mathbb{1}_{\{h\leq d(\xi^u_{s-})\}} \xi^u_{s-} \mathcal{N}'(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h) \\ &\leq R_0 + E_0 + \int_0^t \varsigma(R_s)\mathrm{d}s, \end{aligned}$$

which concludes. Note that we used in particular the fact that the constant χ in the definition (II.1.3) of ρ is larger than 1. □

Corollary II.1.12. *Let φ be a measurable function on $\mathbb{R}_+^*$, such that φ is bounded on any $(0, y)$ with $y > 0$. Suppose also that $\mathbb{E}(E_0) < +\infty$. Then,*

$$\forall T \geq 0, \forall \varepsilon > 0, \exists C > 0, \quad \mathbb{P}\left(\sup_{t\in[0,T\wedge J_\infty)} \sup_{u\in V_t} \varphi(\xi^u_t) \leq C\right) \geq 1 - \varepsilon.$$

Proof. Let φ be measurable and bounded on any $(0,y)$ with $y>0$, $\varepsilon>0$ and $T\geq 0$. We have, for every $y>0$,

$$\begin{aligned}\mathbb{P}\left(\sup_{t\in[0,T\wedge J_\infty]}\sup_{u\in V_t}\varphi(\xi_t^u)\leq ||\varphi||_{\infty,(0,y)}\right) &\geq \mathbb{P}\left(\sup_{t\in[0,T\wedge J_\infty]}\sup_{u\in V_t}\xi_t^u<y\right)\\ &\geq \mathbb{P}\left(\sup_{t\in[0,T\wedge J_\infty]}E_t<y\right)\\ &\geq 1-\frac{1}{y}\mathbb{E}\left(\sup_{t\in[0,T\wedge J_\infty]}E_t\right).\end{aligned}$$

Hence, taking $y=\dfrac{1}{\varepsilon}\left(\mathbb{E}(E_0)+R_{\max}+T||\varsigma||_{\infty,[0,R_{\max}]}\right)$ and $C=||\varphi||_{\infty,(0,y)}$ concludes thanks to Proposition II.1.11. □

II.1.4.2 Definition of the weight function ω

The main goal of this chapter is Theorem II.3.1, which is a tightness result for measure-valued processes in a weighted space of measures. Thus, in the following, we define a weight function ω adapted to the functional parameters b, d, ϕ and ψ for two reasons. First, if such a weight function exists, we shall prove in Section II.1.4.3 that $(\mu_t)_t$ is well-defined on $\mathbb{R}^+$. Then, we will obtain in Section II.1.5 useful martingale properties for our process using the weight function ω. We write $\hbar : x\in\mathbb{R}^+\mapsto x+x^2$, and $\overline{g}: x>0\mapsto \sup_{R\in[0,R_{\max}]}|g(x,R)|$. Remark that for $x>0$, $\overline{g}(x)=\max(\ell(x),\phi(R_{\max})\psi(x)-\ell(x))$.

Assumption II.1.13 (Existence of an appropriate weight function). *There exists $\omega\in\mathcal{C}^1(\mathbb{R}_+^*)$ positive and non-decreasing such that*

- $\exists C_g>0,\forall x>0,\quad \overline{g}(x)(1+\omega'(x))\leq C_g(1+x+\omega(x))$,
- $\exists C_b>0,\forall x>0,\quad b(x)(1+\hbar\left(|\omega(x_0)+\omega(x-x_0)-\omega(x)|\right))\leq C_b(1+x+\omega(x))$,
- $\exists C_d>0,\forall x>0,\quad d(x)\hbar(\omega(x))\leq C_d(1+x+\omega(x))$.

Remarks: If the rates $\overline{g}$, b and d are bounded (which is a classical assumption in the literature [FM04, Tra08, CF15]), the weight function $\omega\equiv 1$ is a possible choice. We present in Example 1 of Section II.6 a choice of unbounded functional parameters, with allometric shapes, and prove that there exists weight functions ω verifying Assumption II.1.13 in this specific context. It is of order x^{κ_1}, respectively x^{κ_2}, in a neighborhood of 0, respectively $+\infty$, with $0\leq\kappa_1\leq\kappa_2\leq 1$ and (κ_1,κ_2) verifying constraints expressed in terms of the allometric coefficients α, β and δ respectively associated to the maximal growth rate $\overline{g}$, the birth rate b and the death rate d (see Lemmas II.6.2 and II.6.3). The typical shape of the weight function ω is shown on Figure II.1. Nevertheless, in general, proving the existence of a weight function ω verifying Assumption II.1.13 could be a difficult problem, similar to the search for Lyapunov functions associated to the extended generator of a Feller process (see for example condition (CD2) in Section 4.1. of [MT93]). Remark that in particular, the third point of Assumption II.1.13 implies that $d\omega$ should be bounded in a neighborhood of 0. Hence, if $d(x)\xrightarrow[x\to 0]{}+\infty$, which is usually the case with allometric parameters, then $\omega(x)\xrightarrow[x\to 0]{}0$ as shown on Figure II.1. We will discuss in Section II.3 how this impacts the interpretation of the tightness result in Theorem II.3.1.

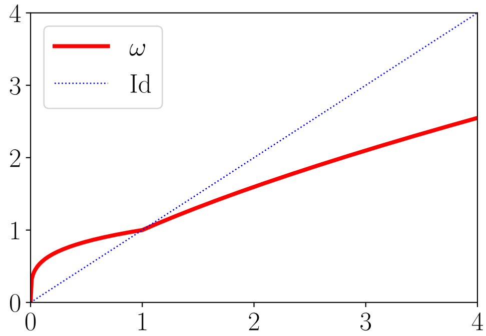


Figure II.1: Possible shape of the weight function ω with an allometric choice of parameters

Finally, recall that $b \equiv 0$ on $(0, x_0]$. Thus, we verify that if ω is Lipschitz continuous on $(1, +\infty)$ (which is equivalent to ω' bounded on $(1, +\infty)$; and again, this is usually the case with allometric parameters), the second point of Assumption II.1.13 is equivalent to the lighter assumption

$$\exists C_b > 0, \forall x > 1, \quad b(x) \leq C_b(1 + x). \tag{II.1.12}$$

Lemma II.1.14. *Assumption II.1.13 is equivalent to the combination of the following properties. First, $\omega \in \mathcal{C}^1(\mathbb{R}_+^*, \mathbb{R}_+^*)$, is non-decreasing, and then*

$$\exists \omega_1 > 0, \forall x > 0,$$
$$\overline{g}(x)(1 + \omega'(x)) + b(x)\left(1 + |\omega(x_0) + \omega(x - x_0) - \omega(x)|\right) + d(x)\omega(x) \leq \omega_1(1 + x + \omega(x)), \tag{II.1.13}$$

and

$$\exists \omega_2 > 0, \forall x > 0, \quad b(x)(\omega(x_0) + \omega(x - x_0) - \omega(x))^2 + d(x)\omega^2(x) \leq \omega_2(1 + x + \omega(x)). \tag{II.1.14}$$

Proof. The fact that Assumption II.1.13 implies (II.1.13) and (II.1.14) is immediate. We can take $\omega_1 = C_g + C_b + C_d$ and $\omega_2 = C_b + C_d$. Conversely, if (II.1.13) and (II.1.14) hold true, we can take $C_g = \omega_1$ and $C_b = C_d = \omega_1 + \omega_2$. □

Remark: Lemma II.1.14 gives us an insight on why Assumption II.1.13 is interesting to obtain martingale properties for our process. This specific choice of a weight function ω shall provide a control for the stochastic integrals appearing in the decomposition of $\langle \mu_t, \varphi_t \rangle$ in Lemma II.1.9, as for classical Lyapunov functions. We will get back to these considerations in Section II.1.5.

II.1.4.3 Proof of the non-accumulation of jump times

For $t \in [0, J_\infty)$, we define $N_t := \langle \mu_t, 1 \rangle$. It represents the number of individuals in the population at time t. We also define $\Omega_t := \langle \mu_t, \omega \rangle$. For $M > 0$, we define the stopping time

$$\tau_M := \inf \left\{ t \in [0, J_\infty), E_t + N_t + \Omega_t \geq M \right\},$$

with the convention $\inf(\varnothing) = +\infty$. Finally, for any positive function w on $\mathbb{R}_+^*$, we write $\varphi \in \mathfrak{B}_w(\mathbb{R}_+^*)$, if $\frac{\varphi}{w}$ is a bounded function on $\mathbb{R}_+^*$.

Definition II.1.15 (**General setting**)**.** *In the following, we denote as 'the general setting', the framework gathering dynamics described in Sections II.1.1 and II.1.2, Assumptions II.1.4, II.1.5 and II.1.13, and the additional assumption*

$$\mathbb{E}(E_0 + N_0 + \Omega_0) < +\infty.$$

We are now ready to prove the main result of this section.

Proposition II.1.16. *Under the general setting, we almost surely have*

$$\tau_M \xrightarrow[M\to+\infty]{} +\infty. \tag{II.1.15}$$

Then, for all $T \geq 0$,

$$\mathbb{E}\left(\sup_{t\in[0,T\wedge J_\infty)} (E_t + N_t + \Omega_t)\right) < +\infty. \tag{II.1.16}$$

This immediately implies that for every $\varphi \in \mathfrak{B}_{1+\mathrm{Id}+\omega}(\mathbb{R}_+^*)$, we have

$$\mathbb{E}\left(\sup_{t\in[0,T\wedge J_\infty)} |\langle \mu_t, \varphi\rangle|\right) < +\infty.$$

Proof. Let $M > 0$ and $T \geq 0$. We use Proposition II.1.11, and apply Lemma II.1.9 to $\varphi : (t, x) \mapsto 1 + \omega(x)$, to obtain for every $0 \leq t < T \wedge \tau_M \wedge J_\infty$,

$$\begin{aligned}
E_t + N_t + \Omega_t \leq{}& R_0 + E_0 + t||\varsigma||_{\infty,[0,R_{\max}]} + N_0 + \Omega_0 + \int_0^t \langle \mu_s, g(., R_s)\omega'\rangle \mathrm{d}s \\
&+ \int_0^t \int_{\mathcal{U}\times\mathbb{R}_+^*} \mathbb{1}_{\{u\in V_{s-}\}} \mathbb{1}_{\{h\leq b(\xi^u_{s-})\}} \\
&\qquad \Big(1 + \omega(x_0) + \omega(\xi^u_{s-} - x_0) - \omega(\xi^u_{s-})\Big) \mathcal{N}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h) \\
&- \int_0^t \int_{\mathcal{U}\times\mathbb{R}_+^*} \mathbb{1}_{\{u\in V_{s-}\}} \mathbb{1}_{\{h\leq d(\xi^u_{s-})\}} \Big(1 + \omega(\xi^u_{s-})\Big) \mathcal{N}'(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h) \\
\leq{}& R_{\max} + E_0 + N_0 + \Omega_0 + T||\varsigma||_{\infty,[0,R_{\max}]} + \int_0^T \langle \mu_s, \overline{g}\omega'\rangle \mathbb{1}_{\{s<\tau_M\wedge J_\infty\}} \mathrm{d}s \\
&+ \int_0^T \int_{\mathcal{U}\times\mathbb{R}_+^*} \mathbb{1}_{\{s<\tau_M\wedge J_\infty\}} \mathbb{1}_{\{u\in V_{s-}\}} \mathbb{1}_{\{h\leq b(\xi^u_{s-})\}} \\
&\qquad \Big(1 + |\omega(x_0) + \omega(\xi^u_{s-} - x_0) - \omega(\xi^u_{s-})|\Big) \mathcal{N}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h).
\end{aligned}$$

We can take the supremum of the left-hand side over $t \in [0, T \wedge \tau_M \wedge J_\infty)$. Then, we take expectations and apply Fubini theorem, which is valid since all integrands are positive. Also, all the expectations are finite thanks to the first and second points of Assumption II.1.13 and by definition of τ_M, so we have a true semi-martingale decomposition of the integrated term against the Poisson point measure $\mathcal{N}$. We obtain

$$\begin{aligned}\mathbb{E}\Big(\sup_{t\in[0,T\wedge\tau_M\wedge J_\infty)}(E_t+N_t+\Omega_t)\Big) &\leq R_{\max}+\mathbb{E}(E_0+N_0+\Omega_0)+T||\varsigma||_{\infty,[0,R_{\max}]}\\ &\quad+(C_g+C_b)\int_0^T\mathbb{E}\left((E_s+N_s+\Omega_s)\mathbb{1}_{\{s<\tau_M\wedge J_\infty\}}\right)\mathrm{d}s\\ &\leq R_{\max}+\mathbb{E}(E_0+N_0+\Omega_0)+T||\varsigma||_{\infty,[0,R_{\max}]}\\ &\quad+(C_g+C_b)\int_0^T\mathbb{E}\left(\sup_{\tau\in[0,s\wedge\tau_M\wedge J_\infty)}(E_\tau+N_\tau+\Omega_\tau)\right)\mathrm{d}s\end{aligned}$$

By Gronwall lemma, we then have

$$\begin{aligned}&\mathbb{E}\left(\sup_{t\in[0,T\wedge\tau_M\wedge J_\infty)}(E_t+N_t+\Omega_t)\right)\\ &\qquad\leq\left(R_{\max}+\mathbb{E}(E_0+N_0+\Omega_0)+T||\varsigma||_{\infty,[0,R_{\max}]}\right)e^{(C_g+C_b)T}.\end{aligned}\tag{II.1.17}$$

Now, as $(\tau_M)_{M\in\mathbb{N}^*}$ is a non-decreasing sequence, to prove (II.1.15), it suffices to show that for all $T\geq0$, $\mathbb{P}(\tau_M\leq T)\xrightarrow[M\to+\infty]{}0$. Because $\mathbb{E}(E_0+N_0+\Omega_0)<+\infty$, this follows from (II.1.17) and

$$\begin{aligned}\mathbb{P}(\tau_M\leq T)&\leq\mathbb{P}\left(\sup_{t\in[0,T\wedge\tau_M\wedge J_\infty)}(E_t+N_t+\Omega_t)\geq M\right)\\ &\leq\frac{1}{M}\mathbb{E}\left(\sup_{t\in[0,T\wedge\tau_M\wedge J_\infty)}(E_t+N_t+\Omega_t)\right).\end{aligned}$$

Finally, to obtain (II.1.16), we use (II.1.15) and apply Fatou lemma to obtain

$$\mathbb{E}\left(\sup_{t\in[0,T\wedge J_\infty)}(E_t+N_t+\Omega_t)\right)\leq\lim_{M\to+\infty}\mathbb{E}\left(\sup_{t\in[0,T\wedge\tau_M\wedge J_\infty)}(E_t+N_t+\Omega_t)\right),$$

which ends the proof by (II.1.17). □

Corollary II.1.17. *Under Assumptions II.1.4, II.1.5 and II.1.13, the process $(\mu_t)_t$ is almost surely well-defined for every $t\geq0$, i.e. $\mathbb{P}(\tau_{\exp}\wedge J_\infty=+\infty)=1$.*

Proof. First, by Corollary II.1.7, we can work under the event $\{\tau_{\exp}=+\infty\}$. Then, if we write ν the law of μ_0, we have that

$$\mathbb{P}_\nu(J_\infty<+\infty)=\int_{\mu_0\in\mathcal{M}_P(\mathbb{R}_+^*)}\mathbb{P}_{\mu_0}(J_\infty<+\infty)\nu(\mathrm{d}\mu_0).$$

Thus, it suffices to show that for any fixed $\mu_0\in\mathcal{M}_P(\mathbb{R}_+^*)$, $\mathbb{P}_{\mu_0}(J_\infty<+\infty)=0$. We fix μ_0 in the following, so $E_0+N_0+\Omega_0$ is deterministic and finite, and we work under the general setting. For every $n\in\mathbb{N}^*$, with the convention $\inf(\varnothing)=+\infty$, we define $B_1:=\inf\{J_n,\ n\geq1,\ J_n^b=J_n\}$, and for $n\geq1$,

$$B_{n+1}:=\inf\{J_m,\ J_m>B_n,\ J_m^b=J_m\}.$$

This is the sequence of successive birth times of the process, and we write $B_\infty := \lim_{n\to+\infty} B_n$. We assess that $B_\infty \leq J_\infty$. Indeed, the only problematic case arises if $B_n = +\infty$ for some $n \in \mathbb{N}^*$, but then only a finite number of jumps can occur after time J_{n-1}, corresponding to the deaths of alive individuals at time J_{n-1}. Hence, we have $B_\infty = J_\infty = +\infty$ in that case. We suppose by contradiction that there exists $T_0 < +\infty$ such that $\mathbb{P}_{\mu_0}(J_\infty \leq T_0) > 0$. Then, $P := \mathbb{P}_{\mu_0}(B_\infty \leq T_0) > 0$, and by Corollary II.1.12 applied to $\varphi \equiv b$, $T \equiv T_0$ and $\varepsilon \equiv P/2$, there exists $C > 0$ such that the event

$$A := \left\{ \sup_{t\in[0,T_0\wedge J_\infty)} \sup_{u\in V_t} b(\xi_t^u) \leq C \right\} \cap \{B_\infty \leq T_0\}$$

occurs with positive probability and we work under this event in the following. Then, we assess that we necessarily have

$$\forall M > 0, \exists n \in \mathbb{N}^*, \quad N_{B_n} \geq M. \tag{II.1.18}$$

Indeed, if not, there exists $M > 0$ such that $N_{B_n} < M$ for all $n \in \mathbb{N}^*$. We could then construct the times of birth jump $(B_n)_{n\geq 1}$ as a subsequence of points given by a Poisson process with intensity CM, whose only accumulation point is almost surely $+\infty$. This cannot be verified on the event $\{B_\infty \leq T_0\}$. Hence, (II.1.18) holds true on the event A, and in that case we would have

$$\mathbb{P}(\forall M > 0, \quad \tau_M \leq B_\infty \leq T_0) > 0.$$

This contradicts the fact that (II.1.15) holds true almost surely by Proposition II.1.16, and ends the proof. □

Remark: We could define an alternative setting, gathering dynamics described in Sections II.1.1 and II.1.2, Assumptions II.1.4 and II.1.5, $\mathbb{E}(E_0 + N_0) < +\infty$, and the additional assumption $b \in \mathfrak{B}_{1+\mathrm{Id}}(\mathbb{R}_+^*)$ (in particular, this alternative setting does not contain Assumption II.1.13). For every $M > 0$, let $\tau'_M := \inf\{t \in [0, J_\infty), E_t + N_t \geq M\}$. Then, with the same techniques as in the proof of Proposition II.1.16, under the alternative setting described above, we verify that $\tau'_M \longrightarrow +\infty$ when M goes to $+\infty$. Thus, similarly to the proof of Corollary II.1.17, we show that almost surely, the process $(\mu_t, R_t)_t$ is well-defined for every $t \geq 0$. One can then wonder why we work under Assumption II.1.13 instead of the previous alternative setting. Importantly, the weight function ω ensures useful martingale properties for our process, and we shall prove it in Section II.1.5 (see Corollary II.1.19). What makes Assumption II.1.13 necessary in this chapter is that we need these martingale properties to obtain the tightness result in Theorem II.3.1.

Finally, we introduce and discuss in Appendix B.1 a stronger assumption that, in addition, makes our process almost surely biologically relevant in the sense of Section I.1.1.2, *i.e.* individual energy never vanishes/explodes, there is no accumulation of jump times, and we add the condition that every individual dies in finite time, in line with Chapter I. For the tightness result in Theorem II.3.1, it is enough to work under Assumptions II.1.4, II.1.5 and II.1.13, and we do so in the following.

II.1.5 Extended generator of the population process

In this section, we work under the general setting, so the process $(\mu_t, R_t)_{t\geq 0}$ is almost surely well-defined thanks to Corollary II.1.17. We shall prove that it is a Jumping Markov

Process (JMP) [JS96], characterized by its extended generator. We begin with martingale properties for our process. We write $\mathfrak{F}$ for the set of functions of the form $F_\varphi : (\mu, r) \in \mathcal{M}_P(\mathbb{R}_+^*) \times [0, R_{\max}] \mapsto F(r, \langle \mu, \varphi \rangle) \in \mathbb{R}$, with $F : (r, x) \mapsto F(r, x)$ in $\mathcal{C}^{1,1}([0, R_{\max}] \times \mathbb{R})$ and $\varphi \in \mathcal{C}^1(\mathbb{R}_+^*)$. For such a function, we write $\partial_r F_\varphi(\mu, r) := \partial_r F(r, \langle \mu, \varphi \rangle)$ and $\partial_x F_\varphi(\mu, r) := \partial_x F(r, \langle \mu, \varphi \rangle)$. Finally, for $F \in \mathcal{C}^{1,1}([0, R_{\max}] \times \mathbb{R})$ and $\varphi \in \mathcal{C}^{1,1}(\mathbb{R}^+ \times \mathbb{R}_+^*)$ (so that for all $t \geq 0$, F_{φ_t} is well-defined in $\mathfrak{F}$), we define

$$\begin{aligned}\mathfrak{M}_{F,\varphi,t} := F_{\varphi_t}(R_t, \mu_t) - & F_{\varphi_0}(R_0, \mu_0) \\ & - \int_0^t \rho(\mu_s, R_s) \partial_r F_{\varphi_s}(R_s, \mu_s) + \langle \mu_s, \Phi_s(R_s, .) \rangle \partial_x F_{\varphi_s}(R_s, \mu_s) \mathrm{d}s \\ & - \int_0^t \int_{\mathbb{R}_+^*} b(x) \Big[F_{\varphi_s}(R_s, \mu_s + \delta_{x_0} + \delta_{x - x_0} - \delta_x) - F_{\varphi_s}(R_s, \mu_s) \Big] \mu_s(\mathrm{d}x) \mathrm{d}s \\ & - \int_0^t \int_{\mathbb{R}_+^*} d(x) \Big[F_{\varphi_s}(R_s, \mu_s - \delta_x) - F_{\varphi_s}(R_s, \mu_s) \Big] \mu_s(\mathrm{d}x) \mathrm{d}s, \quad \text{(II.1.19)}\end{aligned}$$

with Φ associated to φ as in (II.1.10). The process $(\mathfrak{M}_{F,\varphi,t})_{t \geq 0}$ is almost surely well-defined under the general setting by Corollary II.1.7, Lemma II.1.10 and Corollary II.1.17. In the following, quadratic variations of square-integrable martingales are predictable quadratic variation defined as in Theorem 4.2. in [JS+87]. Also, we write $\widetilde{\mathcal{N}}$ and $\widetilde{\mathcal{N}'}$ for the compensated measures associated to the Poisson point measures $\mathcal{N}$ and $\mathcal{N}'$ (*i.e.* $\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h) := \mathcal{N}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h) - \mathrm{d}s\mathrm{d}u\mathrm{d}h$, and the same definition with $\mathcal{N}'$).

Proposition II.1.18. *Under the general setting, let $F \in \mathcal{C}^{1,1}([0, R_{\max}] \times \mathbb{R})$ and $\varphi \in \mathcal{C}^{1,1}(\mathbb{R}^+ \times \mathbb{R}_+^*)$.*

(i) *Assume that for all $t \geq 0$,*

$$\mathbb{E}\left(\int_0^t \int_{\mathbb{R}_+^*} b(x) \Big| F_{\varphi_s}(R_s, \mu_s + \delta_{x_0} + \delta_{x - x_0} - \delta_x) - F_{\varphi_s}(R_s, \mu_s) \Big| \mu_s(\mathrm{d}x) \mathrm{d}s \right) < +\infty,$$

and

$$\mathbb{E}\left(\int_0^t \int_{\mathbb{R}_+^*} d(x) \Big| F_{\varphi_s}(R_s, \mu_s - \delta_x) - F_{\varphi_s}(R_s, \mu_s) \Big| \mu_s(\mathrm{d}x) \mathrm{d}s \right) < +\infty.$$

Then $(\mathfrak{M}_{F,\varphi,t})_{t \geq 0}$ is a $(\mathcal{F}_t)_{t \geq 0}$-martingale.

(ii) *Suppose in addition that for all $t \geq 0$,*

$$\mathbb{E}\left(\int_0^t \int_{\mathbb{R}_+^*} b(x) \Big[F_{\varphi_s}(R_s, \mu_s + \delta_{x_0} + \delta_{x - x_0} - \delta_x) - F_{\varphi_s}(R_s, \mu_s) \Big]^2 \mu_s(\mathrm{d}x) \mathrm{d}s \right) < +\infty,$$

and

$$\mathbb{E}\left(\int_0^t \int_{\mathbb{R}_+^*} d(x) \Big[F_{\varphi_s}(R_s, \mu_s - \delta_x) - F_{\varphi_s}(R_s, \mu_s) \Big]^2 \mu_s(\mathrm{d}x) \mathrm{d}s \right) < +\infty.$$

Then $(\mathfrak{M}_{F,\varphi,t})_{t \geq 0}$ is a square-integrable martingale, with predictable quadratic variation given for all $t \geq 0$ by

$$\begin{aligned}\langle \mathfrak{M}_{F,\varphi} \rangle_t := & \int_0^t \int_{\mathbb{R}_+^*} b(x) \Big[F_{\varphi_s}(R_s, \mu_s + \delta_{x_0} + \delta_{x - x_0} - \delta_x) - F_{\varphi_s}(R_s, \mu_s) \Big]^2 \mu_s(\mathrm{d}x) \mathrm{d}s \\ & + \int_0^t \int_{\mathbb{R}_+^*} d(x) \Big[F_{\varphi_s}(R_s, \mu_s - \delta_x) - F_{\varphi_s}(R_s, \mu_s) \Big]^2 \mu_s(\mathrm{d}x) \mathrm{d}s\end{aligned}$$

Proof. We observe from Lemma II.1.10 that

$$\mathfrak{M}_{F,\varphi,t} = \int_0^t \int_{\mathcal{U}\times\mathbb{R}_+^*} \mathbb{1}_{\{u\in V_{s-}\}}\mathbb{1}_{\{h\leq b(\xi_{s-}^u)\}}\Big(F(R_s, \langle \mu_{s-} + \delta_{x_0} + \delta_{\xi_{s-}^u - x_0} - \delta_{\xi_{s-}^u}, \varphi_s\rangle) - F(R_s, \langle \mu_{s-}, \varphi_s\rangle)\Big)\widetilde{\mathcal{N}}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h)$$
$$+ \int_0^t \int_{\mathcal{U}\times\mathbb{R}_+^*} \mathbb{1}_{\{u\in V_{s-}\}}\mathbb{1}_{\{h\leq d(\xi_{s-}^u)\}}\Big(F(R_s, \langle \mu_{s-} - \delta_{\xi_{s-}^u}, \varphi_s\rangle) - F(R_s, \langle \mu_{s-}, \varphi_s\rangle)\Big)\widetilde{\mathcal{N}'}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h).$$

Then, Proposition II.1.18 follows from classical results from Ikeda and Watanabe on stochastic integrals with respect to Poisson point measures (see p.62 in [IW14]). □

In the following, we define $\mathcal{C}_\omega^{1,1}(\mathbb{R}^+\times\mathbb{R}_+^*)$ the set of functions $\varphi \in \mathcal{C}^{1,1}(\mathbb{R}^+\times\mathbb{R}_+^*)$ such that $\varphi : (t, x) \mapsto \omega(x)$ or

$$\exists C > 0, \forall x > 0, \quad \sup_{t\in\mathbb{R}^+}\left(|\varphi_t(x)| + |\partial_x\varphi_t(x)|\omega(x) + |\partial_t\varphi_t(x)|\frac{\omega(x)}{1 + x + \omega(x)}\right) \leq C\omega(x).$$

Note that the function $\varphi : (t, x) \mapsto \omega(x)$ does not necessarily verify the previous condition (in particular, ω' is not necessarily bounded). However, we include this specific function in $\mathcal{C}_\omega^{1,1}(\mathbb{R}^+\times\mathbb{R}_+^*)$, because we need to be able to apply the following results to this function for the proof of Theorem II.3.1.

Corollary II.1.19. *Under the general setting, let $\varphi \in \mathcal{C}_\omega^{1,1}(\mathbb{R}^+\times\mathbb{R}_+^*)$. Then the process $(\langle \mu_t, \varphi_t\rangle)_{t\geq 0}$ is a semi-martingale, with for all $t \geq 0$, a finite variation part given by*

$$V_{\varphi,t} := \langle \mu_0, \varphi_0\rangle + \int_0^t \langle \mu_s, \Phi_s(R_s, .)\rangle \mathrm{d}s - \int_0^t \int_{\mathbb{R}_+^*} d(x)\varphi_s(x)\mu_s(\mathrm{d}x)\mathrm{d}s + \int_0^t \int_{\mathbb{R}_+^*} b(x)\Big(\varphi_s(x_0) + \varphi_s(x - x_0) - \varphi_s(x)\Big)\mu_s(\mathrm{d}x)\mathrm{d}s,$$

with Φ associated to φ as in (II.1.10), and a square-integrable martingale part $\heartsuit_{\varphi,t} := \langle \mu_t, \varphi_t\rangle - V_{\varphi,t}$ whose predictable quadratic variation is given by

$$\langle \heartsuit_\varphi\rangle_t = \int_0^t \int_{\mathbb{R}_+^*} b(x)\Big[\varphi_s(x_0) + \varphi_s(x - x_0) - \varphi_s(x)\Big]^2 \mu_s(\mathrm{d}x)\mathrm{d}s + \int_0^t \int_{\mathbb{R}_+^*} d(x)\varphi_s^2(x)\mu_s(\mathrm{d}x)\mathrm{d}s.$$

Proof. We apply Proposition II.1.18 to $F : (R, x) \in [0, R_{\max}] \times \mathbb{R} \mapsto x$ and φ. By Lemma II.1.10, it suffices to show that for all $t \geq 0$,

$$\mathbb{E}\left(\int_0^t \int_{\mathbb{R}_+^*} b(x)\Big|\varphi_s(x_0) + \varphi_s(x - x_0) - \varphi_s(x)\Big|\mu_s(\mathrm{d}x)\mathrm{d}s\right) < +\infty,$$

$$\mathbb{E}\left(\int_0^t \int_{\mathbb{R}_+^*} b(x)\Big[\varphi_s(x_0) + \varphi_s(x - x_0) - \varphi_s(x)\Big]^2 \mu_s(\mathrm{d}x)\mathrm{d}s\right) < +\infty,$$

$$\mathbb{E}\left(\int_0^t \int_{\mathbb{R}_+^*} d(x)|\varphi_s(x)|\mu_s(\mathrm{d}x)\mathrm{d}s\right) < +\infty,$$

$$\mathbb{E}\left(\int_0^t \int_{\mathbb{R}_+^*} d(x)\varphi_s^2(x)\mu_s(\mathrm{d}x)\mathrm{d}s\right) < +\infty,$$

$$\mathbb{E}\left(\int_0^t \langle \mu_s, \Phi_s(R_s, .)\rangle \mathrm{d}s\right) < +\infty,$$

with Φ associated to φ as in (II.1.10). All of this follows from the assumption $\varphi \in \mathcal{C}_\omega^{1,1}(\mathbb{R}_+^*)$ (in particular, we use the fact that $||\partial_x \varphi_t||_\infty < +\infty$, so there exists a constant $C > 0$ such that $|\varphi_s(x - x_0) - \varphi_s(x)| \leq C x_0$ for every $s \in [0, t]$ and $x > 0$), Lemma II.1.14 and finally Proposition II.1.16. □

In the following, $\mathcal{C}_c^{1,1}([0, R_{\max}] \times \mathbb{R})$ is the set of $\mathcal{C}^{1,1}([0, R_{\max}] \times \mathbb{R})$ functions with compact support; and similarly, $\mathcal{C}_c^1(\mathbb{R}_+^*)$ is the set of $\mathcal{C}^1(\mathbb{R}_+^*)$ functions with compact support. We write $\mathcal{D}$ for the set of functions $F_\varphi \in \mathfrak{F}$, with $F \in \mathcal{C}_c^{1,1}([0, R_{\max}] \times \mathbb{R})$ and $\varphi \in \mathcal{C}_c^1(\mathbb{R}_+^*)$. We endow $\mathcal{M}_P(\mathbb{R}_+^*)$ with the topology of vague convergence, and $[0, R_{\max}]$ with the usual topology. The space $\mathcal{M}_P(\mathbb{R}_+^*) \times [0, R_{\max}]$ endowed with the product of these two topologies is a Polish space (see Appendix B.2.1 for details). The set $\mathcal{D}$ is composed of bounded continuous functions, and is an algebra that separates points, thus it is a separating class of functions for $\mathcal{M}_P(\mathbb{R}_+^*) \times [0, R_{\max}]$ endowed with the previously described topology (Theorem 4.5 p.113 in [EK86]). Hence, it suffices to describe the extended generator of $(\mu_t, R_t)_{t\geq 0}$ on $\mathcal{D}$ to characterize the distribution of this process. We define the operator $\mathfrak{L}$ such that for every $F_\varphi \in \mathcal{D}$, for every $(\mu, r) \in \mathcal{M}_P(\mathbb{R}_+^*) \times [0, R_{\max}]$,

$$\begin{aligned}
\mathfrak{L}F_\varphi(\mu, r) = \rho(\mu, r)\partial_r F_\varphi(\mu, r) &+ \langle \mu, g(., r)\varphi'(.)\rangle \partial_x F_\varphi(\mu, r) \\
&+ \int_{\mathbb{R}_+^*} b(x)\Big(F_\varphi(\mu + \delta_{x_0} + \delta_{x-x_0} - \delta_x, r) - F_\varphi(\mu, r)\Big)\mu(\mathrm{d}x) \\
&+ \int_{\mathbb{R}_+^*} d(x)\Big(F_\varphi(\mu - \delta_x, r) - F_\varphi(\mu, r)\Big)\mu(\mathrm{d}x).
\end{aligned}$$

Finally, for any $\varphi \in \mathcal{C}^1(\mathbb{R}_+^*)$ we continue to write $(\mathfrak{M}_{F,\varphi,t})_{t\geq 0}$ for the process associated to $(t, x) \mapsto \varphi(x)$ as in (II.1.19).

Proposition II.1.20. *Under the general setting, the process $(\mu_t, R_t)_{t\geq 0}$ is a JMP with extended generator $\mathfrak{L}$, where the domain $D(\mathfrak{L})$ of this generator contains $\mathcal{D}$. Moreover, for every $F_\varphi \in \mathcal{D}$, the process $(\mathfrak{M}_{F,\varphi,t})_{t\geq 0}$ is a square-integrable martingale.*

Proof. We refer to the formalization of JMPs in [JS96]. Under the general setting, from the iterative construction of Section II.1.2, and the strong Markov property for Poisson point processes, $(\mu_t, R_t)_{t\geq 0}$ is a strong Markov process. Furthermore, the filtration $(\mathcal{F}_t)_{t\geq 0}$ is a jumping filtration, associated to the jumping sequence $(J_n)_{n\in\mathbb{N}^*}$, and this defines a JMP. If we fix $t \geq 0$ and $F_\varphi \in \mathcal{D}$, the associated F, $\partial_r F$, $\partial_x F$, φ and φ' are continuous and have compact supports, so the function $s \in [0, t] \mapsto \mathfrak{L}F_\varphi(\mu_s, R_s)$ is almost surely bounded, thus integrable. In particular, for the integral term with b, as for the control of the first two expectations in the proof of Corollary II.1.19, we use the fact that φ' is bounded (hence φ is Lipschitz continuous and we can control quantities of the form $|\varphi(x-x_0)-\varphi(x)|$ for $x > 0$). We also use the second point of Assumption II.1.13 and Proposition II.1.16. The fact that $(\mathfrak{M}_{F,\varphi,t})_{t\geq 0}$ is a martingale then follows from (i) in Proposition II.1.18 applied to F_φ. In the same manner, we verify (ii) in Proposition II.1.18, which entails that $(\mathfrak{M}_{F,\varphi,t})_{t\geq 0}$ is a square-integrable martingale. □

Remark: After the construction of the process $(\mu_t, R_t)_t$, one could already have guessed the shape of the extended generator $\mathfrak{L}$ from Lemma II.1.10, and this is exactly how we proceeded. Then, to justify this expression for $\mathfrak{L}$, we needed to find conditions under which $J_\infty = +\infty$ almost surely, which essentially comes down to control quantities of the form $\mathbb{E}(F_\varphi(\mu_t, R_t))$ for $t \geq 0$. Also, in view of the tightness result in Theorem II.3.1, we wanted to guarantee the existence of square-integrable martingales with explicit predictable quadratic variations. In order to solve these problems, we naturally came up with the conditions expressed in Assumption II.1.13.

II.2 Renormalization of the process

In this section, we work under Assumptions II.1.4, II.1.5 and II.1.13. In Section II.2.1, we define a sequence $\left((\mu_t^K, R_t^K)_{t\geq 0}\right)_{K\in\mathbb{N}^*}$, where every process $\left(\mu_t^K, R_t^K\right)_{t\geq 0}$ is a renormalization of the initial process $(\mu_t, R_t)_{t\geq 0}$ defined in Section II.1.2, and K is a scaling parameter representing the population size of the initial condition, meant to diverge towards $+\infty$. In Section II.2.2, we use the results of Section II.1 to obtain martingale and control properties for the renormalized process $(\mu_t^K, R_t^K)_{t\geq 0}$.

II.2.1 Definition of the renormalized process

We follow a classical procedure, first described in [FM04], and then reproduced in many articles [CFM08, Tra08, CF15, Tch24]. First, for every $K \in \mathbb{N}^*$, we will define an auxiliary process $(\nu_t^K, R_t^K)_{t\geq 0}$, following the exact same construction as in Section II.1.2, but with an inverse conversion efficiency $\chi_K := \chi/K$. Thus, all the results of Section II.1 will apply to $(\nu_t^K, R_t^K)_{t\geq 0}$, simply replacing χ with χ_K. We begin with the definition of the renormalized deterministic flow followed by individual energies between jumps. We consider an initial condition $(\nu_0^K, R_0) \in \mathcal{M}_P(\mathbb{R}_+^*) \times [0, R_{\max}]$ at time 0, which means that there exists $N \in \mathbb{N}$ and $(\xi_0^{u,K})_{1\leq u\leq N}$ in $(\mathbb{R}_+^*)^N$ such that

$$\nu_0^K := \sum_{u=1}^N \delta_{\xi_0^{u,K}},$$

where $\nu_0^K = 0$ if $N = 0$. We write $\left((X_t^{u,K}(\nu_0^K, R_0))_{1\leq u\leq N}, X_t^{\mathfrak{R},K}(\nu_0^K, R_0)\right)$ for a solution to the system of $N+1$ coupled equations

$$\frac{\mathrm{d}X_t^{\mathfrak{R},K}(\nu_0^K, R_0)}{\mathrm{d}t} = \varsigma(X_t^{\mathfrak{R},K}(\nu_0^K, R_0)) - \frac{\chi}{K}\int_{\mathbb{R}_+^*} f(x, X_t^{\mathfrak{R},K}(\nu_0^K, R_0))\tilde{\nu}_t^K(\mathrm{d}x), \tag{II.2.20}$$

$$\frac{\mathrm{d}X_t^{u,K}(\nu_0^K, R_0)}{\mathrm{d}t} = g(X_t^{u,K}(\nu_0^K, R_0), X_t^{\mathfrak{R},K}(\nu_0^K, R_0)) \quad \text{for } 1 \leq u \leq N, \tag{II.2.21}$$

where $\tilde{\nu}_t^K := \sum_{i=1}^N \delta_{X_t^{u,K}(\nu_0^K, R_0)}$, and with initial condition at time 0

$$\begin{aligned} X_0^{\mathfrak{R},K}(\nu_0^K, R_0) &= R_0, \\ X_0^{u,K}(\nu_0^K, R_0) &= \xi_0^{u,K} \quad \text{for } 1 \leq u \leq N. \end{aligned}$$

Note that the system of equations (II.2.20)-(II.2.21) is similar to (II.1.4)-(II.1.5), where we only replace χ by χ/K. With the same arguments as in Proposition II.1.3, we can define for t in a neighborhood of 0, denoted as $[0, t_{\exp}^K(\nu_0^K, R_0))$ the renormalized flow

$$X_t^K(\nu_0^K, R_0) := \left(\sum_{u=1}^{N} \delta_{X_t^{u,K}(\nu_0^K, R_0)}, X_t^{\Re,K}(\nu_0^K, R_0) \right),$$

and it benefits from the same regularity properties as X of Section II.1.2. We also adopt the convention depicted in the remark after Corollary II.1.7 to make sense of the previous notation for $t \geq t_{\exp}^K(\nu_0^K, R_0)$.

Definition II.2.1 (**Renormalized process**). *Let $R_0 \in [0, R_{\max}]$ be a random variable. Let $(\nu_0^K)_{K\geq 1}$ be a sequence of random variables in $\mathcal{M}_P(\mathbb{R}_+^*)$, such that*

$$\sup_{K\geq 1} \left(\frac{1}{K} \mathbb{E}\left(\langle \nu_0^K, 1 + \mathrm{Id} + \omega \rangle \right) \right) < +\infty. \tag{II.2.22}$$

We use the Poisson point measures $\mathcal{N}$ and $\mathcal{N}'$ of Section II.1.2, independent from R_0 and $(\nu_0^K)_{K\geq 1}$. For every $K \geq 1$, the renormalized process $(\mu_t^K, R_t^K)_{t\geq 0}$ with initial condition (μ_0^K, R_0), is given for every $t \geq 0$ by $\mu_t^K := \dfrac{\nu_t^K}{K}$ and

$$\begin{aligned}
(\nu_t^K, R_t^K) = X_t^K&(\nu_0^K, R_0) \\
&+ \int_0^t \int_{\mathcal{U}\times\mathbb{R}_+^*} \mathbb{1}_{\{u\in V_{s-}^K\}} \mathbb{1}_{\{h\leq b(\xi_{s-}^{u,K})\}} [X_{t-s}^K(\nu_{s-}^K + \delta_{x_0} + \delta_{\xi_{s-}^{u,K}-x_0} - \delta_{\xi_{s-}^{u,K}}, R_s^K) \\
&\qquad\qquad - X_{t-s}^K(\nu_{s-}^K, R_s^K)] \mathcal{N}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h) \\
&+ \int_0^t \int_{\mathcal{U}\times\mathbb{R}_+^*} \mathbb{1}_{\{u\in V_{s-}^K\}} \mathbb{1}_{\{h\leq d(\xi_{s-}^{u,K})\}} [X_{t-s}^K(\nu_{s-}^K - \delta_{\xi_{s-}^{u,K}}, R_s^K) \\
&\qquad\qquad - X_{t-s}^K(\nu_{s-}^K, R_s^K)] \mathcal{N}'(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h),
\end{aligned}$$

where for all $t \geq 0$, V_t^K is the set containing alive individuals at time t, and $\xi_t^{u,K}$ are individual energies, defined and actualized over time with the same conventions as in Section II.1.2, simply replacing the flow X by X^K.

Under Assumptions II.1.4, II.1.5 and II.1.13, from the results of Section II.1 (see Corollary II.1.17), for every $K \in \mathbb{N}^*$ and almost surely, the renormalized process $(\mu_t^K, R_t^K)_{t\geq 0}$ is well-defined, *i.e.* individual energies do not vanish/explode and there is no accumulation of jumps in finite time. Our motivation is to keep the same amount of resources and temporal dynamics, but to consider population sizes going to $+\infty$. Intuitively, the way we proceed is to make every individual in the population smaller, and the interaction between individuals via resource consumption proportional to their typical size. The parameter $1/K$ represents the amount of resource consumed by a single individual, and modelling a population of K such individuals leads us back to the temporal dynamics of Section II.1.2. In the case of a chemostat, we can also interpret K as a scaling parameter of the total volume of the vessel where bacteria interact with nutrients (see Section 5.1 in [CF15]).

Definition II.2.2 (**Renormalized setting**). *In the following, we denote as 'the renormalized setting', the framework refering to the previous renormalization. Hence, we work with Assumptions II.1.4, II.1.5, II.1.13, and the condition* (II.2.22)*.*

Remark: Note that for any $K \geq 1$, Assumptions II.1.4, II.1.5 and II.1.13 do not depend on K. In particular, we work with a fixed weight function ω that does not depend on K, and verifies Assumption II.1.13. Moreover, the renormalized setting implies the general setting (the construction of Section II.1.2 accounts for the case $K = 1$).

II.2.2 Properties of the renormalized process

We use the results of Section II.1 to obtain similar properties for the renormalized process $(\mu_t^K, R_t^K)_{t\geq 0}$. We also prove an additional result in Lemma II.2.6.

Lemma II.2.3. *Let $\varphi \in \mathcal{C}^{1,1}(\mathbb{R}^+ \times \mathbb{R}_+^*)$, and $F \in \mathcal{C}^{1,1}([0, R_{\text{in}}] \times \mathbb{R})$. Under Assumptions II.1.4, II.1.5 and II.1.13, we have almost surely, for all $K \in \mathbb{N}^*$, $t \geq 0$,*

$$
\begin{aligned}
F(R_t^K, \langle \mu_t^K, \varphi_t\rangle) = F(R_0, \langle \mu_0^K, \varphi_0\rangle)
&+ \int_0^t \rho(\mu_s^K, R_s^K)\partial_r F(R_s^K, \langle \mu_s^K, \varphi_s\rangle) \\
&\qquad + \langle \mu_s^K, \Phi_s(R_s^K, .)\rangle \partial_x F(R_s^K, \langle \mu_s^K, \varphi_s\rangle)\mathrm{d}s \\
&+ \int_0^t \int_{\mathcal{U}\times\mathbb{R}_+^*} \mathbb{1}_{\{u\in V_{s-}^K\}}\mathbb{1}_{\{h\leq b(\xi_{s-}^{u,K})\}} \\
&\qquad \left(F(R_s^K, \langle \mu_{s-}^K + \frac{1}{K}\left(\delta_{x_0} + \delta_{\xi_{s-}^{u,K}-x_0} - \delta_{\xi_{s-}^{u,K}}\right), \varphi_s\rangle) \right. \\
&\qquad \left. - F(R_s^K, \langle \mu_{s-}^K, \varphi_s\rangle)\right)\mathcal{N}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h) \\
&+ \int_0^t \int_{\mathcal{U}\times\mathbb{R}_+^*} \mathbb{1}_{\{u\in V_{s-}^K\}}\mathbb{1}_{\{h\leq d(\xi_{s-}^{u,K})\}}\left(F(R_s^K, \langle \mu_{s-}^K - \frac{1}{K}\delta_{\xi_{s-}^{u,K}}, \varphi_s\rangle)\right. \\
&\qquad \left. - F(R_s^K, \langle \mu_{s-}^K, \varphi_s\rangle)\right)\mathcal{N}'(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h),
\end{aligned}
$$

with Φ associated to φ as in (II.1.10).

Proof. For $K \in \mathbb{N}^*$ and $t \geq 0$, it suffices to remark that

$$F(R_t^K, \langle \mu_t^K, \varphi_t\rangle) = F\left(R_t^K, \langle \nu_t^K, \frac{1}{K}\varphi_t\rangle\right),$$

and to apply Lemma II.1.10. Remark in particular that we recover the function ρ because we pick the appropriate renormalization of the parameter χ in (II.1.4). The decomposition is almost surely valid for every $t \geq 0$ thanks to Corollary II.1.17. □

In the following, we naturally write $N_t^K := \langle \mu_t^K, 1\rangle$, $E_t^K := \langle \mu_t^K, \mathrm{Id}\rangle$ and $\Omega_t^K := \langle \mu_t^K, \omega\rangle$ for $t \geq 0$ and $K \in \mathbb{N}^*$.

Proposition II.2.4. *Under the renormalized setting, we have for all $T \geq 0$ that*

$$\sup_{K\in\mathbb{N}^*} \mathbb{E}\left(\sup_{t\in[0,T]} (E_t^K + N_t^K + \Omega_t^K)\right) < +\infty.$$

This immediately implies that for every $\varphi \in \mathfrak{B}_{1+\mathrm{Id}+\omega}(\mathbb{R}_+^*)$, we have

$$\sup_{K\in\mathbb{N}^*} \mathbb{E}\left(\sup_{t\in[0,T]} |\langle \mu_t^K, \varphi\rangle|\right) < +\infty.$$

Proof. Let $K \geq 1$ and $T \geq 0$. With the exact same techniques as in the proof of Proposition II.1.16, and thanks to Corollary II.1.17, we obtain similarly as in (II.1.17) that

$$\mathbb{E}\left(\sup_{t\in[0,T]} \langle \nu_t^K, 1 + \mathrm{Id} + \omega\rangle\right) \leq \left(R_{\max} + \mathbb{E}(\langle \nu_0^K, 1 + \mathrm{Id} + \omega\rangle) + T||\varsigma||_{\infty,[0,R_{\max}]}\right)e^{(C_g+C_b)T}.$$

We conclude with (II.2.22) and the fact that that $\mu^K := \nu^K/K$. □

Proposition II.2.5. *Under the renormalized setting, let $\varphi \in \mathcal{C}^{1,1}_{\omega}(\mathbb{R}^+ \times \mathbb{R}^*_+)$ and $K \in \mathbb{N}^*$. Then the process $(\langle \mu_t^K, \varphi_t\rangle)_{t\geq 0}$ is a semi-martingale, with for all $t \geq 0$, a finite variation part given by*

$$\begin{aligned}V_{\varphi,t}^K := \langle \mu_0^K, \varphi_0\rangle + \int_0^t \langle \mu_s^K, \Phi_s(R_s^K, .)\rangle \mathrm{d}s - \int_0^t \int_{\mathbb{R}^*_+} d(x)\varphi_s(x)\mu_s^K(\mathrm{d}x)\mathrm{d}s \\ + \int_0^t \int_{\mathbb{R}^*_+} b(x)\Big(\varphi_s(x_0) + \varphi_s(x - x_0) - \varphi_s(x)\Big)\mu_s^K(\mathrm{d}x)\mathrm{d}s,\end{aligned}$$

with Φ associated to φ as in (II.1.10), and a square-integrable martingale part $\heartsuit_{\varphi,t}^K := \langle \mu_t^K, \varphi_t\rangle - V_{\varphi,t}^K$ whose predictable quadratic variation is given by

$$\begin{aligned}\langle \heartsuit_{\varphi}^K\rangle_t = \frac{1}{K}\Bigg(\int_0^t \int_{\mathbb{R}^*_+} b(x)\Big[\varphi_s(x_0) + \varphi_s(x - x_0) - \varphi_s(x)\Big]^2 \mu_s^K(\mathrm{d}x)\mathrm{d}s \\ + \int_0^t \int_{\mathbb{R}^*_+} d(x)\varphi_s^2(x)\mu_s^K(\mathrm{d}x)\mathrm{d}s\Bigg).\end{aligned}$$

Proof. The proof is similar to the proof of Corollary II.1.19, using Lemma II.2.3 and Proposition II.2.4. □

We end this section with the following results, which will be useful in Section II.4 for the proof of Theorem II.3.1.

Lemma II.2.6. *Let $p > 1$, and under the renormalized setting, assume in addition that*

$$\sup_{K\in\mathbb{N}^*} \mathbb{E}\left((E_0^K + N_0^K + \Omega_0^K)^p\right) < +\infty. \tag{II.2.23}$$

Then, for all $T \geq 0$, we have

$$\sup_{K\in\mathbb{N}^*} \mathbb{E}\left(\sup_{t\in[0,T]} (E_t^K + N_t^K + \Omega_t^K)^p\right) < +\infty.$$

This immediately implies that for every $\varphi \in \mathfrak{B}_{1+\mathrm{Id}+\omega}(\mathbb{R}^*_+)$, we have

$$\sup_{K\in\mathbb{N}^*} \mathbb{E}\left(\sup_{t\in[0,T]} \left|\langle \mu_t^K, \varphi\rangle\right|^p\right) < +\infty.$$

Proof. We fix $T \geq 0$, $K \geq 1$, and apply Lemma II.2.3 to $F : (r, x) \mapsto x^p \mathbb{1}_{\{x>0\}}$ (which is non-negative, non-decreasing on the variable x and $\mathcal{C}^{1,1}([0, R_{\max}] \times \mathbb{R})$ because $p > 1$) and $\varphi : (t, x) \mapsto 1 + x + \omega(x)$, to obtain for every $t \in [0, T]$,

$$
\begin{aligned}
\left(E_t^K + N_t^K + \Omega_t^K\right)^p = &\left(E_0^K + N_0^K + \Omega_0^K\right)^p \\
&+ p \int_0^t \left\langle \mu_s^K, g(., R_s^K)(1+\omega') \right\rangle \left(E_s^K + N_s^K + \Omega_s^K\right)^{p-1} \mathrm{d}s \\
&+ \int_0^t \int_{\mathcal{U} \times \mathbb{R}_+^*} \mathbb{1}_{\{u \in V_{s-}^K\}} \mathbb{1}_{\{h \leq b(\xi_{s-}^{u,K})\}} \\
&\quad \Bigg[F\Bigg(R_s^K, E_{s-}^K + N_{s-}^K + \Omega_{s-}^K \\
&\quad\quad + \frac{1}{K}\left(1 + \omega(x_0) + \omega\left(\xi_{s-}^{u,K} - x_0\right) - \omega\left(\xi_{s-}^{u,K}\right)\right) \Bigg) \\
&\quad\quad - F\left(R_s^K, E_{s-}^K + N_{s-}^K + \Omega_{s-}^K\right) \Bigg] \mathcal{N}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h) \\
&+ \int_0^t \int_{\mathcal{U} \times \mathbb{R}_+^*} \mathbb{1}_{\{u \in V_{s-}^K\}} \mathbb{1}_{\{h \leq d(\xi_{s-}^{u,K})\}} \\
&\quad \Bigg[F\Bigg(R_s^K, E_{s-}^K + N_{s-}^K + \Omega_{s-}^K - \frac{1}{K}\left(1 + \xi_{s-}^{u,K} + \omega\left(\xi_{s-}^{u,K}\right)\right) \Bigg) \\
&\quad\quad - F\left(R_s^K, E_{s-}^K + N_{s-}^K + \Omega_{s-}^K\right) \Bigg] \mathcal{N}'(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h) \\
\leq &\left(E_0^K + N_0^K + \Omega_0^K\right)^p + pC_g \int_0^T \left(E_s^K + N_s^K + \Omega_s^K\right)^p \mathrm{d}s \\
&+ \int_0^T \int_{\mathcal{U} \times \mathbb{R}_+^*} \mathbb{1}_{\{u \in V_{s-}^K\}} \mathbb{1}_{\{h \leq b(\xi_{s-}^{u,K})\}} \\
&\quad \Bigg[\left(E_{s-}^K + N_{s-}^K + \Omega_{s-}^K + \frac{1 + \omega(x_0)}{K} \right)^p \\
&\quad\quad - \left(E_{s-}^K + N_{s-}^K + \Omega_{s-}^K\right)^p \Bigg] \mathcal{N}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}h).
\end{aligned}
$$

We used the fact that ω is positive and non-decreasing, the first point of Assumption II.1.13 and the fact that F is non-decreasing on the variable x. Then, we assess that there exists $C_{p,x_0} > 0$ such that for every $y > 0$,

$$
\left(y + \frac{1 + \omega(x_0)}{K}\right)^p - y^p \leq \frac{C_{p,x_0}}{K}(1 + y^{p-1}).
$$

For $M > 0$, as in Proposition II.1.16, we define the stopping times

$$
\tau_M^K := \inf\left\{t \geq 0,\ E_t^K + N_t^K + \Omega_t^K \geq M\right\},
$$

with the convention $\inf(\varnothing) = +\infty$. We consider the supremum over $[0, T \wedge \tau_M^K]$ of $\left(E_t^K + N_t^K + \Omega_t^K\right)^p$ and then take expectations. In particular, we verify that integrals against $\mathcal{N}$ are true semi-martingales thanks to the definition of the stopping time τ_M^K. We also apply Fubini theorem because all the integrands are positive and this leads to

$$\begin{aligned}
&\mathbb{E}\bigg(\sup_{t\in[0,T\wedge\tau_M^K]}\left(E_t^K+N_t^K+\Omega_t^K\right)^p\bigg)\\
&\qquad\leq \mathbb{E}\left(\left(E_0^K+N_0^K+\Omega_0^K\right)^p\right)+pC_g\int_0^{T\wedge\tau_M^K}\mathbb{E}\left(\left(E_s^K+N_s^K+\Omega_s^K\right)^p\right)\mathrm{d}s\\
&\qquad\quad+\frac{C_{p,x_0}}{K}\int_0^{T\wedge\tau_M^K}\mathbb{E}\left(\langle\nu_{s-}^K,b\rangle\left(1+\left(E_{s-}^K+N_{s-}^K+\Omega_{s-}^K\right)^{p-1}\right)\right)\mathrm{d}s\\
&\qquad\leq \mathbb{E}\left(\left(E_0^K+N_0^K+\Omega_0^K\right)^p\right)+pC_g\int_0^{T\wedge\tau_M^K}\mathbb{E}\left(\left(E_s^K+N_s^K+\Omega_s^K\right)^p\right)\mathrm{d}s\\
&\qquad\quad+C_{p,x_0}C_b\int_0^{T\wedge\tau_M^K}\mathbb{E}\left(E_{s-}^K+N_{s-}^K+\Omega_{s-}^K+\left(E_{s-}^K+N_{s-}^K+\Omega_{s-}^K\right)^p\right)\mathrm{d}s,
\end{aligned}$$

where we used the second point of Assumption II.1.13, and in particular the fact that it implies (II.1.12). Remark that all the expectations are well-defined and finite until time τ_M^K by definition of this stopping time. Finally, we use the fact that for $x\geq 0$, $x+x^p\leq 2(1+x^p)$ to obtain

$$\begin{aligned}
&\mathbb{E}\bigg(\sup_{t\in[0,T\wedge\tau_M^K]}\left(E_t^K+N_t^K+\Omega_t^K\right)^p\bigg)\\
&\qquad\leq \mathbb{E}\left(\left(E_0^K+N_0^K+\Omega_0^K\right)^p\right)+2TC_{p,x_0}C_b\\
&\qquad\quad+\left(pC_g+2C_{p,x_0}C_b\right)\int_0^T\mathbb{E}\left(\sup_{u\in[0,s\wedge\tau_M^K]}\left(E_u^K+N_u^K+\Omega_u^K\right)^p\right)\mathrm{d}s.
\end{aligned}$$

We conclude with Gronwall lemma that

$$\begin{aligned}
&\mathbb{E}\bigg(\sup_{t\in[0,T\wedge\tau_M^K]}\left(E_t^K+N_t^K+\Omega_t^K\right)^p\bigg)\\
&\qquad\leq\left(\mathbb{E}\left(\left(E_0^K+N_0^K+\Omega_0^K\right)^p\right)+2TC_{p,x_0}C_b\right)e^{(pC_g+2C_{p,x_0}C_b)T}.
\end{aligned}$$

By (II.2.23), this upper bound is finite, uniformly on K, and does not depend on M. By the same arguments as in the proof of Proposition II.1.16, the sequence τ_M^K goes to $+\infty$ when $M\to+\infty$, and this ends the proof. □

Corollary II.2.7. *Under the renormalized setting, let $\varphi\in\mathcal{C}_\omega^{1,1}(\mathbb{R}_+^*)$ and $K\in\mathbb{N}^*$. Assume that there exists $p>1$ such that*

$$\sup_{K\in\mathbb{N}^*}\mathbb{E}\left(\left(E_0^K+N_0^K+\Omega_0^K\right)^p\right)<+\infty.$$

Then, for all $t\geq 0$, the family of square-integrable martingales $\left(\heartsuit_{\varphi,t}^K\right)_{K\in\mathbb{N}^*}$ defined in Proposition II.2.5 is uniformly integrable.

***Proof*.** This follows from Lemma II.1.14, Proposition II.2.5, Lemma II.2.6 applied to φ and p, and Proposition 2.2 p.494 in [EK86]. □

II.3 Main theorem and sketch of the proof

In this section, we give our main result in Theorem II.3.1, and the sketch of its proof. We begin with preliminary definitions and consider a function $w : \mathbb{R}_+^* \to \mathbb{R}_+^*$. We write $\mathcal{M}_w(\mathbb{R}_+^*)$ for the set of positive measures μ on $\mathbb{R}_+^*$ such that $\langle \mu, w \rangle < +\infty$. In particular, $\mathcal{M}_P(\mathbb{R}_+^*) \subseteq \mathcal{M}_w(\mathbb{R}_+^*)$. We define $\mathcal{C}_c(\mathbb{R}_+^*)$ the space of continous functions with compact support, and $\mathcal{C}_w(\mathbb{R}_+^*)$ the space of continuous functions f such that $f \in \mathfrak{B}_w(\mathbb{R}_+^*)$. The vague, respectively w-weak, topology on $\mathcal{M}_w(\mathbb{R}_+^*)$ is the finest topology for which the applications $\mu \mapsto \langle \mu, f \rangle$ are continous, with f in $\mathcal{C}_c(\mathbb{R}_+^*)$, respectively in $\mathcal{C}_w(\mathbb{R}_+^*)$. We write $(\mathcal{M}_w(\mathbb{R}_+^*), v)$, respectively $(\mathcal{M}_w(\mathbb{R}_+^*), \mathrm{w})$, when we endow $\mathcal{M}_w(\mathbb{R}_+^*)$ with the vague topology, respectively the w-weak topology.
The latter notation is not standard in the literature and can be seen as a weighted version of the usual weak topology, which corresponds to the case $w \equiv 1$. We introduce it because in Theorem II.3.1, our processes will take values in such weigthed spaces of measures. The w-weak topology is always finer than the vague topology. However, note immediately that depending on the weight function w, the w-weak topology is not necessarily comparable to the usual weak topology. This is why we establish carefully distinct tightness results for both topologies in the sequel (our main result, Theorem II.3.1, is established with respect to the ω-weak topology, whereas an extension of this theorem that includes the usual weak topology is given in Theorem II.5.3). For example, if $w(x) \xrightarrow[x\to 0]{} 0$ and $w(x) \xrightarrow[x\to +\infty]{} +\infty$, a measure $\mu \in \mathcal{M}_w(\mathbb{R}_+^*)$ integrate functions that are negligible compared to a constant function near 0, but dominate bounded functions near $+\infty$. The reader can refer to Appendix B.2.1 for further details on these two topologies, and especially the fact that both spaces $(\mathcal{M}_w(\mathbb{R}_+^*), v)$ and $(\mathcal{M}_w(\mathbb{R}_+^*), \mathrm{w})$ are Polish spaces. We naturally endow $[0, R_{\max}]$ with the usual topology, and $(M_w(\mathbb{R}_+^*), v) \times [0, R_{\max}]$ or $(M_w(\mathbb{R}_+^*), \mathrm{w}) \times [0, R_{\max}]$ with the product topology, and these are again Polish spaces.

For $i = v$ or w, we write $\mathbb{D}([0,T], (\mathcal{M}_w(\mathbb{R}_+^*), i) \times [0, R_{\max}])$, respectively $\mathcal{C}([0,T], (\mathcal{M}_w(\mathbb{R}_+^*), i) \times [0, R_{\max}])$, for the space of càdlàg, respectively continuous, functions from $[0,T]$ to the space $(\mathcal{M}_w(\mathbb{R}_+^*), i) \times [0, R_{\max}]$. These spaces are endowed with the usual Skorokhod topology, hence are Polish spaces (see Appendix B.2.2 for details). Finally, for every $T \geq 0$, we define $\mathcal{C}^{1,1}_{\omega,T}(\mathbb{R}^+ \times \mathbb{R}_+^*)$ the set of functions $\varphi \in \mathcal{C}^{1,1}(\mathbb{R}^+ \times \mathbb{R}_+^*)$ such that

$$\exists C > 0, \forall x > 0, \quad \sup_{t\in[0,T]} \Big(|\varphi_t(x)|\,(1 + d(x)) + |\partial_x \varphi_t(x)|\omega(x) + |\partial_t \varphi_t(x)| \Big) \leq C\omega(x).$$

For technical reasons, just before stating our main theorem, we formulate the following additional assumption on the weight function ω.

Assumption II.3.1. *The functions ω (weight function), b (birth rate) and $\overline{g}$ (maximal speed of energy gain) are such that*

- $\exists c_b > 0, \forall x > 1, \quad b(x) \leq c_b \omega(x),$
- $\exists c_g > 0, \forall x > 0, \quad \overline{g}(x) \leq c_g \omega(x).$

This assumption is natural in a first approach and we use it in particular in Section II.4.4. It is possible to make it less restrictive on the choice of b, $\overline{g}$ and ω, and we present in Section II.5.2 a refined argument to obtain Theorem II.3.1 with a lighter assumption.

Theorem II.3.1. *We work under the renormalized setting and Assumption II.3.1. Let* $\left((\mu_t^K, R_t^K)_{t\geq 0}\right)_{K\in\mathbb{N}^*}$ *the sequence of renormalized processes defined in Section II.2 be such that*

- *there exists a random variable* $\mu_0^* \in \mathcal{M}_\omega(\mathbb{R}_+^*)$ *such that* $(\mu_0^K)_{K\in\mathbb{N}^*}$ *converges in law towards* μ_0^* *in* $(\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w})$,
- *there exists* $p > 1$ *such that* (II.2.23) *holds true.*

Then, for all $T \geq 0$, $\left((\mu_t^K, R_t^K)_{t\in[0,T]} \right)_{K\in\mathbb{N}^*}$ *is tight in* $\mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w}) \times [0, R_{\max}])$. *Any of its accumulation point* $(\mu_t^*, R_t^*)_{t\in[0,T]}$ *is in* $\mathcal{C}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w}) \times [0, R_{\max}])$ *and deterministic conditionnally to* μ_0^*. *Furthermore, it verifies almost surely, for all* $t \in [0,T]$,

$$R_t^* = R_0 + \int_0^t \rho(R_s^*, \mu_s^*)\mathrm{d}s \tag{II.3.24}$$

and for every $\varphi \in \mathcal{C}^{1,1}_{\omega,T}(\mathbb{R}^+ \times \mathbb{R}_+^*)$,

$$\langle \mu_t^*, \varphi_t \rangle = \langle \mu_0^*, \varphi_0 \rangle + \int_0^t \int_{\mathbb{R}_+^*} \Big(\Phi_s(R_s^*, x) + b(x)(\varphi_s(x_0) + \varphi_s(x - x_0) - \varphi_s(x)) - d(x)\varphi_s(x) \Big) \mu_s^*(\mathrm{d}x)\mathrm{d}s, \tag{II.3.25}$$

with Φ *associated to* φ *as in* (II.1.10).

On the one hand, the tightness result of Theorem II.3.1 is an important contribution to the study of individual-based models as in Section II.1.2 with unbounded growth, birth and/or death rates. In the literature, when the previously mentioned rates are bounded, one of the main technical point of the proof is to provide uniform (on K and $t \in [0,T]$) bounds and martingale properties for quantities of the form $\langle \mu_t^K, 1 \rangle$. With Lemma II.2.3 in mind, one can relate $\langle \mu_t^K, 1 \rangle$ and quantities of the form $\langle \mu_t^K, b \rangle$ and $\langle \mu_t^K, d \rangle$. When rates are bounded, $\langle \mu_t^K, b \rangle$ and $\langle \mu_t^K, d \rangle$ are themselves controlled by $C\langle \mu_t^K, 1 \rangle$ with a constant $C > 0$. One classically deduces a functional equation verified by $\langle \mu_t^K, 1 \rangle$ and concludes with Gronwall lemma. It is then possible to make sense of limiting quantities of the form $\langle \mu_t^*, b \rangle$ and $\langle \mu_t^*, d \rangle$. With unbounded rates, the previous technique does not work anymore, and it is even possible that for $t \in [0,T]$, quantities of the form $\langle \mu_t^*, b \rangle$ and $\langle \mu_t^*, d \rangle$ are infinite. Hence, instead of controlling quantities of the form $\langle \mu_t^K, 1 \rangle$, we search for a function ω such that $\langle \mu_t^K, \omega \rangle$ is related to $\langle \mu_t^K, b\omega \rangle$ and $\langle \mu_t^K, d\omega \rangle$, and the latter quantities are themselves controlled by $C\langle \mu_t^K, \omega \rangle$ with a constant $C > 0$. Also, we want to be able to define $\langle \mu_t^*, b\omega \rangle$ and $\langle \mu_t^*, d\omega \rangle$ as finite quantities. The constraints that have to be verified by ω are expressed in Assumptions II.1.13 and II.3.1. We thus work in a weighted space with respect to the function ω and apply the same procedure as in the classical case. We recover a classical result of tightness for measure-valued processes, initiated in [FM04], but without *a priori* bounds on the growth, birth and/or death rates.

On the other hand, there is a price to pay to obtain this general tightness result. It holds true only in the weighted space $\mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w}) \times [0, R_{\max}])$, with a weight function ω verifying Assumptions II.1.13 and II.3.1. In particular, if $d(x) \xrightarrow[x\to 0]{} +\infty$, then $\omega(x) \xrightarrow[x\to 0]{} 0$. For biological reasons, this is the typical case we want to investigate if we think of an unbounded death rate (the Metabolic Theory of Ecology assumes a death rate of the form $x > 0 \mapsto x^{-\delta}$ with $\delta > 0$ [MM19]). The tightness result of Theorem II.3.1 is weaker 'near 0' than a tightness in $\mathbb{D}([0,T],(\mathcal{M}_1(\mathbb{R}_+^*),\mathrm{w})\times[0,R_{\max}])$ (*i.e.* with the usual weak topology on $\mathcal{M}_1(\mathbb{R}_+^*)$), in the sense that Equation (II.3.25) is not valid for $\varphi \equiv 1$, because this function does not converge to 0 at 0. Still, it is possible that $\omega(x) \xrightarrow[x\to+\infty]{} +\infty$ (see Example 1 for an example with allometric rates), so that the tightness result of Theorem II.3.1 is stronger 'near $+\infty$' than a tightness with the usual weak topology on $\mathcal{M}_1(\mathbb{R}_+^*)$, in the sense that Equation (II.3.25) is valid for functions φ going to $+\infty$ near $+\infty$.

Note that classical results depicted in the literature (Theorem 5.3. in [FM04], Corollary 3.3. in [Tra08], Theorem 5.2 in [CF15]) establish convergence in law and not only tightness of sequences of similar renormalizations as the one described in Section II.2.1. In our context with unbounded rates, we did not succeed in proving that there exists a unique measure solution to the system (II.3.24)-(II.3.25), which is the missing argument to improve our tightness result to a convergence result. We discuss the difficulties encountered for this proof of uniqueness in Section II.5.1. Also, in Section II.5.3, we provide an extension of Theorem II.3.1 to a tightness result in a broader set of measure-valued processes, with additional regularity and control assumptions on the solutions to (II.3.24)-(II.3.25).

Finally, remark that for a given initial condition μ_0^*, Equations (II.3.24) and (II.3.25) are the weak formulation of a PDE system, and Theorem II.3.1 provides the existence of measure solutions to this problem. We will discuss in Section III.1.1 of Chapter III under which assumptions there exists regular function solutions to this PDE system. The reader can already consider the system (III.1.4), (III.1.5), (III.1.6) to have a clearer idea of the kind of deterministic PDE we obtain in the large population limit for our renormalized process.

Sketch of the proof of Theorem II.3.1:

In the following, we work under the assumptions of Theorem II.3.1 and fix $T \geq 0$. For any $K \in \mathbb{N}^*$, we write $\mathscr{L}^K$ for the law of the process $\left((\mu_t^K, R_t^K)_{t\in[0,T]} \right)_{K\in\mathbb{N}^*}$. Every $\mathscr{L}^K$ is a probability measure on $\mathbb{D}([0,T],\mathcal{M}_\omega(\mathbb{R}_+^*)\times[0,R_{\max}])$. Note that $\mathscr{L}^K$ does not depend on the choice of the topology on $\mathcal{M}_\omega(\mathbb{R}_+^*)$, if we choose among the vague topology or the ω-weak topology (see Lemma B.2.10 in Appendix B.2.2). Our aim in the following proof is first to prove the tightness of $(\mathscr{L}^K)_{K\in\mathbb{N}^*}$ in $\mathbb{D}([0,T],(\mathcal{M}_\omega(\mathbb{R}_+^*),\mathrm{w})\times[0,R_{\max}])$, and then to characterize any accumulation point with (II.3.24)-(II.3.25). We divide the proof in four steps.

- First, in Section II.4.1, we show that $(\mathscr{L}^K)_{K\in\mathbb{N}^*}$ is tight in $\mathbb{D}([0,T],(\mathcal{M}_\omega(\mathbb{R}_+^*),v)\times[0,R_{\max}])$. Remark that $\mathcal{M}_\omega(\mathbb{R}_+^*)$ is endowed with the vague topology at this step. We extend a criterion of Roelly [Roe86] to our weighted space of measures (see Theorem II.4.1), which reduces the problem to proving the tightness of a sequence in $\mathbb{D}([0,T],\mathbb{R})$. To do so, we use a criterion of Aldous and Rebolledo [JM86] and Proposition II.2.5.

- In Section II.4.2, we prove that any limit of a subsequence of $\left((\mu_t^K, R_t^K)_{t\in[0,T]} \right)_{K\in\mathbb{N}^*}$ converging in law in $\mathbb{D}([0,T],(\mathcal{M}_\omega(\mathbb{R}_+^*),v)\times[0,R_{\max}])$ is in $\mathcal{C}([0,T],(\mathcal{M}_\omega(\mathbb{R}_+^*),\mathrm{w})\times$

$[0, R_{\max}])$ (note that the limit is continuous for $\mathcal{M}_\omega(\mathbb{R}_+^*)$ endowed with the ω-weak topology). We use a criterion from [EK86].

- Thanks to the continuity of the limit, in Section II.4.3, we extend a result of Méléard and Roelly [MR93] to our weighted space of measures (see Theorem II.4.2), and prove that $(\mathscr{L}^K)_{K\in\mathbb{N}^*}$ is tight in $\mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w}) \times [0, R_{\max}])$. In particular, we use the previous step to control the finite variation and martingale parts of $\langle \mu_t^K, \omega\rangle$ for $K \geq 1$ and $t \in [0,T]$.

- Finally in Section II.4.4, we characterize the limit (μ^*, R^*) of any converging subsequence of our process in $\mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w}) \times [0, R_{\max}])$, still written $(\mu^K, R^K)_{K\geq 1}$, with Equations (II.3.24) and (II.3.25). It is precisely at this step that we use the additional Assumption II.3.1, to be able to control, uniformly on $K \geq 1$ and $t \in [0,T]$, quantities of the form $\mathbb{E}\left(\langle \mu_t^K - \mu_t^*, b + \overline{g}\rangle\right)$.

II.4 Proof of Theorem II.3.1

We follow the sketch of the proof highlighted in Section II.3.

II.4.1 Proof of the tightness of $\left(\mathscr{L}^K\right)_{K\in\mathbb{N}^*}$ in $\mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), v)\times[0, R_{\max}])$

First, we prove the tightness of $\left(\mathscr{L}_\mu^K\right)_{K\in\mathbb{N}^*}$, where for $K \in \mathbb{N}^*$, $\mathscr{L}_\mu^K$ is the law of $(\mu_t^K)_{t\in[0,T]}$. We give the following criterion of tightness in $\mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), v))$, which is an extension of Theorem 2.1 in [Roe86] to weighted spaces of measures. For every $f \in \mathfrak{B}_\omega(\mathbb{R}_+^*)$, we define the projection

$$\begin{array}{rccc} \pi_f: & \mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), v)) & \longrightarrow & \mathbb{D}([0,T], \mathbb{R}) \\ & (\mu_t)_{t\in[0,T]} & \longmapsto & (\langle \mu_t, f\rangle)_{t\in[0,T]}. \end{array}$$

Also, we write $f \in \mathcal{C}_0(\mathbb{R}_+^*)$, if f is continous, $f(x) \xrightarrow[x\to 0]{} 0$ and $f(x) \xrightarrow[x\to+\infty]{} 0$.

Theorem II.4.1. *Let $T \geq 0$, $\left(P^K\right)_{K\in\mathbb{N}}$ be a sequence of probability measures on the space $\mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), v))$, and D be a dense countable subset of $\mathcal{C}_0(\mathbb{R}_+^*)$ for the topology of uniform convergence. Assume that for all $f \in D\cup\{\omega\}$, $\left(\pi_f * P^K\right)_{K\in\mathbb{N}}$ is a tight sequence of probability measures on $\mathbb{D}([0,T], \mathbb{R})$, where $\pi_f * P^K$ is the usual pushforward of P^K by π_f. Then $\left(P^K\right)_{K\in\mathbb{N}}$ is tight on $\mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), v))$.*

We prove Theorem II.4.1 in Appendix B.3.1.

Lemma II.4.1. *There exists a countable set $D \subseteq \mathcal{C}_c^\infty(\mathbb{R}_+^*)$, such that D is dense in $\mathcal{C}_0(\mathbb{R}_+^*)$ for the topology of uniform convergence.*

Proof. The space $\mathbb{R}_+^*$ is a locally compact metric space, hence $\mathcal{C}_0(\mathbb{R}_+^*)$ is separable, *i.e.* there exists a sequence $(f_n)_{n\in\mathbb{N}}$ dense in $\mathcal{C}_0(\mathbb{R}_+^*)$ for the topology of uniform convergence. Furthermore, for every $n \in \mathbb{N}$, there exists a sequence $(\varphi_{n,k})_{k\in\mathbb{N}}$ of $\mathcal{C}_c^\infty(\mathbb{R}_+^*)$ functions, converging uniformly towards f_n, precisely because $f_n \in \mathcal{C}_0(\mathbb{R}_+^*)$. Then, the set $D := (\varphi_{n,k})_{k\in\mathbb{N}, n\in\mathbb{N}}$ is dense in $\mathcal{C}_0(\mathbb{R}_+^*)$ for the topology of uniform convergence. □

Thus, thanks to Theorem II.4.1 and Lemma II.4.1, it suffices to show that for all $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+^*) \cup \{\omega\}$, $\left(\pi_\varphi * \mathscr{L}_\mu^K\right)_{K\in\mathbb{N}}$ is a tight sequence of probability measures on $\mathbb{D}([0,T], \mathbb{R})$. Let φ be such a function, and note that in particular, $\varphi \in \mathfrak{B}_\omega(\mathbb{R}_+^*)$ and φ is bounded. Then by Proposition II.2.5, applied to $(t,x)\mapsto\varphi(x)$, for $t\in[0,T]$, we have the semi-martingale decomposition

$$\langle\mu_t^K,\varphi\rangle = V_{\varphi,t}^K + \heartsuit_{\varphi,t}^K,$$

where $(\heartsuit_{\varphi,t}^K)_{t\geq 0}$ is a square-integrable martingale. The tightness of $\left(\pi_\varphi * \mathscr{L}_\mu^K\right)_{K\in\mathbb{N}}$ in $\mathbb{D}([0,T],\mathbb{R})$ is proven thanks to a criterion from Aldous and Rebolledo ([JM86], Corollary 2.3.3). It suffices to show that

1. For every $t\in[0,T]$, the sequence of laws of $(\langle\mu_t^K,\varphi\rangle)_{K\geq 1}$ is tight in $\mathbb{R}$.

2. For every $t\in[0,T]$, for every $\varepsilon>0$, for every $\eta>0$, there exists $\delta>0$ and $K_0\geq 1$, such that for every sequence of stopping times $(S_K,T_K)_{K\in\mathbb{N}^*}$ such that $S_K\leq T_K\leq t$ for all $K\in\mathbb{N}^*$, we have

$$\sup_{K\geq K_0}\mathbb{P}\left(|\langle\heartsuit_\varphi^K\rangle_{T_K}-\langle\heartsuit_\varphi^K\rangle_{S_K}|\geq\eta, T_K\leq S_K+\delta\right)\leq\varepsilon, \tag{II.4.26}$$

$$\sup_{K\geq K_0}\mathbb{P}\left(|V_{\varphi,T_K}^K-V_{\varphi,S_K}^K|\geq\eta, T_K\leq S_K+\delta\right)\leq\varepsilon. \tag{II.4.27}$$

First, we use Markov inequality to obtain, for any $t\in[0,T]$, $M>0$, $K\in\mathbb{N}^*$,

$$\begin{aligned}\mathbb{P}(|\langle\mu_t^K,\varphi\rangle|\geq M) &\leq \frac{1}{M}\mathbb{E}(|\langle\mu_t^K,\varphi\rangle|)\\ &\leq \frac{1}{M}\sup_{K\in\mathbb{N}^*}\mathbb{E}(\sup_{t\in[0,T]}|\langle\mu_t^K,\varphi\rangle|).\end{aligned}$$

Remark that $\varphi\in\mathfrak{B}_\omega(\mathbb{R}_+^*)$, so Proposition II.2.4 entails that the sequence of laws of $(\langle\mu_t^K,\varphi\rangle)_{K\geq 1}$ is tight in $\mathbb{R}$. Then, we fix $t\in[0,T]$, $\delta>0$ and $(S_K,T_K)_{K\in\mathbb{N}^*}$ a sequence of stopping times such that $S_K\leq T_K\leq t$ for all $K\in\mathbb{N}^*$. By Lemma II.1.14, Proposition II.2.5, and using the fact that $\varphi\in\mathcal{C}_c^\infty(\mathbb{R}_+^*)\cup\{\omega\}$, so φ is bounded or $\varphi=\omega$, there exists a constant $C>0$ such that for $K\geq 1$,

$$\begin{aligned}&\mathbb{E}\left(|\langle\heartsuit_\varphi^K\rangle_{T_K}-\langle\heartsuit_\varphi^K\rangle_{S_K}|\mathbb{1}_{\{|T_K-S_K|\leq\delta\}}\right)\\ &\qquad= \frac{1}{K}\mathbb{E}\Bigg(\mathbb{1}_{\{|T_K-S_K|\leq\delta\}}\int_{S_K}^{T_K}\int_{\mathbb{R}_+^*}\Bigg[b(x)\Big(\varphi(x_0)+\varphi(x-x_0)-\varphi(x)\Big)^2\\ &\qquad\qquad + d(x)\varphi^2(x)\Bigg]\mu_s^K(\mathrm{d}x)\mathrm{d}s\Bigg)\\ &\qquad\leq \frac{C}{K}\mathbb{E}\left(\mathbb{1}_{\{|T_K-S_K|\leq\delta\}}\int_{S_K}^{T_K}\left\langle\mu_s^K,1+\mathrm{Id}+\omega\right\rangle\mathrm{d}s\right)\\ &\qquad\leq \frac{C}{K}\delta\sup_{K\in\mathbb{N}^*}\mathbb{E}\left(\sup_{t\in[0,T]}\left(E_t^K+N_t^K+\Omega_t^K\right)\right).\end{aligned}$$

By Proposition II.2.4 and using Markov inequality, for every $\varepsilon>0$ and $\eta>0$, we can thus find δ and K_0 such that (II.4.26) holds true uniformly on the choice of the sequence $(S_K,T_K)_{K\in\mathbb{N}^*}$. Similarly, we use Lemma II.1.14, Proposition II.2.5, the fact that $\varphi\in$

$\mathcal{C}_c^\infty(\mathbb{R}_+^*) \cup \{\omega\}$, so φ and φ' are bounded and $\varphi \in \mathfrak{B}_\omega(\mathbb{R}_+^*)$, or $\varphi \equiv \omega$, to assess that

$$
\begin{aligned}
&\mathbb{E}\left(|V^K_{\varphi,T_K} - V^K_{\varphi,S_K}|\mathbb{1}_{\{|T_K-S_K|\leq\delta\}}\right)\\
&\qquad \leq \mathbb{E}\Bigg(\mathbb{1}_{\{|T_K-S_K|\leq\delta\}}\int_{S_K}^{T_K}\int_{\mathbb{R}_+^*}\Big\{\overline{g}(x)|\varphi'(x)| + d(x)|\varphi(x)|\\
&\qquad\qquad\qquad + b(x)\Big|\varphi(x_0)+\varphi(x-x_0)-\varphi(x)\Big|\Big\}\mu_s^K(\mathrm{d}x)\mathrm{d}s\Bigg)\\
&\qquad \leq \omega_1\mathbb{E}\Bigg(\mathbb{1}_{\{|T_K-S_K|\leq\delta\}}\int_{S_K}^{T_K}\int_{\mathbb{R}_+^*}\Big(1+x+\omega(x)\Big)\mu_s^K(\mathrm{d}x)\mathrm{d}s\Bigg)\\
&\qquad \leq \omega_1\delta\sup_{K\in\mathbb{N}^*}\mathbb{E}\left(\sup_{t\in[0,T]}\left(E_t^K+N_t^K+\Omega_t^K\right)\right),
\end{aligned}
$$

and we conclude in the same manner. Now, let us show the tightness of $\left(\mathscr{L}_R^K\right)_{K\in\mathbb{N}^*}$, where for $K\in\mathbb{N}^*$, $\mathscr{L}_R^K$ is the law of $(R_t^K)_{t\in[0,T]}$. We use a simpler criterion of Aldous, without decomposing R_t^K into a finite variation part and a martingale part. First, for every $t\geq 0$, for every $K\in\mathbb{N}^*$, then $R_t^K\in[0,R_{\max}]$, so for every $t\in[0,T]$, $(R_t^K)_{K\geq 1}$ is tight in $\mathbb{R}$. Then from Theorem 16.10. in [Bil99], it suffices to show that for every $t\in[0,T]$, for every $\varepsilon>0$, for every $\eta>0$, there exists $\delta>0$ and $K_0\geq 1$, such that for every sequence of stopping times $(S_K,T_K)_{K\in\mathbb{N}^*}$ with $S_K\leq T_K\leq t$ for all $K\in\mathbb{N}^*$, we have

$$
\sup_{K\geq K_0}\mathbb{P}\left(|R^K_{T_K}-R^K_{S_K}|\geq\eta, T_K\leq S_K+\delta\right)\leq\varepsilon.
$$

Let us fix $t\in[0,T]$, $\delta>0$, $K\geq 1$ and a sequence of stopping times as defined previously, then we have by (II.1.3) that

$$
\begin{aligned}
&\mathbb{E}\left(|R^K_{T_K}-R^K_{S_K}|\mathbb{1}_{\{T_K\leq S_K+\delta\}}\right)\\
&\qquad\qquad = \mathbb{E}\left(\left|\int_{S_K}^{T_K}\varsigma(R_s^K)-\chi\langle\mu_s^K,f(.,R_s^K)\rangle\mathrm{d}s\right|\mathbb{1}_{\{T_K\leq S_K+\delta\}}\right)\\
&\qquad\qquad \leq \delta||\varsigma||_{\infty,[0,R_{\max}]}+\mathbb{E}\left(\chi\int_{S_K}^{T_K}\langle\mu_s^K,\overline{g}\rangle\mathrm{d}s\ \mathbb{1}_{\{T_K\leq S_K+\delta\}}\right)\\
&\qquad\qquad \leq \delta\left(||\varsigma||_{\infty,[0,R_{\max}]}+\chi C_g\sup_{K\in\mathbb{N}^*}\mathbb{E}\left(\sup_{t\in[0,T]}\left(E_t^K+N_t^K+\Omega_t^K\right)\right)\right),
\end{aligned}
$$

where we used the fact that for every $R\geq 0$ and $x>0$, $|f(x,R)|\leq\overline{g}(x)$ and Assumption II.1.13. This concludes by Markov inequality and Proposition II.2.4. At this step, we have shown the tightness of $\left(\mathscr{L}^K\right)_{K\in\mathbb{N}^*}$ in $\mathbb{D}([0,T],(\mathcal{M}_\omega(\mathbb{R}_+^*),v)\times[0,R_{\max}])$. Then, we can use Prokhorov theorem, because $\mathbb{D}([0,T],(\mathcal{M}_\omega(\mathbb{R}_+^*),v)\times[0,R_{\max}])$ is metrizable (see Theorem 5.1. in [Bil99]). This theorem states that we can extract a subsequence, still denoted as $\left(\left(\mu_t^K,R_t^K\right)_{t\in[0,T]}\right)_{K\in\mathbb{N}^*}$ for the sake of simplicity, that converges in law towards some $(\mu_t^*,R_t^*)_{t\in[0,T]}$ in $\mathbb{D}([0,T],(\mathcal{M}_\omega(\mathbb{R}_+^*),v)\times[0,R_{\max}])$.

II.4.2 Continuity of accumulation points

In this section, we show the continuity of the limit $(\mu_t^*,R_t^*)_{t\in[0,T]}$ highlighted at the end of Section II.4.1, which is essential in the use of Theorem II.4.2 in Section II.4.3, and in the identification of the limit in Section II.4.4.

Lemma II.4.2. *Any limit* $(\mu_t^*, R_t^*)_{t\in[0,T]}$ *of a subsequence of* $\left((\mu_t^K, R_t^K)_{t\in[0,T]} \right)_{K\in\mathbb{N}^*}$ *converging in law in* $\mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), v)\times[0,R_{\max}])$ *is in* $\mathcal{C}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w})\times[0,R_{\max}])$.

Proof. For the sake of simplicity, we write again $\left((\mu_t^K, R_t^K)_{t\in[0,T]} \right)_{K\in\mathbb{N}^*}$ for the converging subsequence. We begin with the continuity of $(\mu_t^*)_{t\in[0,T]}$. From Theorem 10.2 p.148 in [EK86], it suffices to show that almost surely,

$$\sup_{t\in[0,T]} \mathrm{d}_P^\omega(\mu_t^K, \mu_{t-}^K) \xrightarrow[K\to+\infty]{} 0,$$

with d_P^ω the ω-Prokhorov distance defined in Proposition B.2.2 of Appendix B.2.1. By Lemma B.2.5 in Appendix B.2.1, it suffices to show that this convergence holds true with the ω-Fortet-Mourier distance d_{FM}^ω (see Definition B.2.4). Let $\mathcal{D} := (\varphi_n)_{n\in\mathbb{N}}$ be a countable and dense subset of the set $\left\{\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+^*),\ ||\varphi/\omega||_\infty \le 1,\ ||\varphi'||_\infty \le 1\right\}$ for the topology of uniform convergence, which exists by Lemma II.4.1. It suffices to show that almost surely

$$\sup_{t\in[0,T]} \mathrm{d}_{\mathrm{FM}}^\omega(\mu_t^K, \mu_{t-}^K) = \sup_{t\in[0,T]}\sup_{n\in\mathbb{N}} |\langle \mu_t^K - \mu_{t-}^K, \varphi_n\rangle| = \sup_{n\in\mathbb{N}}\sup_{t\in[0,T]} |\langle \mu_t^K - \mu_{t-}^K, \varphi_n\rangle| \xrightarrow[K\to+\infty]{} 0.$$

Without loss of generality, we can prove that this convergence holds true in L^1, because this implies almost sure convergence up to extraction, and our argument using Theorem 10.2 p.148 in [EK86] remains true up to extracting a new subsequence that still converges in law towards $(\mu_t^*)_{t\in[0,T]}$ in $\mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), v)\times[0,R_{\max}])$. Furthermore, for $K\ge 1$ and $n\ge 0$, we have the domination

$$\sup_{t\in[0,T]} |\langle \mu_t^K - \mu_{t-}^K, \varphi_n\rangle| \le 2 \sup_{t\in[0,T]} \Omega_t^K.$$

The right-hand side above does not depend on n, and its expectation is bounded uniformly on K by Proposition II.2.4. Hence, by this domination argument, it suffices to show that

$$\mathbb{E}\left(\sup_{n\in\mathbb{N}}\sup_{t\in[0,T]} |\langle \mu_t^K - \mu_{t-}^K, \varphi_n\rangle|\right) = \sup_{n\in\mathbb{N}} \mathbb{E}\left(\sup_{t\in[0,T]} \left|\langle \mu_t^K - \mu_{t-}^K, \varphi_n\rangle\right|\right) \xrightarrow[K\to+\infty]{} 0.$$

Let us consider any $\varphi\in\mathcal{C}_c^\infty(\mathbb{R}_+^*)$ with $||\varphi/\omega||_\infty\le 1$ and $||\varphi'||_\infty \le 1$. For $t\in[0,T]$ and $K\ge 1$, we have from the decomposition of Proposition II.2.5 that

$$\langle \mu_t^K - \mu_{t-}^K, \varphi\rangle = V_{\varphi,t}^K - V_{\varphi,t-}^K + \heartsuit_{\varphi,t}^K - \heartsuit_{\varphi,t-}^K = \heartsuit_{\varphi,t}^K - \heartsuit_{\varphi,t-}^K,$$

because $t\in[0,T]\mapsto V_{\varphi,t}^K$ is continuous. Hence, we obtain

$$\sup_{t\in[0,T]} |\langle \mu_t^K - \mu_{t-}^K, \varphi\rangle| \le 2 \sup_{t\in[0,T]} \left|\heartsuit_{\varphi,t}^K\right|.$$

Then, we use Doob maximal inequality for square-integrable martingales to obtain

$$\mathbb{E}\left(\sup_{t\in[0,T]} \left|\heartsuit_{\varphi,t}^K\right|\right) \le 4\mathbb{E}\left(\langle \heartsuit_\varphi^K\rangle_T\right) \le \frac{4(C_b(\omega(x_0)+x_0)^2 + C_d)}{K}\mathbb{E}\left(E_T^K + N_T^K + \Omega_T^K\right),$$

by Proposition II.2.5, the fact that $||\varphi/\omega||_\infty\le 1$ and $||\varphi'||_\infty\le 1$, and Assumption II.1.13 (in particular, we use (II.1.14) in Lemma II.1.14). This upper bound is uniform on φ

and converges to 0 when $K \to +\infty$ thanks to Proposition II.2.4, which concludes for the continuity of $(\mu_t^*)_{t\in[0,T]}$. Finally, we use again Theorem 10.2 p.148 in [EK86] for the continuity of $(R_t^*)_{t\in[0,T]}$, it suffices to show that

$$\sup_{t\in[0,T]} \left|R_t^K - R_{t-}^K\right| \xrightarrow[K\to+\infty]{} 0,$$

which is immediate because every $(R_t^K)_{t\in[0,T]}$ is continuous by construction.

□

Remark: In the previous proof, note that we need to use the ω-Fortet-Mourier distance with an additional control on the differential of the considered test functions compared to the ω-total variation distance (see Definition B.2.3) that is classically used in similar contexts (see Lemma 5.7 in [CF15]). This is essentially because we need to control the term $\varphi(x - x_0) - \varphi(x)$.

II.4.3 Proof of the tightness of $\left(\mathscr{L}^K\right)_{K\in\mathbb{N}^*}$ in $\mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w})\times[0, R_{\max}])$

Considering Section II.4.1, it suffices to prove the tightness of $\left(\mathscr{L}_\mu^K\right)_{K\in\mathbb{N}^*}$ in the space $\mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w}))$. To this end, we extend Théorème 3. in [MR93] to weighted spaces of measures.

Theorem II.4.2. *Let w be any positive and continuous function on $\mathbb{R}_+^*$, $((\nu_t^K)_{t\in[0,T]})_{K\in\mathbb{N}}$ be a sequence of processes in $\mathbb{D}([0,T], (\mathcal{M}_w(\mathbb{R}_+^*), \mathrm{w})$ and $(\nu_t^*)_{t\in[0,T]}$ a process in the space $\mathcal{C}([0,T], (\mathcal{M}_w(\mathbb{R}_+^*), \mathrm{w}))$. Then, the following assertions are equivalent*

(i) *$((\nu_t^K)_{t\in[0,T]})_{K\in\mathbb{N}}$ converges in law towards $(\nu_t^*)_{t\in[0,T]}$ in $\mathbb{D}([0,T], (\mathcal{M}_w(\mathbb{R}_+^*), \mathrm{w}))$.*

(ii) *$((\nu_t^K)_{t\in[0,T]})_{K\in\mathbb{N}}$ converges in law towards $(\nu_t^*)_{t\in[0,T]}$ in $\mathbb{D}([0,T], (\mathcal{M}_w(\mathbb{R}_+^*), v))$, and $((\langle\nu_t^K, w\rangle)_{t\in[0,T]})_{K\in\mathbb{N}}$ converges in law towards $(\langle\nu_t^*, w\rangle)_{t\in[0,T]}$ in $\mathbb{D}([0,T], \mathbb{R})$.*

We provide the proof of this result in Appendix B.3.2, and use it in the following with the converging subsequence still denoted as $(\mu^K)_{K\geq1}$ constructed in Section II.4.1, its limit μ^* and $w \equiv \omega$. Note that we could also have formulated Theorem II.4.2 in terms of tightness of the sequences instead of convergence. We proved in Section II.4.1, using Aldous and Rebolledo criterion, that $((\langle\mu_t^K, \omega\rangle)_{t\in[0,T]})_{K\in\mathbb{N}}$ is tight in $\mathbb{D}([0,T], \mathbb{R})$. Hence, also from Section II.4.1, extracting again if necessary and using Prokhorov theorem, we can find a subsequence of $\left((\mu_t^K, R_t^K)_{t\in[0,T]} \right)_{K\in\mathbb{N}^*}$ that converges in law towards some $(\mu_t^*, R_t^*)_{t\in[0,T]}$ in $\mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), v) \times [0, R_{\max}])$, such that $((\langle\mu_t^K, \omega\rangle)_{t\in[0,T]})_{K\in\mathbb{N}}$ converges in law towards some $(x_t^*)_{t\in[0,T]}$ in $\mathbb{D}([0,T], \mathbb{R})$. By Lemma II.4.2, we also know that these limits are continuous. Hence, to use Theorem II.4.2 and conclude this section, it remains to show that $(x_t^*)_{t\in[0,T]}$ and $(\langle\mu_t^*, \omega\rangle)_{t\in[0,T]}$ have same law. Using Skorokhod representation theorem (Theorem 6.7. p.70 in [Bil99]) if necessary, we assume that the previous convergences hold true almost surely. By a straightforward approximation argument by test functions as in the proof of Theorem II.4.2 in Appendix B.3.2, we then obtain almost surely

$$\forall t \in [0,T], \quad x_t^* \geq \langle\mu_t^*, \omega\rangle.$$

Furthermore, if $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence of $\mathcal{C}_c^\infty(\mathbb{R}_+^*)$ positive functions that converges pointwise to ω such that $\varphi_n \le \omega$ for $n \in \mathbb{N}$, we obtain that for every $t \in [0,T]$

$$\begin{aligned}\mathbb{E}\left(x_t^* - \langle \mu_t^*, \omega\rangle\right) &= \mathbb{E}\left(\lim_{K\to+\infty}\lim_{n\to+\infty}\left(\langle \mu_t^K, \varphi_n\rangle - \langle \mu_t^*, \varphi_n\rangle\right)\right)\\ &\le \lim_{K\to+\infty}\lim_{n\to+\infty}\mathbb{E}\left(\langle \mu_t^K, \varphi_n\rangle - \langle \mu_t^*, \varphi_n\rangle\right)\\ &= \lim_{n\to+\infty}\lim_{K\to+\infty}\mathbb{E}\left(\langle \mu_t^K, \varphi_n\rangle - \langle \mu_t^*, \varphi_n\rangle\right) = 0,\end{aligned}$$

where we first used the convergence of $((\langle \mu_t^K, \omega\rangle)_{t\in[0,T]})_{K\in\mathbb{N}}$ towards $(x_t^*)_{t\in[0,T]}$ in $\mathbb{D}([0,T],\mathbb{R})$, then Fatou lemma, followed by a domination argument (because every φ_n is dominated by ω and we have the uniform control in K of Proposition II.2.4), and finally used the convergence of $\left((\mu_t^K)_{t\in[0,T]}\right)_{K\in\mathbb{N}^*}$ towards $(\mu_t^*)_{t\in[0,T]}$ in $\mathbb{D}([0,T],(\mathcal{M}_\omega(\mathbb{R}_+^*),v))$ and domination by ω again. Hence, for every $t\in[0,T]$, we obtain almost surely $x_t^* = \langle \mu_t^*, \omega\rangle$, which concludes by continuity of the considered functions by Lemma II.4.2.

II.4.4 Characterization of accumulation points

The aim of this section is to prove the upcoming Proposition II.4.5, which characterizes the law of the limit of any subsequence of $\left((\mu_t^K, R_t^K)_{t\in[0,T]}\right)_{K\in\mathbb{N}^*}$ converging in law in $\mathbb{D}([0,T],(\mathcal{M}_\omega(\mathbb{R}_+^*),\mathrm{w})\times[0,R_{\max}])$. We begin with preliminary deterministic results, writing $\mathcal{C}_\omega^{0,0}([0,T]\times\mathbb{R}_+^*)$ for the set of continous functions on $[0,T]\times\mathbb{R}_+^*$ such that

$$\exists C>0, \forall s\in[0,T], \forall x>0 \quad |\varphi_s(x)| \le C\omega(x).$$

The following lemma presents a classical result that holds true in a more general context (see Problem 26. p.153 in [EK86] or Proposition 2.4 p.339 in [JS+87]). For the sake of completeness, and because the classical case does not consider the ω-weak topology, we propose a proof valid in our particular context.

Lemma II.4.3. *For every $t\in[0,T]$, for every $\varphi \in \mathcal{C}_\omega^{0,0}([0,T]\times\mathbb{R}_+^*)$, the functions*

$$\Sigma_t : \left\{\begin{array}{l}\mathbb{D}([0,T],(\mathcal{M}_\omega(\mathbb{R}_+^*),\mathrm{w})) \longrightarrow \mathbb{R}\\ \mu \longmapsto \sup\limits_{s\in[0,T]}\langle \mu_s, \omega\rangle\end{array}\right. \qquad \text{and}$$

$$\zeta_{\varphi,t} : \left\{\begin{array}{l}\mathbb{D}([0,T],(\mathcal{M}_\omega(\mathbb{R}_+^*),\mathrm{w})) \longrightarrow \mathbb{R}\\ \mu \longmapsto \displaystyle\int_0^t \langle \mu_s, \varphi_s\rangle \mathrm{d}s\end{array}\right.$$

are continuous at every point $\mu \in \mathcal{C}([0,T],(\mathcal{M}_\omega(\mathbb{R}_+^*),\mathrm{w}))$.

Proof. Let us consider $\nu := (\nu_s)_{s\in[0,T]} \in \mathcal{C}([0,T],(\mathcal{M}_\omega(\mathbb{R}_+^*),\mathrm{w}))$ and $((\nu_s^K)_{s\in[0,T]})_{K\ge1}$ that converges towards ν in $\mathbb{D}([0,T],(\mathcal{M}_\omega(\mathbb{R}_+^*),\mathrm{w}))$. As the limit is continuous, this convergence holds true on $\mathbb{D}([0,T],(\mathcal{M}_\omega(\mathbb{R}_+^*),\mathrm{w}))$ for the topology of uniform convergence (see Lemma B.2.8 in Appendix B.2.2), which translates into

$$\sup_{s\in[0,T]} d_P^\omega(\nu_s^K,\nu_s) \xrightarrow[K\to+\infty]{} 0, \tag{II.4.28}$$

with d_P^ω the ω-Prokhorov distance defined in Proposition B.2.2, which metrizes the space $(\mathcal{M}_\omega(\mathbb{R}_+^*),\mathrm{w})$. We fix $t\in[0,T]$ and first prove the result for Σ_t (*i.e.* we want to show that $\left|\Sigma_t(\nu^K) - \Sigma_t(\nu)\right| \xrightarrow[K\to+\infty]{} 0$). For $s\in[0,T]$, let us write $\delta_s := d_P^\omega(\nu_s^K,\nu_s)$. We have by definition (see Proposition B.2.2 in Appendix B.2.1) that for every A open subset of $\mathbb{R}_+^*$ and $K \geq 1$,

$$\omega * \nu_s^K(A) \leq \omega * \nu_s(A^{\delta_s}) + \delta_s,$$
$$\omega * \nu_s(A) \leq \omega * \nu_s^K(A^{\delta_s}) + \delta_s,$$

with $A^{\delta_s} := \{y \in \mathbb{R}_+^*, \inf_{x \in A} |x-y| < \delta_s\}$ and $\omega * \mu$ the usual pushforward of μ by ω. Applying this with $A = \mathbb{R}_+^*$, so that $A^{\delta_s} = A$, we obtain finally

$$\left|\langle \nu_s^K - \nu_s, \omega\rangle\right| = |\omega * \nu_s^K(\mathbb{R}_+^*) - \omega * \nu_s(\mathbb{R}_+^*)| \leq \delta_s = d_P^\omega(\nu_s^K, \nu_s).$$

This entails

$$\left|\Sigma_t(\nu^K) - \Sigma_t(\nu)\right| \leq \sup_{s \in [0,T]} \left|\langle \nu_s^K - \nu_s, \omega\rangle\right| \leq \sup_{s \in [0,T]} d_P^\omega(\nu_s^K, \nu_s),$$

which concludes by (II.4.28). Let us now consider $\varphi \in \mathcal{C}_\omega^{0,0}([0,T] \times \mathbb{R}_+^*)$ and show that $\lim_{K \to +\infty} \left|\zeta_{\varphi,t}(\nu^K) - \zeta_{\varphi,t}(\nu)\right| = 0$. For $K \geq 1$, we have that

$$\left|\zeta_{\varphi,t}(\nu^K) - \zeta_{\varphi,t}(\nu)\right| \leq \int_0^T \left|\langle \nu_s^K - \nu_s, \varphi_s\rangle\right| \mathrm{d}s.$$

For every $s \in [0,t]$, we have that $d_P^\omega(\nu_s^K, \nu_s) \xrightarrow[K \to +\infty]{} 0$, so by definition of the ω-weak topology and because $\varphi_s \in \mathcal{C}_\omega(\mathbb{R}_+^*)$, we then have $\left|\langle \nu_s^K - \nu_s, \varphi_s\rangle\right| \xrightarrow[K \to +\infty]{} 0$. Furthermore, as $\varphi \in \mathcal{C}_\omega^{0,0}([0,T] \times \mathbb{R}_+^*)$, and by continuity of Σ_t and convergence of $((\nu_s^K)_{s \in [0,T]})_{K \geq 1}$ towards ν in $\mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w}))$, there exists constants $C > 0$ and $C' > 0$ such that

$$\sup_{s \in [0,T]} \left|\langle \nu_s^K - \nu_s, \varphi_s\rangle\right| \leq C \left(\sup_{s \in [0,T]} \langle \nu_s, \omega\rangle + \sup_{K \geq 1} \sup_{s \in [0,T]} \langle \nu_s^K, \omega\rangle \right) \leq C' \sup_{s \in [0,T]} \langle \nu_s, \omega\rangle < +\infty,$$

Hence, by dominated convergence, we have

$$\begin{aligned} \lim_{K \to +\infty} \left|\zeta_{\varphi,t}(\nu^K) - \zeta_{\varphi,t}(\nu)\right| &\leq \lim_{K \to +\infty} \int_0^T \left|\langle \nu_s^K - \nu_s, \varphi\rangle\right| \mathrm{d}s \\ &= \int_0^T \left(\lim_{K \to +\infty} \left|\langle \nu_s^K - \nu_s, \varphi\rangle\right| \right) \mathrm{d}s = 0. \end{aligned}$$

□

Lemma II.4.4. *For every $t \in [0,T]$, for every $\varphi \in \mathcal{C}_\omega^{0,0}([0,T] \times \mathbb{R}_+^*)$ and ϕ continous on $[0, R_{\max}]$, the function*

$$\Xi_{\varphi,\phi,t} : \left\{ \begin{array}{l} \mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w}) \times [0, R_{\max}]) \longrightarrow \mathbb{R} \\ (\mu, R) \longmapsto \displaystyle\int_0^t \phi(R_s)\langle \mu_s, \varphi_s\rangle \mathrm{d}s \end{array} \right.$$

is continuous at every point $(\mu, R) \in \mathcal{C}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w}) \times [0, R_{\max}])$.

Proof. Let (μ, R) and (ν, R') be elements of $\mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w}) \times [0, R_{\max}])$. We compute that

$$\begin{aligned}
&\left|\Xi_{\varphi,\phi,t}\left((\mu, R)\right) - \Xi_{\varphi,\phi,t}\left((\nu, R')\right)\right| \\
&\qquad \leq \int_0^t \left|\left(\phi\left(R_s\right) - \phi\left(R'_s\right)\right)\langle \nu_s, \varphi_s\rangle\right| \mathrm{d}s + \int_0^t \left|\phi\left(R_s\right)\langle \mu_s - \nu_s, \varphi_s\rangle\right| \mathrm{d}s \\
&\qquad \leq \int_0^t \left|\phi\left(R_s\right) - \phi\left(R'_s\right)\right| \sup_{\tau\in[0,T]} \langle \nu_\tau, \omega\rangle \mathrm{d}s \\
&\qquad\qquad + ||\phi||_{\infty,[0,R_{\max}]} \int_0^t \left|\langle \mu_s - \nu_s, \varphi_s\rangle\right| \mathrm{d}s.
\end{aligned}$$

Using the previous upper bound and the fact that ϕ is uniformly continuous on $[0, R_{\max}]$, the reader can check than the same techniques as in Lemma II.4.3 lead to the result. $\square$

In the following, for every $\varphi \in \mathcal{C}^{1,1}_{\omega,T}(\mathbb{R}^+ \times \mathbb{R}_+^*)$, for every $\nu \in \mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w}))$, and for every $t \in [0,T]$, we define

$$\begin{aligned}
\Psi_t(\nu) := \langle \nu_t, \varphi_t\rangle - \langle \nu_0, \varphi_0\rangle - \int_0^t \int_{\mathbb{R}_+^*} \Big(&\Phi_s(R_s^\nu, x) \\
&+ b(x)(\varphi_s(x_0) + \varphi_s(x - x_0) - \varphi_s(x)) - d(x)\varphi_s(x)\Big)\nu_s(\mathrm{d}x)\mathrm{d}s,
\end{aligned}$$

with Φ associated to φ as in (II.1.10), and R^ν defined by $R_0^\nu = R_0$ and for $t \in [0,T]$,

$$\frac{\mathrm{d}R_t^\nu}{\mathrm{d}t} = \rho(R_t^\nu, \nu_t). \tag{II.4.29}$$

Proposition II.4.5. *Under the renormalized setting, let $T \geq 0$ and we keep the notation $\left((\mu_t^K, R_t^K)_{t\in[0,T]}\right)_{K\in\mathbb{N}^*}$ for a subsequence of our renormalized sequence of processes, converging in law towards $(\mu_t^*, R_t^*)_{t\in[0,T]}$ in $\mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w}) \times [0, R_{\max}])$. Then, almost surely, for every $t \in [0,T]$ and $\varphi \in \mathcal{C}^{1,1}_{\omega,T}(\mathbb{R}^+ \times \mathbb{R}_+^*)$,*

$$\Psi_t(\mu^*) = 0.$$

Furthermore, $R^* = R^{\mu^*}$ almost surely, with R^{μ^*} defined as in (II.4.29). We thus conclude that almost surely, $(\mu_t^*, R_t^*)_{t\in[0,T]}$ verifies Equations (II.3.24) and (II.3.25).

Proof. Let us fix $\varphi \in \mathcal{C}^{1,1}_{\omega,T}(\mathbb{R}^+ \times \mathbb{R}_+^*)$, $K \in \mathbb{N}^*$. We verify that we can always define a function $\tilde{\varphi} \in \mathcal{C}^{1,1}_{\omega}(\mathbb{R}^+ \times \mathbb{R}_+^*)$ with $\tilde{\varphi} \equiv \varphi$ on $[0,T] \times \mathbb{R}_+^*$. The results of Proposition II.2.5 are then valid for $\tilde{\varphi}_t$ for $t \geq 0$, hence for φ_t for any $t \in [0,T]$. In the following, we fix $t \in [0,T]$ and to simplify the notations, we write φ instead of $\tilde{\varphi}$. We divide the proof in three steps. First, we reduce the problem of showing that $\Psi_t(\mu^*) = 0$ almost surely to proving that $(\Psi_t(\mu^K))_{K\in\mathbb{N}^*}$ converges in law towards $\Psi_t(\mu^*)$ in $\mathbb{R}$. Then, we prove the convergence in law of $(\Psi_t(\mu^K))_{K\in\mathbb{N}^*}$ (essentially by using the additional Assumption II.3.1, the uniform control in Proposition II.2.4 and Lemmas II.4.3 and II.4.4) towards a randow variable defined in the same manner as $\Psi_t(\mu^*)$, but replacing R^{μ^*} with R^*. Finally, we prove that almost surely, $R^* = R^{\mu^*}$, which concludes the proof.

Step 1: Reduction of the problem of showing that $\Psi_t(\mu^*)=0$ almost surely

We consider $\Psi_t(\mu^K)=\heartsuit^K_{\varphi,t}$ defined in Proposition II.2.5. By Proposition II.2.5 applied to φ, by assumptions on φ and by Lemma II.1.14, there exists a constant $C>0$ such that

$$\mathbb{E}(|\Psi_t(\mu^K)|^2)=\mathbb{E}(|\heartsuit^K_{\varphi,t}|^2)=\mathbb{E}(\langle\heartsuit^K_{\varphi}\rangle_t)\leq\frac{C}{K}\sup_{K\in\mathbb{N}^*}\mathbb{E}\left(\sup_{t\in[0,T]}(E^K_t+N^K_t+\Omega^K_t)\right),$$

which entails with Proposition II.2.4 that $\mathbb{E}(|\Psi_t(\mu^K)|)\xrightarrow[K\to+\infty]{}0$, because $\mathbb{E}(|\Psi_t(\mu^K)|)^2\leq\mathbb{E}(|\Psi_t(\mu^K)|^2)$ by Jensen inequality. Hence, it suffices to show that $\mathbb{E}(|\Psi_t(\mu^K)|)\xrightarrow[K\to+\infty]{}\mathbb{E}(|\Psi_t(\mu^*)|)$ to conclude that $\Psi_t(\mu^*)=0$ almost surely. By Corollary II.2.7, the family of square-integrable martingales $\big(\Psi_t(\mu^K)\big)_{K\in\mathbb{N}^*}$ is uniformly integrable. Then, by Proposition 2.3 p.494 in [EK86], it suffices to show that $(\Psi_t(\mu^K))_{K\in\mathbb{N}^*}$ converges in law towards $\Psi_t(\mu^*)$ in $\mathbb{R}$.

Step 2: Convergence in law of $(\Psi_t(\mu^K))_{K\in\mathbb{N}^*}$ towards $\Psi_t(\mu^*)$

By definition, R^{μ^K} coincide with R^K and we write

$$\Psi_t(\mu^K)=\langle\mu^K_t,\varphi_t\rangle-\langle\mu^K_0,\varphi_0\rangle-\int_0^t\phi\left(R^K_s\right)\langle\mu^K_s,\psi(x)\partial_x\varphi_s(x)\rangle\mathrm{d}s$$
$$+\int_0^t\left\langle\mu^K_s,d(x)\varphi_s(x)-b(x)\Big(\varphi_s(x_0)+\varphi_s(x-x_0)-\varphi_s(x)\Big)-\partial_t\varphi_s(x)+\ell(x)\partial_x\varphi_s(x)\right\rangle\mathrm{d}s,$$

with ϕ, ψ and ℓ defined in Section II.1.1, related to respectively the functional response, the growth rate and the metabolic rate. Then, for $s\in[0,T]$ and $x>0$, let us define

$$\mho_{\varphi,s}(x):=d(x)\varphi_s(x)-b(x)\Big(\varphi_s(x_0)+\varphi_s(x-x_0)-\varphi_s(x)\Big)-\partial_t\varphi_s(x)+\ell(x)\partial_x\varphi_s(x),$$

which is well-defined because $b\equiv0$ on $(0,x_0)$, and we obtain

$$\Psi_t(\mu^K)=\langle\mu^K_t,\varphi_t\rangle-\langle\mu^K_0,\varphi_0\rangle+\int_0^t\langle\mu^K_s,\mho_{\varphi,s}\rangle\mathrm{d}s-\int_0^t\phi\left(R^K_s\right)\langle\mu^K_s,\psi(x)\partial_x\varphi_s\rangle\mathrm{d}s,\tag{II.4.30}$$

First, by Lemma II.4.2, the limit $(\mu^*_t)_{t\in[0,T]}$ is in $\mathcal{C}([0,T],(\mathcal{M}_\omega(\mathbb{R}^*_+),\mathrm{w}))$. Hence, from Theorem 7.8 p.131 in [EK86], and because $\big((\mu^K_t)_{t\in[0,T]}\big)_{K\in\mathbb{N}^*}$ converges in law towards $(\mu^*_t)_{t\in[0,T]}$ in $\mathbb{D}([0,T],(\mathcal{M}_\omega(\mathbb{R}^*_+),\mathrm{w}))$, for every $t\in[0,T]$, the marginal distribution $(\mu^K_t)_{K\in\mathbb{N}^*}$ converges in law towards $\delta_{\mu^*_t}$ in $(\mathcal{M}_\omega(\mathbb{R}^*_+),\mathrm{w})$. By definition of the ω-weak topology (which makes every $\mu\in\mathcal{M}_\omega(\mathbb{R}^*_+)\mapsto\langle\mu,\varphi\rangle$ continuous if $|\varphi|\in\mathcal{C}_\omega(\mathbb{R}^*_+)$) and assumption on φ, we obtain that

$$\big(\langle\mu^K_t,\varphi_t\rangle-\langle\mu^K_0,\varphi_0\rangle\big)_{K\in\mathbb{N}^*}\text{ converges in law towards }\langle\mu^*_t,\varphi_t\rangle-\langle\mu^*_0,\varphi_0\rangle\text{ in }\mathbb{R}.$$

Then, with the assumption $\varphi\in\mathcal{C}^{1,1}_{\omega,T}(\mathbb{R}^+\times\mathbb{R}^*_+)$, Assumption II.1.13 and the additional Assumption II.3.1, we verify that $\mho_\varphi\in\mathcal{C}^{0,0}_\omega([0,T]\times\mathbb{R}^*_+)$ (in particular, we use the fact that $\partial_s\varphi_s$ is bounded, uniformly on $s\in[0,T]$ to control the term $\varphi_s(x-x_0)-\varphi(x)$; the fact that $b(x)\leq c_b\omega(x)$ for $x>1$ thanks to Assumption II.3.1; and the fact that $d\varphi_s/\omega$ is bounded, uniformly on $s\in[0,T]$). Hence, by Lemma II.4.3, thanks to the mapping theorem (see Theorem 2.7. page 21 in [Bil99]) and because $\big((\mu^K_t)_{t\in[0,T]}\big)_{K\in\mathbb{N}^*}$ converges in law towards $(\mu^*_t)_{t\in[0,T]}$ in $\mathbb{D}([0,T],(\mathcal{M}_\omega(\mathbb{R}^*_+),\mathrm{w}))$ where the limit μ^* is continuous, we obtain that

$$\left(\int_0^t \langle \mu_s^K, \mho_{\varphi,s}\rangle \mathrm{d}s\right)_{K\in\mathbb{N}^*} \text{ converges in law towards } \int_0^t \langle \mu_s^*, \mho_{\varphi,s}\rangle \mathrm{d}s \text{ in } \mathbb{R}.$$

Finally, by assumption on φ and Assumption II.3.1, we have that $\psi\partial_x\varphi \in \mathcal{C}^{0,0}_\omega([0,T]\times\mathbb{R}_+^*)$, and the function ϕ is continuous on $[0, R_{\max}]$ by assumption (see again Section II.1.1 for the definition of the functions ψ and ϕ). By Lemma II.4.4, the mapping theorem again, and the fact that $\big((\mu_t^K, R_t^K)_{t\in[0,T]}\big)_{K\in\mathbb{N}^*}$ converges in law towards $(\mu_t^*, R_t^*)_{t\in[0,T]}$ in $\mathbb{D}([0,T],(\mathcal{M}_\omega(\mathbb{R}_+^*),\mathrm{w})\times[0,R_{\max}])$, we obtain that

$$\int_0^t \phi\left(R_s^K\right)\langle \mu_s^K, \psi\partial_x\varphi_s\rangle \mathrm{d}s \text{ converges in law towards } \int_0^t \phi\left(R_s^*\right)\langle \mu_s^*, \psi\partial_x\varphi_s\rangle \mathrm{d}s.$$

Hence, to conclude the whole proof, it suffices to show that $(R_t^*)_{t\in[0,T]}$ and $(R_t^{\mu^*})_{t\in[0,T]}$ are equal almost surely.

Step 3: Proof of $R^* = R^{\mu^*}$ almost surely

For every $\nu \in \mathbb{D}([0,T],(\mathcal{M}_\omega(\mathbb{R}_+^*),\mathrm{w})$, R^ν is entirely determined by ν with (II.4.29). Indeed, there exists a function $\mathcal{R} : \mathbb{D}([0,T],(\mathcal{M}_\omega(\mathbb{R}_+^*),\mathrm{w}) \mapsto \mathcal{C}([0,T],[0,R_{\max}])$ such that $\mathcal{R}(\nu) = R^\nu$, and it is precisely given for $t\in[0,T]$ by the functional equation

$$\mathcal{R}(\nu)_t = R_0 + \int_0^t \Big(\varsigma(\mathcal{R}(\nu)_s) - \chi\langle \nu_s, f(.,\mathcal{R}(\nu)_s)\rangle\Big)\mathrm{d}s.$$

Let us first prove that $\mathcal{R}$ is continuous at every ν continuous. We take $\nu := (\nu_s)_{s\in[0,T]} \in \mathcal{C}([0,T],(\mathcal{M}_\omega(\mathbb{R}_+^*),\mathrm{w}))$ and a sequence $((\nu_s^K)_{s\in[0,T]})_{K\geq 1}$ that converges towards ν in the space $\mathbb{D}([0,T],(\mathcal{M}_\omega(\mathbb{R}_+^*),\mathrm{w}))$, and we aim to show that $\sup_{t\in[0,T]} \left|\mathcal{R}(\nu^K)_t - \mathcal{R}(\nu)_t\right| \xrightarrow[K\to+\infty]{} 0$.

For $t\in[0,T]$ and $K\geq 1$, we compute

$$\begin{aligned}
&|\mathcal{R}(\nu^K)_t - \mathcal{R}(\nu)_t| \\
&= \left|\int_0^t \Big[\varsigma(\mathcal{R}(\nu^K)_s) - \varsigma(\mathcal{R}(\nu)_s) - \chi\left(\phi(\mathcal{R}(\nu^K)_s)\langle \nu_s^K,\psi\rangle - \phi(\mathcal{R}(\nu)_s)\langle \nu_s,\psi\rangle\right)\Big]\mathrm{d}s\right| \\
&\leq ||\varsigma'||_{\infty,[0,R_{\max}]}\int_0^t |\mathcal{R}(\nu^K)_s - \mathcal{R}(\nu)_s|\mathrm{d}s + \chi\int_0^t \left|\phi(\mathcal{R}(\nu^K)_s) - \phi(\mathcal{R}(\nu)_s)\right|\langle \nu_s^K,\psi\rangle \mathrm{d}s \\
&\qquad + \chi||\phi||_{\infty,[0,R_{\max}]}\int_0^t \left|\langle \nu_s^K - \nu_s,\psi\rangle\right|\mathrm{d}s \\
&\leq \left(||\varsigma'||_{\infty,[0,R_{\max}]} + \chi k T\sup_{K\geq 1}\sup_{u\in[0,T]}\left(E_u^K + N_u^K + \Omega_u^K\right)\right)\int_0^T \sup_{\tau\in[0,s]}|\mathcal{R}(\nu^K)_\tau - \mathcal{R}(\nu)_\tau|\mathrm{d}s \\
&\qquad + \chi||\phi||_{\infty,[0,R_{\max}]}T\sup_{s\in[0,T]}\left|\langle \nu_s^K - \nu_s,\psi\rangle\right|,
\end{aligned}$$

where we used the fact that ϕ is Lipschitz continuous (see (II.1.1)), that $\psi \leq \overline{g}$ and Assumption II.1.13. Then, by Proposition II.2.4, the previous upper bound is almost surely finite and independent of t so it is an upper bound for $\sup_{s\in[0,t]}\left|\mathcal{R}(\nu^K)_s - \mathcal{R}(\nu)_s\right|$ and by Gronwall's lemma, there exists a constant $C'>0$ such that

$$\begin{aligned}
\sup_{t\in[0,T]}|\mathcal{R}(\nu^K)_t - \mathcal{R}(\nu)_t| &\leq \chi||\phi||_{\infty,[0,R_{\max}]}T\sup_{s\in[0,T]}\left|\langle \nu_s^K - \nu_s,\psi\rangle\right|e^{C'T} \\
&\leq \chi||\phi||_{\infty,[0,R_{\max}]}Te^{C'T}\sup_{s\in[0,T]}\eth_\psi(\nu_s^K,\nu_s),
\end{aligned}$$

where $\eth_\psi$ is a distance that is topologically equivalent to d_P^ψ (see Corollary B.3.3). Furthermore, by Assumption II.3.1 and because $\psi \leq \overline{g}$, then $\psi \in \mathcal{C}_\omega(\mathbb{R}_+^*)$. We also assume that $((\nu_s^K)_{s\in[0,T]})_{K\geq 1}$ converges towards ν in $\mathbb{D}([0,T],(\mathcal{M}_\omega(\mathbb{R}_+^*),\mathrm{w}))$, we thus obtain by Corollary B.3.4 that $((\nu_s^K)_{s\in[0,T]})_{K\geq 1}$ converges towards ν in $\mathbb{D}([0,T],(\mathcal{M}_\psi(\mathbb{R}_+^*),\mathrm{w}))$, which implies by continuity of the limit that

$$\sup_{s\in[0,T]} \eth_\psi(\nu_s^K,\nu_s) \xrightarrow[K\to+\infty]{} 0.$$

This ends the proof of the fact that $\mathcal{R}$ is continuous at every ν continuous. Now, let us get back to the proof that $R^* = R^{\mu^*}$ almost surely. We consider $(\varphi_n)_{n\in\mathbb{N}}$ a sequence of $\mathcal{C}_c^\infty(\mathbb{R})$ positive functions that converges pointwise towards $\mathbb{1}_{\{0\}}$ and such that $\varphi_n(0)=1$ for every $n\geq 0$. By the mapping theorem and convergence in law of $(\mu^K,R^K)_{K\geq 1}$ towards (μ^*,R^*), we have that for every $n\in\mathbb{N}$, for every $t\in[0,T]$,

$$\lim_{K\to+\infty} \mathbb{E}\left(\varphi_n\left(R_t^K - R_t^{\mu^K}\right)\right) = \lim_{K\to+\infty} \mathbb{E}\left(\varphi_n\left(R_t^K - \mathcal{R}\left(\mu_t^K\right)\right)\right) = \mathbb{E}\left(\varphi_n\left(R_t^* - R_t^{\mu^*}\right)\right). \tag{II.4.31}$$

On the one hand, for every $K\geq 1$, we have by definition that $R^K = R^{\mu^K}$ so that the left-most term in (II.4.31) is constant equal to 1. On the other hand, taking the limit $n\to+\infty$ in the right-most term gives $\mathbb{E}\left(\mathbb{1}_{\{0\}}\left(R_t^* - R_t^{\mu^*}\right)\right)$ (dominated convergence holds trivially because everything is bounded), so we conclude that for every $t\in[0,T]$, we almost surely have $R_t^* = R_t^{\mu^*}$, which concludes thanks to the continuity of the considered functions. □

II.5 Extensions of Theorem II.3.1

In this section, we fix $T\geq 0$ and present three lines of research to extend the tightness result in Theorem II.3.1. First in Section II.5.1, we present the difficulties encountered for showing that there exists a unique measure solution to the system (II.3.24)-(II.3.25). If this uniqueness holds true, the tightness result of Theorem II.3.1 is in fact a convergence in law towards the unique limit identified by (II.3.24)-(II.3.25). Then in Section II.5.2, we provide alternative assumptions on the functions b, $\overline{g}$ and ω to relax the restrictive Assumption II.3.1 and obtain the same tightness result in $\mathbb{D}([0,T],(\mathcal{M}_\omega(\mathbb{R}_+^*),\mathrm{w})\times[0,R_{\max}])$, using the fact that we have uniform bounds on quantities of the form $\mathbb{E}\left((N_t^K+E_t^K+\Omega_t^K)^p\right)$ with $p>1$ (see Lemma II.2.6). In particular, this will allow us to work with a broader set of allometric parameters in Section II.6 (see Lemma II.6.3). Finally in Section II.5.3, we aim for a stronger conclusion presented in Theorem II.5.3, where the tightness holds true in $\mathbb{D}([0,T],(\mathcal{M}_{1+\mathrm{Id}+\omega}(\mathbb{R}_+^*),\mathrm{w})\times[0,R_{\max}])$. Remark that this attempt of extension aims to replace ω with $1+\mathrm{Id}+\omega$, thus is an amelioration of Theorem II.3.1, only if ω is dominated by $1+\mathrm{Id}$ in a neighborhood of 0 or $+\infty$. This is precisely the case in the allometric case presented in Example 1 of Section II.6 (see also Figure II.1), which is the main biological motivation of this work. We begin with providing additional results and use Theorem II.4.2 to reduce the problem to proving the tightness of the sequence of laws of $\left((\langle\mu_t^K,1+\mathrm{Id}\rangle)_{t\in[0,T]}\right)_{K\in\mathbb{N}^*}$ in $\mathbb{D}([0,T],\mathbb{R})$. We further give additional assumptions on the limit μ^* to obtain this tightness, reducing our probabilistic problem to the deterministic study of the weak formulation of a PDE.

II.5.1 Uniqueness of a solution to (II.3.24)-(II.3.25)

In this section, we want to investigate under which condition a solution $(\mu_t^*, R_t^*)_{t\in[0,T]}$ to (II.3.24) and (II.3.25) (such a solution exists by Theorem II.3.1) is unique conditionally to μ_0^*. If uniqueness holds true for any fixed initial condition μ_0^*, we deduce that the law of an accumulation point of $\big((\mu_t^K, R_t^K)_{t\in[0,T]}\big)_{K\in\mathbb{N}^*}$ in $\mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w}) \times [0, R_{\max}])$ is a mixture between the law of μ_0^* and the solutions we provide for every μ_0^*, so is unique in law. Hence, without loss of generality, we work with μ_0^* fixed in the following, and everything becomes deterministic as the only source of randomness of a solution to (II.3.24) and (II.3.25) is precisely the law of μ_0^*. First, let us explain how this uniqueness result would allow us to improve the tightness result of Theorem II.3.1. The following basic idea is at the root of our reasoning.

Lemma II.5.1 (Subsubsequences lemma). *Let (X, d) be a metric space, $x \in X$ and $(x_K)_{K\in\mathbb{N}^*}$ a sequence of elements of X. Suppose that any subsequence of $(x_K)_{K\in\mathbb{N}^*}$ admits a subsequence that converges to x. Then, $(x_K)_{K\in\mathbb{N}^*}$ converges towards x.*

Proof. This is a simple reasoning by contradiction. □

Corollary II.5.2. *We work under the assumptions of Theorem II.3.1 and assume that there exists a unique solution $(\mu_t^*, R_t^*)_{t\in[0,T]}$ to (II.3.24) and (II.3.25). Then, for all $T \geq 0$, $\left((\mu_t^K, R_t^K)_{t\in[0,T]} \right)_{K\in\mathbb{N}^*}$ converges in law in $\mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w}) \times [0, R_{\max}])$ towards $(\mu_t^*, R_t^*)_{t\in[0,T]}$.*

Proof. With the same sketch of proof of Theorem II.3.1, we can show that any subsequence of $(\mathscr{L}^K)_{K\in\mathbb{N}^*}$ is tight in $\mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w}) \times [0, R_{\max}])$, so admits a converging subsequence by Prokhorov theorem. Furthermore, any accumulation point verifies (II.3.24) and (II.3.25), so is uniquely determined and equal to $(\mu_t^*, R_t^*)_{t\in[0,T]}$ by assumption. We conclude with Lemma II.5.1. □

Contrary to previous works in the literature (see Step 3. in the proof of Theorem 5.3. in [FM04], Proposition 3.2. in [Tra08], Step 1 of the proof of Theorem 5.2 in [CF15]), we did not succeed in proving that there exists a unique solution to (II.3.24)-(II.3.25) in our general case with unbounded birth and death rates. In the following, we present for the sake of exhaustivity a proof of the uniqueness under additional restictive assumptions (see Proposition II.5.7), and then highlight the steps of this proof that are lacking in the general case. We begin with a preliminary and general probabilistic result.

Lemma II.5.3. *We work under the assumptions of Theorem II.3.1 and consider a subsequence of $\left((\mu_t^K, R_t^K)_{t\in[0,T]} \right)_{K\in\mathbb{N}^*}$ that converges in law towards $(\mu_t^*, R_t^*)_{t\in[0,T]}$ in the space $\mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w}) \times [0, R_{\max}])$. Let $\varphi : (t,x) \in \mathbb{R}^+ \times \mathbb{R}_+^* \mapsto \varphi_t(x)$ be continuous and such that*

$$\exists C > 0, \forall x > 0, \quad \sup_{t\in[0,T]} |\varphi_t(x)| \leq C(1 + x + \omega(x)).$$

Then, we have

$$\mathbb{E}\left(\sup_{t\in[0,T]} |\langle \mu_t^*, \varphi_t \rangle| \right) < +\infty.$$

Proof. Let $T \geq 0$, for the sake of simplicity, we still denote as $\left((\mu_t^K, R_t^K)_{t\in[0,T]}\right)_{K\in\mathbb{N}^*}$ the converging subsequence in the assumptions of Lemma II.5.3. Without loss of generality, we can show that

$$\mathbb{E}\left(\sup_{t\in[0,T]} |\langle \mu_t^*, 1 + \mathrm{Id} + \omega\rangle|\right) < +\infty.$$

There exists an increasing sequence $(\varphi_n)_{n\in\mathbb{N}}$ of $\mathcal{C}_c^\infty(\mathbb{R}_+^*)$ functions that converges simply towards $1 + \mathrm{Id} + \omega$, so that by monotone convergence, for every $t \in [0, T]$,

$$\langle \mu_t^*, 1 + \mathrm{Id} + \omega\rangle = \lim_{n\to+\infty} \langle \mu_t^*, \varphi_n\rangle \quad \text{and} \quad \forall x > 0, \quad \sup_{n\in\mathbb{N}} |\varphi_n(x)| \leq 1 + x + \omega(x).$$

By convergence of $(\mu^K)_{K\in\mathbb{N}^*}$ towards μ^* and by the Skorokhod representation theorem (Theorem 6.7. p.70 in [Bil99]), there exists a probability space Ω and random variables $(\nu^K)_{K\geq 1}$ and ν defined on Ω with values in $\mathbb{D}([0, T], (\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w}))$, such that

- $\forall K \geq 1$, ν^K and μ^K have same law,
- ν and μ^* have same law,
- $(\nu^K)_{K\geq 1}$ converges almost surely towards ν in $\mathbb{D}([0, T], (\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w}))$.

Moreover for every $n \in \mathbb{N}$, $\varphi_n \in \mathcal{C}_{\omega,T}^{1,1}(\mathbb{R}^+ \times \mathbb{R}_+^*)$ because it is in $\mathcal{C}_c^\infty(\mathbb{R}_+^*)$, so for every $n \in \mathbb{N}$ and $t \in [0, T]$ we have almost surely $|\langle \nu_t^*, \varphi_n\rangle| = \lim_{K\to+\infty} |\langle \nu_t^K, \varphi_n\rangle|$ and we obtain by the previous equalities

$$\begin{aligned}
\mathbb{E}\left(\sup_{t\in[0,T]} |\langle \mu_t^*, 1 + \mathrm{Id} + \omega\rangle|\right) &= \mathbb{E}\left(\sup_{t\in[0,T]} \lim_{n\to+\infty} |\langle \nu_t, \varphi_n\rangle|\right) \\
&= \mathbb{E}\left(\sup_{t\in[0,T]} \lim_{n\to+\infty} \lim_{K\to+\infty} |\langle \nu_t^K, \varphi_n\rangle|\right) \\
&\leq C\mathbb{E}\left(\sup_{t\in[0,T]} \lim_{K\to+\infty} \left(E_t^K + N_t^K + \Omega_t^K\right)\right) \\
&\leq C\mathbb{E}\left(\lim_{K\to+\infty} \sup_{t\in[0,T]} \left(E_t^K + N_t^K + \Omega_t^K\right)\right) \\
&\leq C \sup_{K\geq 1} \mathbb{E}\left(\sup_{t\in[0,T]} \left(E_t^K + N_t^K + \Omega_t^K\right)\right),
\end{aligned}$$

where we used Fatou lemma, and this concludes by Proposition II.2.4. □

Lemma II.5.3 states that the accumulation points of $\left((\mu_t^K)_{t\in[0,T]}\right)_{K\in\mathbb{N}^*}$ integrate every continuous function in $\mathfrak{B}_{1+\mathrm{Id}+\omega}(\mathbb{R}_+^*)$ for every $t \geq 0$. Hence, to obtain the uniqueness result in Corollary II.5.2, we will assume without loss of generality in the proof of Proposition II.5.7 below that a measure solution to (II.3.24)-(II.3.25) satisfies this integrability property for $t \geq 0$. Let us now introduce some notations. For any $z > 0$ and $t_0 \geq 0$, we consider the following equation with unknown function $F_{t_0,z}$ and condition at t_0:

$$\begin{cases} F'_{t_0,z}(t) = g(F_{t_0,z}(t), \wp(t)), \\ F_{t_0,z}(t_0) = z, \end{cases}$$

where $\wp$ is any continuous and non-negative function on $\mathbb{R}^+$. With the same arguments as in the proof of Proposition II.1.3, for every $z > 0$ and $t_0 \geq 0$, there exists a unique local positive solution $F_{t_0,z}(.)$ to the previous equation (we omit the dependence in $\wp$ in the following). We will then work with the following assumption.

Assumption II.5.4. *We assume that the flow $(t_0, z, t) \mapsto F_{t_0,z}(t)$ is globally well-defined on $\mathbb{R}^+ \times \mathbb{R}_+^* \times \mathbb{R}^+$. Furthermore, we assume that there exists a constant $C > 0$ such that for every $T \geq 0$*

- $\forall t_0 \in [0,T], \forall z > 0, \forall t \in [0,T], \quad e^{-CT} \leq F_{t_0,z}(t) \leq e^{CT}$,
- $\forall t_0 \in [0,T], \forall z > 0, \forall t \in [0,T], \quad |\partial_z F_{t_0,z}(t)| \leq e^{CT}$.

Remark: Assumption II.5.4 is verified for instance if $\psi \equiv \mathrm{Id}$ and $\ell \equiv \mathrm{Id}/2$ (*i.e.* $g : (x,R) \mapsto (\phi(R) - 1/2)x$). This corresponds to a linear growth, which is the allometric case with a coefficient $\alpha = 1$. More generally, we verify Assumption II.5.4 if $\overline{g}$ is sublinear on $\mathbb{R}_+^*$ (*i.e.* $\overline{g}/\mathrm{Id}$ is bounded on $\mathbb{R}_+^*$). However, it is not verified if $\overline{g}$ is a power function with an exponent $\alpha \neq 1$, which will yet be the case in the allometric case in Example 1 of Section II.6.

Lemma II.5.5. *Under Assumption II.5.4, for every $(s,z,t) \in \mathbb{R}^+ \times \mathbb{R}_+^* \times \mathbb{R}^+$, we have*

$$\partial_s F_{s,z}(t) + g(z, \wp(s))\partial_z F_{s,z}(t) = 0.$$

Proof. Let $(s,z,t) \in \mathbb{R}^+ \times \mathbb{R}_+^* \times \mathbb{R}^+$, then for every $u \in \mathbb{R}^+$, the flow F verifies the classical identity

$$F_{s,z}(t) = F_{u,F_{s,z}(u)}(t).$$

We can differentiate this expression with respect to u and obtain by chain rule

$$\begin{aligned} 0 &= \partial_s F_{u,F_{s,z}(u)}(t) + \partial_u F_{s,z}(u) \times \partial_z F_{u,F_{s,z}(u)}(t) \\ &= \partial_s F_{u,F_{s,z}(u)}(t) + g(F_{s,z}(u), \wp(u))\partial_z F_{u,F_{s,z}(u)}(t), \end{aligned}$$

where we used the definition of $F_{s,z}$. This entails the result with $u = s$. □

Proposition II.5.6. *Under Assumption II.5.4, for every $t \in [0,T]$, for every $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+^*)$, there exists a function $y^t : (s,z) \in \mathbb{R}^+ \times \mathbb{R}_+^* \mapsto y_s^t(z)$ which is a solution to the following transport equation*

$$\begin{cases} \partial_s y_s^t(z) + g(z, \wp(s))\partial_z y_s^t(z) = 0, \\ y_t^t(z) = \varphi(z). \end{cases}$$

Proof. We adapt a generalisation of the characteristic method described by Evans (see Theorem 1 and 2 in Section 3.2. of [Eva22]). Let us fix $t \in [0,T]$, we can define for every $(s,z) \in \mathbb{R}^+ \times \mathbb{R}_+^*$,

$$y_s^t(z) = y^t(s,z) := \varphi(F_{s,z}(t)),$$

and for $R \in \mathbb{R}^+$, we define

$$Y(z,R) := \partial_s y_s^t(z) + g(z,R)\partial_z y_s^t(z).$$

For every $(s,z) \in \mathbb{R}^+ \times \mathbb{R}_+^*$, we obtain

$$Y(z, \wp(s)) = \Big(\partial_s F_{s,z}(t) + g(z, \wp(s))\partial_z F_{s,z}(t)\Big)\varphi'(F_{s,z}(t)) = 0,$$

by Lemma II.5.5. Furthermore, $y_t^t(z) = \varphi(F_{t,z}(t)) = \varphi(z)$, which ends the proof. □

Once this preliminary work is done, we are ready to state the main result of this section, but yet valid only for bounded rates.

Proposition II.5.7. *Under Assumption II.5.4, if the birth and death rates b and d are bounded and in $\mathcal{C}^1(\mathbb{R}_+^*)$, if b' and d' are bounded, if $\psi \in \mathcal{C}^1(\mathbb{R}_+^*)$ and ψ and ψ' are bounded, then there exists a unique solution $(\mu_t^*, R_t^*)_{t\in[0,T]}$ in $\mathcal{C}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w}) \times [0, R_{\max}])$ to (II.3.24)-(II.3.25).*

***Proof*.** We consider $(\mu_t, R_t)_{t\in[0,T]}$ and $(\mu_t', R_t')_{t\in[0,T]}$ in $\mathcal{C}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w}) \times [0, R_{\max}])$ two solutions to (II.3.24) and (II.3.25), and in the following, we aim to show that

$$(\mu_t, R_t)_{t\in[0,T]} = (\mu_t', R_t')_{t\in[0,T]}.$$

By Lemma II.5.3, without loss of generality, we can assume that there exists $\mathcal{S}_T > 0$, such that for every $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+^*)$ with $||\varphi||_\infty \leq 1$,

$$\sup_{t\in[0,T]} \Big[|\langle \mu_t, \varphi\rangle| + |\langle \mu_t', \varphi\rangle| \Big] \leq \mathcal{S}_T < +\infty. \tag{II.5.32}$$

In particular, the total variation distance distance $d_{\mathrm{TV}}(\mu_t, \mu_t')$ (see Appendix B.2.1) is well-defined and uniformly bounded on $t \in [0,T]$, hence the Fortet-Mourier distance $d_{\mathrm{FM}}(\mu_t, \mu_t')$ (see Definition B.2.4) verifies the same property. In the following, let us fix $t \in [0,T]$ and consider $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+^*)$ such that $||\varphi||_\infty + ||\varphi'||_\infty \leq 1$. We apply Proposition II.5.6 with for all $s \geq 0$, $\wp(s) = R_s$. This defines a function y^t in $\mathcal{C}^{1,1}_{\omega,T}(\mathbb{R}^+ \times \mathbb{R}_+^*)$, because it is $\mathcal{C}^{1,1}(\mathbb{R}^+ \times \mathbb{R}_+^*)$ and for every $s \in [0,T]$, $z \mapsto y_s^t(z)$ has compact support (by Assumption II.5.4 and the fact that φ has compact support), so we can apply (II.3.25) to y^t. With the notations in the proof of Proposition II.5.6 with for all $s \geq 0$, $\wp(s) = R_s$, we also have $\varphi \equiv y_t^t$, $Y(x, R_s) = 0$ and $Y(x, R_s') = Y(x, R_s') - Y(x, R_s)$ for every $x > 0$ and $s \in [0,T]$, so we obtain the following expressions:

$$\begin{aligned}
\langle \mu_t, \varphi\rangle =& \langle \mu_t, y_t^t\rangle \\
=& \langle \mu_0^*, y_0^t\rangle \\
&+ \int_0^t \int_{\mathbb{R}^+} \bigg(Y(x, R_s) + b(x) \left(y_s^t(x_0) + y_s^t(x - x_0) - y_s^t(x)\right) - d(x) y_s^t(x) \bigg) \mu_s(\mathrm{d}x)\mathrm{d}s \\
=& \langle \mu_0^*, y_0^t\rangle + \int_0^t \int_{\mathbb{R}^+} \bigg(b(x) \left(y_s^t(x_0) + y_s^t(x - x_0) - y_s^t(x)\right) - d(x) y_s^t(x) \bigg) \mu_s(\mathrm{d}x)\mathrm{d}s,
\end{aligned}$$

and

$$\begin{aligned}
\langle \mu_t', \varphi\rangle =& \langle \mu_t', y_t^t\rangle \\
=& \langle \mu_0^*, y_0^t\rangle \\
&+ \int_0^t \int_{\mathbb{R}^+} \bigg(Y(x, R_s') + b(x) \left(y_s^t(x_0) + y_s^t(x - x_0) - y_s^t(x)\right) - d(x) y_s(x) \bigg) \mu_s'(\mathrm{d}x)\mathrm{d}s \\
=& \langle \mu_0^*, y_0^t\rangle \\
&+ \int_0^t \int_{\mathbb{R}^+} \bigg(\left\{ g(x, R_s') - g(x, R_s)\right\} \partial_x y_s^t(x) \\
&\qquad + b(x) \left(y_s^t(x_0) + y_s^t(x - x_0) - y_s^t(x)\right) - d(x) y_s(x) \bigg) \mu_s'(\mathrm{d}x)\mathrm{d}s.
\end{aligned}$$

This entails

$$
\begin{aligned}
&|\langle \mu'_t - \mu_t, \varphi\rangle| \\
&\qquad \leq k\int_0^t \int_{\mathbb{R}+} \left|R'_s - R_s\right| \psi(x) \left|\partial_x y_s^t(x)\right| \mu'_s(\mathrm{d}x)\mathrm{d}s \\
&\qquad\quad + \int_0^t \left| \int_{\mathbb{R}+} \Big(b(x)\left(y_s^t(x_0) + y_s^t(x-x_0) - y_s^t(x)\right) - d(x) y_s^t(x)\Big)(\mu'_s - \mu_s)(\mathrm{d}x)\right|\mathrm{d}s,
\end{aligned}
$$

where we used (II.1.1). First, by Assumption II.5.4 and assumptions on φ and ψ, there exists a constant C_ψ such that $||\psi||_\infty + ||\psi'||_\infty \leq C_\psi$, so that

$$
\forall x > 0, \quad \psi(x) \left|\partial_x y_s^t(x)\right| = \psi(x)\left|\partial_x F_{s,x}(t) \varphi'(F_{s,x}(t))\right| \leq C_\psi e^{CT}.
$$

Then, for $x > 0$, we define

$$
\imath(x) := b(x)\left(y_s^t(x_0) + y_s^t(x-x_0) - y_s^t(x)\right) - d(x) y_s^t(x).
$$

By assumptions on b and d, by Assumption II.5.4, assumption on φ and regularity of the flow F, then $\imath \in \mathcal{C}^1(\mathbb{R}_+^*)$ and we verify that there exists constants $\overline{b} > 0$ and $\overline{d} > 0$ such that for all $x > 0$,

$$
|\imath(x)| \leq 3\overline{b} + \overline{d},
$$

and

$$
\left|\imath'(x)\right| \leq (3 + 2e^{CT})\overline{b} + (1 + e^{CT})\overline{d}.
$$

We finally obtain, using (II.5.32),

$$
\begin{aligned}
|\langle \mu'_t - \mu_t, \varphi\rangle| &\leq k\mathcal{S}_T C_\psi e^{CT} \int_0^t \left|R'_s - R_s\right| \mathrm{d}s \\
&\quad + \left((6 + 2e^{CT})\overline{b} + (2 + e^{CT})\overline{d}\right)\int_0^t d_{\mathrm{FM}}(\mu'_s, \mu_s)\mathrm{d}s.
\end{aligned}
$$

Finally, the previous reasoning is valid for every $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+^*)$ such that $||\varphi||_\infty + ||\varphi'||_\infty \leq 1$, so

$$
\begin{aligned}
d_{\mathrm{FM}}(\mu'_t, \mu_t) &\leq k\mathcal{S}_T C_\psi e^{CT} \int_0^t \left|R'_s - R_s\right| \mathrm{d}s \\
&\quad + \left((6 + 2e^{CT})\overline{b} + (2 + e^{CT})\overline{d}\right)\int_0^t d_{\mathrm{FM}}(\mu'_s, \mu_s)\mathrm{d}s.
\end{aligned}
$$

Furthermore, remark that by (II.1.3) and (II.3.24), for $t \in [0, T]$,

$$
\begin{aligned}
|R'_t - R_t| &= \left|\int_0^t \varsigma(R'_s) - \varsigma(R_s) + \chi(\langle \mu'_s, f(., R'_s)\rangle - \langle \mu_s, f(., R_s)\rangle)\mathrm{d}s\right| \\
&\leq ||\varsigma||_{\infty,[0,R_{\max}]} \int_0^t |R'_s - R_s|\mathrm{d}s \\
&\quad + \chi \left|\int_0^t \langle \mu'_s, f(., R'_s) - f(., R_s)\rangle - \langle \mu_s - \mu'_s, f(., R_s)\rangle \mathrm{d}s\right| \\
&\leq (||\varsigma||_{\infty,[0,R_{\max}]} + \chi k C_\psi \mathcal{S}_T)\int_0^t |R'_s - R_s|\mathrm{d}s + \chi C_\psi ||\phi||_{\infty,[0,R_{\max}]} \int_0^t d_{\mathrm{FM}}(\mu_s, \mu'_s)\mathrm{d}s,
\end{aligned}
$$

where we used again (II.1.1), the assumption on ψ and (II.5.32). By Gronwall lemma applied to $t \mapsto |R'_t - R_t| + d_{\mathrm{FM}}(\mu'_t, \mu_t)$, we obtain that for all $t \in [0, T]$,

$$|R'_t - R_t| + d_{\mathrm{FM}}(\mu'_t, \mu_t) = 0.$$

These are positive quantities, so for all $t \in [0, T]$, we obtain $(\mu_t, R_t) = (\mu'_t, R'_t)$, which concludes. □

Remark: Although Proposition II.5.7 provides the missing uniqueness argument to improve the tightness result of Theorem II.3.1 to the convergence result of Corollary II.5.2, it is only valid under restrictive assumptions on the rates b, d and ψ (and in particular, they are bounded). There are two main difficulties to overcome in the general case with unbounded rates.

- First, in the general case, the flow $(s, t, z) \mapsto F_{s,z}(t)$ is only locally well-defined on an open set $\mathfrak{V} \subseteq \mathbb{R}^+ \times \mathbb{R}^*_+ \times \mathbb{R}^+$. This is because with a possibly unbounded growth rate g, this flow can explode or reach 0 in finite time (we will take this fact into account in Section III.1.1). In addition to the lack of global definition of the flow, we do not benefit from any control as in Assumption II.5.4 in general.

- Then, in the proof of Proposition II.5.7, we use the Fortet-Mourier distance to show that μ_t and μ'_t coincide for every $t \geq 0$. Hence, we pick test functions φ verifying $||\varphi||_\infty + ||\varphi'||_\infty \leq 1$, and we search for an upper bound for $|\langle \mu'_t - \mu_t, \varphi\rangle|$ uniform on φ. We provide an upper bound with integral terms, where integrands are functions that verify the same conditions as φ (*i.e.* regularity and $||\varphi||_\infty + ||\varphi'||_\infty \leq 1$), up to a multiplicative constant, and conclude with Gronwall lemma. The main technical point here is to verify that the integrands we obtain verify the same constraints as φ, because they depend themselves on φ and the flow F. In the case of unbounded rates, apart from the technical difficulties of well-definition of the flow, we did not succeed in finding a control condition on φ or its derivatives that we are able to recover for the previously mentioned integrands. This is left for future work.

II.5.2 Obtaining Theorem II.3.1 with less restrictive assumptions on the functional parameters b, ψ and ω

In this section, we provide an alternative assumption in order to replace the restrictive Assumption II.3.1 in Theorem II.3.1. Remark that we use Assumption II.3.1 in Section II.4.4 (charecterization of accumulation ponts) in order to control quantities of the form $\displaystyle\int_0^t \langle \mu^K_s, \varphi_s\rangle \mathrm{d}s$ for $K \geq 1$, $t \in [0, T]$ and $\varphi \in \mathcal{C}^{0,0}_\omega([0, T] \times \mathbb{R}^*_+)$, where the functions b and $\overline{g}$ naturally intervene in the conditions verified by φ. Considering Lemma II.4.3 for example, at first sight, it was very natural for us to ask for these functions to be controlled by ω (*i.e.* to work with Assumption II.3.1). However, the uniform controls in Proposition II.2.4 and Lemma II.2.6 encourage us to introduce the following assumption.

Assumption II.5.8. *There exists $\eta \in (0, 1)$ such that*

- $\exists \tilde{c}_b > 0, \forall x > 1, \quad b(x) \leq \tilde{c}_b(\varpi(x) + \omega(x))$,

- $\exists \tilde{c}_g > 0, \forall x > 0, \quad \overline{g}(x) \leq \tilde{c}_g(\varpi(x) + \omega(x))$,

where $\varpi : x > 0 \mapsto (x - x_0)^{1-\eta}\mathbb{1}_{x - x_0 > 0}$. In addition, we ask for $1/\omega$ to be bounded in a neighborhood of $+\infty$.

Remark: In particular, the first two conditions of Assumption II.5.8 are less restrictive on b and ψ than in Assumption II.3.1, if $\omega(x)/x^{1-\eta}$ goes to 0 when x goes to $+\infty$. We will thus be less restrictive than Assumption II.3.1 in the allometric case, where we will prove that such a η exists in Lemma II.6.2 (see also Figure II.1). Also, the additional condition on $1/\omega$ is not a restrictive thing to ask if we think again of the allometric setting highlighted in Example 1 of Section II.6, because in that case, we can even construct ω such that $\omega(x) \xrightarrow[x\to+\infty]{} +\infty$.

Theorem II.5.1. *Under the assumptions of Theorem II.3.1, replacing Assumption II.3.1 with Assumption II.5.8, the conclusions of Theorem II.3.1 hold true.*

Before giving the proof of Theorem II.5.1, we begin with a preliminary important result. As Assumption II.3.1 does not intervene in Sections II.4.1, II.4.2 and II.4.3, we fix $T \geq 0$ in the following and can consider a subsequence, still denoted as $\left((\mu_t^K, R_t^K)_{t\in[0,T]} \right)_{K\in\mathbb{N}^*}$, converging in law towards $(\mu_t^*, R_t^*)_{t\in[0,T]}$ in $\mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w}) \times [0, R_{\max}])$.

Proposition II.5.9. *We work under the assumptions of Theorem II.3.1, replacing Assumption II.3.1 with Assumption II.5.8. Then, the following convergence in law in the space* $\mathbb{D}([0,T],\mathbb{R})$ *holds true*

$$\left(\langle \mu_t^K, \varpi\rangle\right)_{t\in[0,T]} \xrightarrow[K\longrightarrow+\infty]{} \left(\langle \mu_t^*, \varpi\rangle\right)_{t\in[0,T]}.$$

Proof. Using the Skorokhod representation theorem, we assume that the convergence of the sequence $\left((\mu_t^K, R_t^K)_{t\in[0,T]} \right)_{K\in\mathbb{N}^*}$ towards $(\mu_t^*, R_t^*)_{t\in[0,T]}$ in $\mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w}) \times [0, R_{\max}])$ is almost sure in the following. We then aim to show that

$$\mathbb{E}\left(\sup_{t\in[0,T]} \left| \langle \mu_t^K - \mu_t^*, \varpi\rangle \right| \right) \xrightarrow[K\longrightarrow+\infty]{} 0.$$

We consider $M > 0$ and for $t \in [0,T]$, $K \geq 1$, we write

$$\langle \mu_t^K, \varpi\rangle = \left\langle \mu_t^K, \varpi \mathbb{1}_{\{\varpi\leq M\omega\}} \right\rangle + \left\langle \mu_t^K, \varpi \mathbb{1}_{\{\varpi> M\omega\}} \right\rangle.$$

Hence, we have, as ϖ is non-negative,

$$\begin{aligned} \mathbb{E}\left(\sup_{t\in[0,T]} \left| \langle \mu_t^K - \mu_t^*, \varpi\rangle \right| \right) &\leq \mathbb{E}\left(\sup_{t\in[0,T]} \left| \left\langle \mu_t^K - \mu_t^*, \varpi \mathbb{1}_{\{\varpi\leq M\omega\}} \right\rangle \right| \right) \\ &+ \mathbb{E}\left(\sup_{t\in[0,T]} \left\langle \mu_t^*, \varpi \mathbb{1}_{\{\varpi> M\omega\}} \right\rangle \right) + \sup_{K\geq 1} \mathbb{E}\left(\sup_{t\in[0,T]} \left\langle \mu_t^K, \varpi \mathbb{1}_{\{\varpi> M\omega\}} \right\rangle \right). \end{aligned} \tag{II.5.33}$$

Let us focus on the integrand of the right-most term for a fixed $t \in [0,T]$ and use Hölder inequality with $p := \frac{1}{1-\eta}$ and $q := 1/\eta$ to obtain

$$\begin{aligned} \left\langle \mu_t^K, \varpi \mathbb{1}_{\{\varpi> M\omega\}} \right\rangle &= \left\langle \mu_t^K, \frac{\varpi}{\omega^{1/q}} \omega^{1/q} \mathbb{1}_{\{\varpi> M\omega\}} \right\rangle \\ &\leq \left\langle \mu_t^K, \frac{\varpi^p}{\omega^{p/q}} \right\rangle^{1/p} \times \left\langle \mu_t^K, \omega \mathbb{1}_{\{\varpi> M\omega\}} \right\rangle^{1/q} \end{aligned}$$

Then, by Assumption II.5.8, $1/\omega$ is bounded on a neighborhood of $+\infty$. This associated to the facts that $p/q > 0$, ω is continuous and $\varpi \equiv 0$ on $(0, x_0)$ entails that there exists a constant $C > 0$ with $\dfrac{\varpi^p(x)}{\omega^{p/q}(x)} \leq Cx$ for every $x > 0$ (we also use the definition of ϖ in Assumption II.5.8). Also, there exists a constant $C' > 0$ such that $\omega \mathbb{1}_{\{\varpi > M\omega\}} \leq \varpi/M \leq C' \times \mathrm{Id}/M$ and $1/p + 1/q = 1$, so we finally obtain

$$\left\langle \mu_t^K, \varpi \mathbb{1}_{\{\varpi > M\omega\}} \right\rangle \leq \frac{C' C^{1/p}}{M^{1/q}} \left\langle \mu_t^K, \mathrm{Id} \right\rangle.$$

Remark that we obtain the exact same bound, replacing K with $*$, so that

$$\begin{aligned} &\mathbb{E}\left(\sup_{t\in[0,T]} \left\langle \mu_t^*, \varpi \mathbb{1}_{\{\varpi > M\omega\}} \right\rangle \right) + \sup_{K\geq 1} \mathbb{E}\left(\sup_{t\in[0,T]} \left\langle \mu_t^K, \varpi \mathbb{1}_{\{\varpi > M\omega\}} \right\rangle \right) \\ &\qquad \leq \frac{C' C^{1/p}}{M^{1/q}} \left[\mathbb{E}\left(\sup_{t\in[0,T]} \left\langle \mu_t^*, \mathrm{Id} \right\rangle \right) + \sup_{K\geq 1} \mathbb{E}\left(\sup_{t\in[0,T]} \left\langle \mu_t^K, \mathrm{Id} \right\rangle \right) \right], \end{aligned} \tag{II.5.34}$$

which converges to 0 when $M \longrightarrow +\infty$ thanks to Proposition II.2.4 and Lemma II.5.3. In the following, we thus fix $\varepsilon > 0$ and $M_0 > 0$ such that (II.5.34)$< \varepsilon/2$. The function $x > 0 \mapsto \varpi(x)\mathbb{1}_{\{\varpi \leq M_0\omega\}}$ is positive and dominated by $M_0\omega$. Hence, by the almost sure convergence of $\left((\mu_t^K, R_t^K)_{t\in[0,T]} \right)_{K\in\mathbb{N}^*}$ towards $(\mu_t^*, R_t^*)_{t\in[0,T]}$ in $\mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w}) \times [0, R_{\max}])$, we obtain

$$\mathbb{E}\left(\sup_{t\in[0,T]} \left| \left\langle \mu_t^K - \mu_t^*, \varpi \mathbb{1}_{\{\varpi \leq M_0\omega\}} \right\rangle \right| \right) \xrightarrow[K\longrightarrow+\infty]{} 0,$$

so that there exists by (II.5.33) some $K_0 \geq 1$ such that for $K \geq K_0$,

$$\mathbb{E}\left(\sup_{t\in[0,T]} \left| \langle \mu_t^K - \mu_t^*, \varpi \rangle \right| \right) \leq \varepsilon,$$

which ends the proof since this is valid for every $\varepsilon > 0$. □

Proof of Theorem II.5.1:

The missing points to obtain the conclusion of Theorem II.3.1 with Assumption II.5.8 instead of Assumption II.3.1 are the following:

(*i*) $\left(\int_0^t \langle \mu_s^K, \mho_{\varphi,s} \rangle \mathrm{d}s \right)_{K\in\mathbb{N}^*}$ converges in law towards $\int_0^t \langle \mu_s^*, \mho_{\varphi,s} \rangle \mathrm{d}s$ in $\mathbb{R}$,

(*ii*) $\left(\int_0^t \langle \mu_s^K, \psi \partial_x \varphi_s \rangle \mathrm{d}s \right)_{K\in\mathbb{N}^*}$ converges in law towards $\int_0^t \langle \mu_s^*, \psi \partial_x \varphi_s \rangle \mathrm{d}s$ in $\mathbb{R}$,

with the notations of Section II.4.4. Indeed, if we verify the previous points, this allows us to characterize the limit as in Section II.4.4. Now, under Assumption II.5.8, both functions $\mho_{\varphi,s}$ and $\psi \partial_x \varphi_s$ are dominated by $\varpi + \omega$ (in particular, we control the term $b(x)\varphi(x_0)$) up to a multiplicative constant. We proved in Section II.4.1, using Aldous and Rebolledo criterion, that $((\langle \mu_t^K, \omega \rangle)_{t\in[0,T]})_{K\in\mathbb{N}}$ converges in law in $\mathbb{D}([0,T], \mathbb{R})$ towards $(\langle \mu_t^*, \omega \rangle)_{t\in[0,T]}$. With Proposition II.5.9, this entails the convergence in law of $((\langle \mu_t^K, \varpi + \omega \rangle)_{t\in[0,T]})_{K\in\mathbb{N}}$ in $\mathbb{D}([0,T], \mathbb{R})$ towards $(\langle \mu_t^*, \varpi + \omega \rangle)_{t\in[0,T]}$. With the same techniques as in the proof of Lemma II.4.3, we thus obtain (*i*) and (*ii*), which concludes the proof.

□

II.5.3 Additional results and extension of Theorem II.3.1 with additional assumptions on the accumulation points μ^*

First, let us motivate the formulation of the main conjecture of this section, which is Conjecture II.5.2. Let $T \geq 0$, thanks to Lemma II.5.3, we have that for every $t \in [0,T]$, any accumulation point μ_t^* integrates a broader set of functions than $\mathfrak{B}_\omega(\mathbb{R}_+^*)$ (by an approximation argument, we add any continuous function in $\mathfrak{B}_{1+\mathrm{Id}+\omega}(\mathbb{R}_+^*)$). Thus, for any continuous function $\varphi \in \mathfrak{B}_{1+\mathrm{Id}+\omega}(\mathbb{R}_+^*)$ and $t \geq 0$, even if we cannot give an explicit expression for $\langle \mu_t^*, \varphi\rangle$ as in (II.3.25), we can compute this quantity as the limit

$$\lim_{n\to+\infty} \lim_{K\to+\infty} \langle \mu_t^K, \varphi_n\rangle,$$

where $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence of $\mathcal{C}_c^\infty(\mathbb{R}_+^*)$ that converges pointwise towards φ. The expression of every $\langle \mu_t^K, \varphi_n\rangle$ is explicit and given in Proposition II.2.5. For instance, we are able to compute $\langle \mu_t^*, 1\rangle$, or $\langle \mu_t^*, \mathrm{Id}\rangle$ for any $t \geq 0$ (at least numerically), and we know that these quantities are finite. In fact, we even verify that we can adapt the reasoning of Section II.4.1 to obtain the tightness as a process in the space $\mathbb{D}([0,T], (\mathcal{M}_{1+\mathrm{Id}+\omega}(\mathbb{R}_+^*), v) \times [0, R_{\max}])$ (with $\mathcal{M}_{1+\mathrm{Id}+\omega}(\mathbb{R}_+^*)$ endowed with the vague convergence). This raises a natural question: can we hope for the following extension of Theorem II.3.1?

Conjecture II.5.2. *For all $T \geq 0$, the sequence of processes $\left((\mu_t^K, R_t^K)_{t\in[0,T]} \right)_{K\in\mathbb{N}^*}$ is tight in $\mathbb{D}([0,T], (\mathcal{M}_{1+\mathrm{Id}+\omega}(\mathbb{R}_+^*), \mathrm{w}) \times [0, R_{\max}])$.*

There are two difficulties to overcome, summarized in the following lemma. Without loss of generality, we fix μ_0^* in the following.

Lemma II.5.10. *We work under the assumptions of Theorem II.3.1, and make the three following additional assumptions.*

1. *The sequence $(\mu_0^K)_{K\in\mathbb{N}^*}$ converges in law towards μ_0^* in $(\mathcal{M}_{1+\mathrm{Id}+\omega}(\mathbb{R}_+^*), \mathrm{w})$.*

2. *The sequence of laws of $\left((\langle \mu_t^K, 1+\mathrm{Id}\rangle)_{t\in[0,T]} \right)_{K\in\mathbb{N}^*}$ is tight in $\mathbb{D}([0,T], \mathbb{R})$.*

3. *Any accumulation point $(\mu_t^*)_{t\in[0,T]}$ is in $\mathcal{C}([0,T], (\mathcal{M}_{1+\mathrm{Id}+\omega}(\mathbb{R}_+^*), \mathrm{w}))$.*

Then, Conjecture II.5.2 holds true.

Proof. We obviously need point 1. to be able to extend the tightness in Conjecture II.5.2 with the same initial condition. Then, the only missing argument to obtain the conclusion of Conjecture II.5.2 in the previous proof of Theorem II.3.1 lies in the use of Theorem II.4.2 in Section II.4.3 with $w \equiv 1 + \mathrm{Id} + \omega$, to prove the tightness of $\left((\mu_t^K, R_t^K)_{t\in[0,T]} \right)_{K\in\mathbb{N}^*}$ in $\mathbb{D}([0,T], (\mathcal{M}_{1+\mathrm{Id}+\omega}(\mathbb{R}_+^*), \mathrm{w}) \times [0, R_{\max}])$. Thus, we check if under the assumptions of Lemma II.5.10 we can use Theorem II.4.2 for this broader set of measure-valued processes. The tightness of $\left((\mu_t^K, R_t^K)_{t\in[0,T]} \right)_{K\in\mathbb{N}^*}$ in $\mathbb{D}([0,T], (\mathcal{M}_{1+\mathrm{Id}+\omega}(\mathbb{R}_+^*), v) \times [0, R_{\max}])$ (with the vague convergence) is straightforward by Theorem II.3.1, because $\mathcal{C}_c^\infty(\mathbb{R}_+^*) \subseteq \mathfrak{B}_\omega(\mathbb{R}_+^*)$. We also already proved that $(\langle \mu_t^K, \omega\rangle)_{K\in\mathbb{N}^*}$ is tight in $\mathbb{D}([0,T], \mathbb{R})$, and that $(R_t^*)_{t\in[0,T]}$ is continuous. The missing arguments to be able to use Theorem II.4.2 are therefore those introduced in points 2. and 3. of Lemma II.5.10. □

Lemma II.5.11. *Assume that* $d \times (1 + \mathrm{Id}^2) \in \mathfrak{B}_{1+\mathrm{Id}+\omega}(\mathbb{R}_+^*)$. *Then, points 2. and 3. of Lemma II.5.10 hold true. As a corollary, under the assumptions of Theorem II.3.1, under this additional assumption on* d*, and if point 1. of Lemma II.5.10 holds true, then Conjecture II.5.2 holds true.*

Proof. With this additionnal assumption on d and Assumption II.1.13, and writing $\nabla := 1 + \mathrm{Id} + \omega$, we verify that

- $\exists C_g' > 0, \forall x > 0, \quad \overline{g}(x)(1 + \nabla'(x)) \leq C_g' \nabla(x)$,
- $\exists C_b' > 0, \forall x > 0, \quad b(x)(1 + \hbar(|\nabla(x_0) + \nabla(x - x_0) - \nabla(x)|)) \leq C_b' \nabla(x)$,
- $\exists C_d' > 0, \forall x > 0, \quad d(x)\hbar(\nabla(x)) \leq C_d' \nabla(x)$.

We thus have replaced ω with ∇ in Assumption II.1.13, and in particular can use it to control the finite variation and martingale parts of $\langle \mu_t^K, \nabla \rangle$ for every $t \in [0, T]$ and $K \geq 1$. We let the reader check that then, the Aldous and Rebolledo criterion for the tightness of $\left((\langle \mu_t^K, 1 + \mathrm{Id} \rangle)_{t \in [0,T]} \right)_{K \in \mathbb{N}^*}$ we used in Section II.4.1 apply in the exact same manner; and that with the same techniques as in Lemma II.4.2, we verify that any accumulation point verifies $\mu^* \in \mathcal{C}([0, T], (\mathcal{M}_{1+\mathrm{Id}+\omega}(\mathbb{R}_+^*), \mathrm{w}))$, which ends the proof. $\square$

Remark: In particular, if the death rate d is bounded, we verify the additional assumption in Lemma II.5.11. Hence, if we adopt the classical framework depicted in the literature where growth, birth and death rates are bounded, Theorem II.5.2 extends the usual tightness result to a tightness in the weighted space $\mathbb{D}([0, T], (\mathcal{M}_{1+\mathrm{Id}+\omega}(\mathbb{R}_+^*), \mathrm{w}) \times [0, R_{\max}])$. In particular, for any accumulation point μ^*, Equation (II.3.25) is valid for $\varphi : (t, x) \mapsto x$, *i.e.* we have an explicit expression of $\langle \mu_t^*, \mathrm{Id} \rangle$ for $t \in [0, T]$ (in the case of a mass-structured model, this represents the total biomass for the limiting system described by μ^* at time t), which is a new result for individual-based models constructed as in Section II.1.2, with bounded rates.

In fact, the assumption on d in Lemma II.5.11 implies that d is bounded in a neighborhood of 0. Even if it entails Theorem II.5.2 thanks to Lemma II.5.10, we chose not to present it directly in Assumption II.1.13. Indeed, our main motivation is to obtain a general tightness result, also valid for unbounded rates. In particular, we will see in Section II.6 that if we consider the allometric setting with $\alpha \in (0, 1)$, by Lemma II.6.1, we would have a death rate of the form $x > 0 \mapsto x^{-\delta}$ with $\delta > 0$, which is not bounded in a neighborhood of 0. In the following, we present an approach to show the tightness of $(\langle \mu_t^K, 1 + \mathrm{Id} \rangle)_{K \in \mathbb{N}^*}$ in $\mathbb{D}([0, T], \mathbb{R})$ without further assumptions on d. We fix $T \geq 0$ in the following and can consider a subsequence, still denoted as $\left((\mu_t^K, R_t^K)_{t \in [0,T]} \right)_{K \in \mathbb{N}^*}$, converging in law towards $(\mu_t^*, R_t^*)_{t \in [0,T]}$ in $\mathbb{D}([0, T], (\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w}) \times [0, R_{\max}])$. We begin with a result in expectation.

Proposition II.5.12. *We work under the assumptions of Theorem II.3.1, and assume in addition that*

- *the sequence* $(\mu_0^K)_{K \in \mathbb{N}^*}$ *converges in law towards* μ_0^* *in* $(\mathcal{M}_{1+\mathrm{Id}+\omega}(\mathbb{R}_+^*), \mathrm{w})$,
- *the function* $t \in [0, T] \mapsto \mathbb{E}\left(\langle \mu_t^*, 1 + \mathrm{Id} \rangle\right) \in \mathbb{R}$ *and the function* $t \in [0, T] \mapsto \int_0^t \mathbb{E}\left(\langle \mu_s^*, g(., R_s^*) + b \rangle\right) \mathrm{d}s$ *are continuous.*

Then, we have

$$\sup_{t\in[0,T]} \left|\mathbb{E}\left(\langle \mu_t^K, 1+\mathrm{Id}\rangle\right) - \mathbb{E}\left(\langle \mu_t^*, 1+\mathrm{Id}\rangle\right)\right| \xrightarrow[K\to+\infty]{} 0.$$

Proof. For $K \in \mathbb{N}$, we write $\vartheta^K : t \in [0,T] \mapsto \mathbb{E}\left(\langle \mu_t^K, 1+\mathrm{Id}\rangle\right)$ and $\vartheta^* : t \in [0,T] \mapsto \mathbb{E}\left(\langle \mu_t^*, 1+\mathrm{Id}\rangle\right)$. These functions are deterministic and well-defined from $[0,T]$ to $\mathbb{R}$ thanks to Proposition II.2.4 and Lemma II.5.3. Our aim is to show that $(\vartheta^K)_{K\in\mathbb{N}^*}$ converges uniformly towards ϑ^* as real-valued functions defined on $[0,T]$. Using Theorem II.3.1 and Skorokhod representation theorem if necessary, we assume that $(\mu^K)_{K\in\mathbb{N}^*}$ converges almost surely towards μ^* in $\mathbb{D}([0,T],(\mathcal{M}_\omega(\mathbb{R}_+^*),\mathrm{w}))$. We divide the proof in two steps: first, we show the pointwise convergence of $(\vartheta^K)_{K\in\mathbb{N}^*}$ towards ϑ^*; and then we use a continuity and monotony argument and Dini first theorem to conclude to the uniform convergence of $(\vartheta^K)_{K\in\mathbb{N}^*}$ towards ϑ^*.

Step 1: Pointwise convergence of $(\vartheta^K)_{K\in\mathbb{N}^*}$ towards ϑ^*

We fix $t \in [0,T]$. For any $M > 0$, we define $\mathbf{1}_M$ a $\mathcal{C}^\infty(\mathbb{R}_+^*)$ function such that $0 \le \mathbb{1}_M \le 1$, $\mathbf{1}_M(x) = 1$ for $x \in (0, M/2)$ and $\mathbb{1}_M(x) = 0$ if $x \ge M$. Then, for $K \ge 1$ and $\natural \in \{*, K\}$ we write

$$\langle \mu_t^\natural, 1+\mathrm{Id}\rangle = \langle \mu_t^\natural, 1+\mathrm{Id}\rangle \mathbf{1}_M(\langle \mu_t^\natural, 1+\mathrm{Id}\rangle) + \langle \mu_t^\natural, 1+\mathrm{Id}\rangle(1-\mathbf{1}_M(\langle \mu_t^\natural, 1+\mathrm{Id}\rangle)), \quad \text{(II.5.35)}$$

where all terms are well-defined and almost surely finite thanks to Proposition II.2.4 and Lemma II.5.3. With the same techniques as in the proof of Lemma II.5.3, and using (II.2.23) and Lemma II.2.6, we let the reader check that there exists $p > 1$ such that

$$\mathbb{E}\left(\sup_{t\in[0,T]} |\langle \mu_t^*, 1+\mathrm{Id}\rangle^p|\right) + \sup_{K\in\mathbb{N}^*} \mathbb{E}\left(\sup_{t\in[0,T]} |\langle \mu_t^K, 1+\mathrm{Id}\rangle^p|\right) =: \mathfrak{C} < +\infty.$$

Remark that for all $x > 0$, $1 - \mathbf{1}_M(x) \le \mathbb{1}_{x>M/2}$. Hence, using Hölder inequality with $p > 1$, and then Markov inequality that for any $K \ge 1$ and $\natural \in \{*, K\}$, we obtain

$$\begin{aligned}\mathbb{E}\left(\langle \mu_t^\natural, 1+\mathrm{Id}\rangle(1-\mathbf{1}_M(\langle \mu_t^\natural, 1+\mathrm{Id}\rangle))\right) &\le \mathbb{E}\left(\langle \mu_t^\natural, 1+\mathrm{Id}\rangle^p\right)^{1/p} \mathbb{E}\left(\mathbb{1}_{\langle \mu_t^\natural, 1+\mathrm{Id}\rangle > M/2}\right)^{(p-1)/p} \\ &\le \frac{2^{p-1}\mathbb{E}\left(\langle \mu_t^\natural, 1+\mathrm{Id}\rangle^p\right)}{M^{p-1}} \\ &\le \frac{2^{p-1}\mathfrak{C}}{M^{p-1}},\end{aligned}$$

and this upper bound goes to 0 when M goes to 0. Thus, in the following, we fix $\varepsilon > 0$ and choose $M_0 > 0$ such that $2^{p-1}\mathfrak{C}/M_0^{p-1} \le \varepsilon/3$. In that case, we obtain with the decomposition in (II.5.35) and the previous upper bound,

$$\begin{aligned}\left|\mathbb{E}\left(\langle \mu_t^K, 1+\mathrm{Id}\rangle\right) - \mathbb{E}\left(\langle \mu_t^*, 1+\mathrm{Id}\rangle\right)\right| \le \frac{2\varepsilon}{3} + \Big|&\mathbb{E}\left(\langle \mu_t^K, 1+\mathrm{Id}\rangle \mathbf{1}_{M_0}(\langle \mu_t^K, 1+\mathrm{Id}\rangle)\right) \\ &- \mathbb{E}\left(\langle \mu_t^*, 1+\mathrm{Id}\rangle \mathbf{1}_{M_0}(\langle \mu_t^*, 1+\mathrm{Id}\rangle)\right)\Big|. \quad \text{(II.5.36)}\end{aligned}$$

Now, let us consider $(\varphi_n)_{n\in\mathbb{N}}$ an increasing sequence of $\mathcal{C}_c^\infty(\mathbb{R}_+^*)$ positive functions that converges pointwise towards $1+\mathrm{Id}$. Because $(\mu^K)_{K\in\mathbb{N}^*}$ converges almost surely towards

μ^* in $\mathbb{D}([0,T],(\mathcal{M}_\omega(\mathbb{R}_+^*),\mathrm{w}))$, and the functions φ_n are in $\mathcal{C}_c^\infty(\mathbb{R}_+^*)$, we obtain in particular that $(\langle \mu_t^K, \varphi_n\rangle)_{K\geq 1}$ converges almost surely towards $\langle \mu_t^*, \varphi_n\rangle$ for every $n\in\mathbb{N}$. Then by continuity of $x>0 \mapsto x\mathbf{1}_{M_0}(x)$ and dominated convergence (we can dominate all the integrands by M_0), we obtain

$$\begin{aligned}\mathbb{E}\left(\langle \mu_t^*, 1+\mathrm{Id}\rangle \mathbf{1}_{M_0}(\langle \mu_t^*, 1+\mathrm{Id}\rangle)\right) &= \lim_{n\to+\infty}\lim_{K\to+\infty} \mathbb{E}\left(\langle \mu_t^K, \varphi_n\rangle \mathbf{1}_{M_0}(\langle \mu_t^K, \varphi_n\rangle)\right)\\ &= \lim_{K\to+\infty}\lim_{n\to+\infty} \mathbb{E}\left(\langle \mu_t^K, \varphi_n\rangle \mathbf{1}_{M_0}(\langle \mu_t^K, \varphi_n\rangle)\right)\\ &= \lim_{K\to+\infty} \mathbb{E}\left(\langle \mu_t^K, 1+\mathrm{Id}\rangle \mathbf{1}_{M_0}(\langle \mu_t^K, 1+\mathrm{Id}\rangle)\right),\end{aligned}$$

Hence, for K high enough, we obtain thanks to (II.5.36) that

$$|\vartheta_t^K - \vartheta_t^*| \leq \varepsilon,$$

which concludes the proof of the pointwise convergence of $(\vartheta^K)_{K\in\mathbb{N}^*}$ towards ϑ^* since this is valid for any $\varepsilon>0$.

Step 2: Continuity and monotony argument to use Dini first theorem

In addition to the pointwise convergence proved in Step 1, we also have $\vartheta^*\in\mathcal{C}([0,T],\mathbb{R})$ by assumption. For $K\in\mathbb{N}^*$ and $t\in[0,T]$, we first write thanks to Lemma II.2.3,

$$\begin{aligned}\langle \mu_t^K, 1+\mathrm{Id}\rangle = \;& \langle \mu_0^K, 1+\mathrm{Id}\rangle + \int_0^t \langle \mu_s^K, g(.,R_s^K)\rangle \mathrm{d}s\\ &+ \frac{1}{K}\int_0^t\int_{\mathcal{U}\times\mathbb{R}_+^*} \mathbb{1}_{\{u\in V_{s-}^K\}}\mathbb{1}_{\{h\leq b(\xi_{s-}^{u,K})\}}\mathcal{N}(\mathrm{d}s,\mathrm{d}u,\mathrm{d}h)\\ &- \frac{1}{K}\int_0^t\int_{\mathcal{U}\times\mathbb{R}_+^*} \mathbb{1}_{\{u\in V_{s-}^K\}}\mathbb{1}_{\{h\leq d(\xi_{s-}^{u,K})\}}\left(1+\xi_{s-}^{u,K}\right)\mathcal{N}'(\mathrm{d}s,\mathrm{d}u,\mathrm{d}h).\end{aligned} \quad \text{(II.5.37)}$$

Thanks to Assumption II.3.1, b/ω is bounded, so that we can use similar arguments as in Proposition II.2.5 to give a semi-martingale decomposition of the integrated term with respect to $\mathcal{N}$ in (II.5.37). To show that the same holds true for the integrated term with respect to $\mathcal{N}'$, we isolate this term in Equation (II.5.37) and use the same argument as in Proposition II.1.18 (importantly, we use Assumption II.3.1 which states that $(b+\overline{g})/\omega$ is bounded, and then Proposition II.2.4 to control the expectation of the integrated terms uniformly on K). Once we identified (II.5.37) as a true semi-martingale decomposition, we take expectations and obtain

$$\begin{aligned}\vartheta_t^K &= \vartheta_0^K + \int_0^t \mathbb{E}\left(\langle \mu_s^K, g(.,R_s^K)+b\rangle\right)\mathrm{d}s - \int_0^t \mathbb{E}\left(\langle \mu_s^K, d\times(1+\mathrm{Id})\rangle\right)\mathrm{d}s\\ &= \vartheta_0^K + I^K(t) + D^K(t),\end{aligned}$$

where $I^K: t\in[0,T]\mapsto \int_0^t \mathbb{E}\left(\langle \mu_s^K, g(.,R_s^K)+b\rangle\right)\mathrm{d}s$ is a non-decreasing function of t and $D^K: t\in[0,T]\mapsto -\int_0^t \mathbb{E}\left(\langle \mu_s^K, d\times(1+\mathrm{Id})\rangle\right)\mathrm{d}s$ is a non-increasing function of t. Remark that $D^K(t)$ is well-defined and finite for all $t\geq 0$, by Fubini theorem and because it is equal to $\vartheta_t^K - \vartheta_0^K - I^K(t)$, which is a well-defined function thanks to Proposition II.2.4.

Let $t \in [0,T]$, then by Assumption II.3.1, $(b+\overline{g})/\omega$ is bounded, so by Theorem II.3.1, Lemma II.4.4 and domination by $C\omega$ with $C>0$ a constant independent of K and t, we obtain that $(I_t^K)_{K\in\mathbb{N}^*}$ converges to $I_t^* := \int_0^t \mathbb{E}\left(\langle \mu_s^*, g(.,R_s^*)+b\rangle\right)\mathrm{d}s$. Furthermore, we assumed that $I^* : t\in[0,T] \mapsto I_t^*$ is in $\mathcal{C}([0,T],\mathbb{R})$. By Dini first theorem, we deduce that $(I^K)_{K\in\mathbb{N}^*}$ converges uniformly towards I^*.

Now, by the pointwise convergence of $(\vartheta^K)_{K\geq 1}$ towards ϑ^*, the previous pointwise convergence of $(I^K)_{K\geq 1}$ and our assumption on the convergence of the initial conditions $(\mu_0^K)_{K\geq 1}$, we obtain that $D^K(t) = \vartheta_t^K - \vartheta_0^K - I^K(t)$ converges pointwise towards $t\in[0,T] \mapsto \vartheta_t^* - \vartheta_0^* - I^*(t)$. Furthermore, this limit is continuous and the functions D^K are non-increasing. By Dini first theorem again, $(D^K)_{K\in\mathbb{N}^*}$ converges uniformly towards $\vartheta^* - \vartheta_0^* - I^*$, and this ends the proof. □

Proposition II.5.12 is not enough to obtain the tightness of $(\langle \mu_t^K, 1+\mathrm{Id}\rangle)_{K\in\mathbb{N}^*}$ in $\mathbb{D}([0,T],\mathbb{R})$. We introduce additional assumptions to work without expectations. Let us introduce some notations, for $K\geq 1$ and $t\in[0,T]$, we write thanks to Lemma II.2.3,

$$\langle \mu_t^K, 1+\mathrm{Id}\rangle = A_t^K - B_t^K,$$

with

$$A_t^K := \langle \mu_0^K, 1+\mathrm{Id}\rangle + \int_0^t \langle \mu_s^K, g(., R_s^K)\rangle \mathrm{d}s + \int_0^t \int_{\mathcal{U}\times\mathbb{R}_+^*} \mathbb{1}_{\{u\in V_{s-}^K\}} \mathbb{1}_{\left\{h\leq b(\xi_{s-}^{u,K})\right\}} \mathcal{N}(\mathrm{d}s,\mathrm{d}u,\mathrm{d}h) \tag{II.5.38}$$

and

$$B_t^K := \int_0^t \int_{\mathcal{U}\times\mathbb{R}_+^*} \mathbb{1}_{\{u\in V_{s-}^K\}} \mathbb{1}_{\{h\leq d(\xi_{s-}^{u,K})\}} \left(1+\xi_{s-}^{u,K}\right) \mathcal{N}'(\mathrm{d}s,\mathrm{d}u,\mathrm{d}h).$$

For every $t\in[0,T]$, we also define

$$A_t := \langle \mu_0^*, 1+\mathrm{Id}\rangle + \int_0^t \langle \mu_s^*, g(.,R_s^*) + b(.)\rangle \mathrm{d}s.$$

For every $\varepsilon\in(0,1)$, we can construct $\mathrm{Reg}_\varepsilon \in \mathcal{C}_c^\infty(\mathbb{R}_+^*)$ such that $0\leq \mathrm{Reg}_\varepsilon \leq 1$, $\mathrm{supp}(\mathrm{Reg}_\varepsilon) \subseteq (2\varepsilon, 1/2\varepsilon)$ and $\mathrm{Reg}_\varepsilon \equiv 1$ on $(\varepsilon, 1/\varepsilon)$. We then define for $t\in[0,T]$, $K\geq 1$ and $\varepsilon\in(0,1)$,

$$\beta_t^{K,\varepsilon} := \int_0^t \int_{\mathcal{U}\times\mathbb{R}_+^*} \mathbb{1}_{\{u\in V_{s-}^K\}} \mathbb{1}_{\left\{h\leq d(\xi_{s-}^{u,K})\right\}} \left(1+\xi_{s-}^{u,K}\right) \mathrm{Reg}_\varepsilon\left(\xi_{s-}^{u,K}\right) \mathcal{N}'(\mathrm{d}s,\mathrm{d}u,\mathrm{d}h), \tag{II.5.39}$$

and

$$\beta_t^\varepsilon := \int_0^t \langle \mu_s^*, d\times(1+\mathrm{Id})\mathrm{Reg}_\varepsilon\rangle \mathrm{d}s,$$

and all these quantities are almost surely finite thanks to Proposition II.2.4, the fact that Reg_ε has compact support for any $\varepsilon>0$ for $\beta_t^{K,\varepsilon}$, and Lemma II.5.3 for β_t^ε. For $t\in[0,T]$, we finally write

$$\beta_t := \int_0^t \langle \mu_s^*, d\times(1+\mathrm{Id})\rangle \mathrm{d}s.$$

Note that at this step, this quantity is possibly finite or infinite. We begin with a preliminary result.

Lemma II.5.13. *Under the assumptions of Theorem II.3.1, for every $\varepsilon > 0$, a subsequence of $(\beta^{K,\varepsilon})_{K\in\mathbb{N}^*}$ converges in law in $\mathbb{D}([0,T],\mathbb{R})$ towards $(\beta^{\varepsilon}_t)_{t\in[0,T]}$.*

If in addition the sequence $(\mu^K_0)_{K\in\mathbb{N}^*}$ converges in law towards μ^*_0 in $(\mathcal{M}_{1+\mathrm{Id}+\omega}(\mathbb{R}^*_+),\mathrm{w})$, then a subsequence of $\left(A^K\right)_{K\in\mathbb{N}^*}$ converges in law in $\mathbb{D}([0,T],\mathbb{R})$ towards $(A_t)_{t\in[0,T]} \in \mathcal{C}([0,T],\mathbb{R})$.

Proof. We fix $\varepsilon > 0$, and we know that $d\times(1+\mathrm{Id})\mathrm{Reg}_\varepsilon \in \mathcal{C}^\infty_c(\mathbb{R}^*_+)$, so by similar arguments as in Proposition II.2.5, we can give a semi-martingale decomposition of the integrated term against $\mathcal{N}'$ in (II.5.39), using the associated compensated Poisson point measure. The quadratic variation associated to the martingale term is

$$\frac{1}{K}\int_0^t \langle \mu^K_s, d\times(1+\mathrm{Id})^2\mathrm{Reg}^2_\varepsilon\rangle \mathrm{d}s,$$

so that this martingale term converges in L^1 to 0 when K goes to $+\infty$ by similar arguments as in Step 1. of the proof of Proposition II.4.5, and the fact that Reg_ε has compact support. Then, with Theorem II.3.1, Lemma II.4.3 and the mapping theorem, we verify that the finite variation part converges in law to $(\beta^\varepsilon_t)_{t\in[0,T]}$, which ends the proof of the first part of Lemma II.5.13.

The proof of the second point is similar, but importantly, we use Assumption II.3.1 (*i.e.* the fact that $(b+\overline{g})/\omega$ is bounded) and the uniform control of Proposition II.2.4 to show that we have a true martingale decomposition in (II.5.38), and to show that the quadratic variation associated to the martingale part converges in L^1 to 0. Then, for the convergence of the finite variation part towards $(A_t)_{t\in[0,T]}$, we use our assumption on $(\mu^K_0)_{K\geq 1}$, and then Theorem II.3.1, Lemma II.4.4 and the mapping theorem to conclude for the integrated terms. The fact that $(A_t)_{t\in[0,T]}$ is continuous follows from the facts that $\mu^* \in \mathcal{C}([0,T],(\mathcal{M}_\omega(\mathbb{R}^*_+),\mathrm{w}))$ (see Theorem II.3.1) and $(b+\overline{g})/\omega$ is bounded. □

We are now ready to state the main result of this section.

Theorem II.5.3. *We work under the assumptions of Theorem II.3.1, and assume in addition that*

- *(i) the sequence $(\mu^K_0)_{K\in\mathbb{N}^*}$ converges in law towards μ^*_0 in $(\mathcal{M}_{1+\mathrm{Id}+\omega}(\mathbb{R}^*_+),\mathrm{w})$,*
- *(ii) for any accumulation point μ^*, we have: $\forall t\in[0,T],\quad \beta_t < +\infty$,*
- *(iii) any accumulation point $(\mu^*_t)_{t\in[0,T]}$ is in $\mathcal{C}([0,T],(\mathcal{M}_{1+\mathrm{Id}+\omega}(\mathbb{R}^*_+),\mathrm{w}))$.*

Then, Conjecture II.5.2 holds true.

Proof. For every $k\in\mathbb{N}^*$, we write $\varepsilon_k := 1/2^k$. Without loss of generality, extracting if necessary, from Lemma II.5.13, we use the Skorokhod representation theorem and consider that we work in a probability space where the following convergences hold true almost surely:

- $(\mu^K_0)_{K\in\mathbb{N}^*}$ converges towards μ^*_0,
- $\forall k\geq 1$, $(\beta^{K,\varepsilon_k})_{K\in\mathbb{N}^*}$ converges towards $(\beta^{\varepsilon_k}_t)_{t\in[0,T]}$,
- $\left(A^K\right)_{K\in\mathbb{N}^*}$ converges towards $(A_t)_{t\in[0,T]}$.

Note that for the use of Skorokhod representation theorem, we carefully consider a countable number of processes. Obtaining another almost sure convergence result in this new probability space will translate into a convergence in law for the initial process that we consider. For $t \in [0,T]$, $K \geq 1$ and $k \geq 1$, we verify that

$$\beta_t^{K,\varepsilon_k} \leq B_t^K = A_t^K - \langle \mu_t^K, 1 + \mathrm{Id} \rangle \leq A_t^K - \langle \mu_t^K, (1+\mathrm{Id})\mathrm{Reg}_{\varepsilon_k} \rangle.$$

First, we take the limit $K \to +\infty$ and obtain almost surely

$$\beta_t^{\varepsilon_k} \leq \liminf_{K \to +\infty} B_t^K \leq \limsup_{K \to +\infty} B_t^K \leq A_t - \langle \mu_t^*, (1+\mathrm{Id})\mathrm{Reg}_{\varepsilon_k} \rangle.$$

Then, by (ii) and Lemma II.5.3, we can take the limit $k \to +\infty$ and obtain almost surely

$$\beta_t \leq \liminf_{K \to +\infty} B_t^K \leq \limsup_{K \to +\infty} B_t^K \leq A_t - \langle \mu_t^*, (1+\mathrm{Id}) \rangle.$$

Furthermore, we can write for all $t \in [0,T]$ that

$$\langle \mu_t^*, (1+\mathrm{Id}) \rangle = A_t + \beta_t,$$

using an approximation argument, Equation (II.3.25), Lemma II.5.3 and (ii). This entails that for every $t \in [0,T]$, $\lim_{K \to +\infty} B_t^K = \beta_t$. In addition to this pointwise convergence, we assess that the limit $(\beta_t)_{t\in[0,T]} \in \mathcal{C}([0,T]), \mathbb{R})$. It is given by (iii), Lemma II.5.13 and the equality $\beta_t = \langle \mu_t^*, (1+\mathrm{Id}) \rangle - A_t$. Finally, every $(B_t^K)_{t\in[0,T]}$ is an increasing function, so by Dini first theorem, we obtain that, in the probability space given by the Skorokhod representation theorem, we have almost surely that $(B^K)_{K\geq 1}$ converges uniformly towards β in $\mathbb{D}([0,T]), \mathbb{R})$. This implies that both sequences of laws of $(A^K)_{K\geq 1}$ and $(B^K)_{K\geq 1}$ are tight, thus $(\langle \mu^K, 1+\mathrm{Id} \rangle = A^K - B^K))_{K\geq 1}$ is tight and this ends the proof by Lemma II.5.10. □

Remark: Interestingly, note that conditionally to the initial condition μ_0^*, the additional assumptions we introduce in Theorem II.5.3 depend only on the limit μ^*, which is a deterministic object, and is characterized by (II.3.24) and (II.3.25). Hence, we transposed our probabilistic questioning into the study of the solution to the weak formulation of a PDE system. The question of verifying assumptions (i), (ii) and (iii) in Theorem II.5.3 remains open, but could be settled with a deterministic approach, which is not our area of expertise.

II.6 Application to a new setting with allometric functional parameters, and link with existing models

In this section, with Example 1, we first introduce a new setting that does not exist yet in the literature, with allometric functional parameters. We show that this allometric case falls within the framework of the general setting depicted in Section II.1, *i.e.* we prove that there exists a weight function ω verifying Assumption II.1.13 when the functional parameters of our model are allometric. Thus, the tightness result of Theorem II.3.1 is valid in this allometric case, and we will provide numerical illustrations in Section III.4.2. Then, with Example 2 and Example 3, we draw links between the general setting and specific frameworks already existing in the literature.

Example 1 (Allometric setting): We give a fundamental example, with allometric scalings for the metabolic, growth, birth and death rates. To the best of our knowledge, this case has not been studied with an individual-based approach so far. It is motivated by our biological interests, and in the following, we will refer to it as the 'allometric setting'. For every $x > 0$ and $R \geq 0$, we set:

1. $\ell(x) := C_\alpha x^\alpha$,
2. $b(x) := \mathbb{1}_{x>x_0} C_\beta x^\beta$,
3. $f(x, R) := \phi(R) C_\gamma x^\gamma$ (*i.e.* $\psi(x) = C_\gamma x^\gamma$),
4. $d(x) := C_\delta x^\delta$,

with $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $(C_\alpha, C_\beta, C_\gamma, C_\delta) \in \mathbb{R}_+^*$. If f and g are two functions on $\mathbb{R}_+^*$ with g positive, we write $f \mathrel{\mathring{=}}_0 g$, respectively $f \mathrel{\mathring{=}}_\infty g$, if $f(x) = g(x)$ on a neighborhood of 0, respectively a neighborhood of $+\infty$. We aim to choose a weight function $\omega \in \mathcal{C}^1(\mathbb{R}_+^*)$ such that there exists $0 \leq \kappa_1 \leq \kappa_2$ with

- $\exists C_1 > 0, \quad \omega(x) \mathrel{\mathring{=}}_0 C_1 x^{\kappa_1}$,
- $\exists C_2 > 0, \quad \omega(x) \mathrel{\mathring{=}}_\infty C_2 x^{\kappa_2}$,

so that ω is non-decreasing. We verify easily that we can construct such functions for any $0 \leq \kappa_1 \leq \kappa_2$, and we say in the following that ω has an allometric form if it verifies the previous conditions. In the upcoming lemmas, we investigate under which conditions on the allometric coefficients we verify Assumptions II.1.4, II.1.5, II.1.13 and II.3.1.

Lemma II.6.1. *Under the allometric setting, we have*

$$(\textit{Assumptions II.1.4 and II.1.5}) \Leftrightarrow (\delta \leq \alpha - 1,\ \max(\beta, \delta) \geq \alpha - 1,\ \gamma = \alpha \textit{ and } C_\gamma > C_\alpha).$$

Proof. We work under the allometric setting. We have that

$$\forall x > 0, \quad \psi(x) - \ell(x) = C_\gamma x^\gamma - C_\alpha x^\alpha,$$

so Lemma II.1.2 implies that $\gamma = \alpha$ and $C_\gamma > C_\alpha$. Then, Assumption II.1.4 implies that $\delta \leq \alpha - 1$, and $\max(\beta, \delta) \geq \alpha - 1$ follows from Assumption II.1.5. The converse implication is straightforward. □

Lemma II.6.2. *Under the allometric setting, under Assumptions II.1.4 and II.1.5, suppose that the weight function ω has an allometric form with $0 \leq \kappa_1 \leq \kappa_2$.*

- *If $\alpha - 3 \leq \delta < -1$, then Assumption II.1.13 is equivalent to*

$$\kappa_1 = \kappa_2 = -\delta, \quad \alpha \in [0, 1] \quad \text{and} \quad \beta \leq 2 + \delta;$$

- *If $-1 \leq \delta \leq 0$, then Assumption II.1.13 is equivalent to*

$$-\delta \leq \kappa_1 \leq \kappa_2 \leq \frac{1 - \delta}{2} \leq 1, \quad \alpha \in [0, 1], \quad \text{and} \quad \beta \leq 1.$$

- *If $\delta > 0$ or $\delta < \alpha - 3$, then Assumption II.1.13 cannot be verified.*

These constraints on the allometic coefficients are illustrated on Figure II.2 and Figure II.3.

Proof. Under Assumptions II.1.4 and II.1.5, by Lemma II.6.1, we work with $\delta \le \alpha - 1$, $\max(\beta, \delta) \ge \alpha - 1$, $\gamma = \alpha$ and $C_\gamma > C_\alpha$, and an allometric form for ω with $0 \le \kappa_1 \le \kappa_2$. First, we suppose that Assumption II.1.13 holds true. If we consider the different points of Assumption II.1.13, it suffices to study the associated inequalities on a neighborhood of 0 and on a neighborhood of $+\infty$, since all the considered functions are continuous. Under the allometric setting, the third point of Assumption II.1.13 gives

$$\forall x > 0, \quad C_\delta x^\delta(\omega(x) + \omega^2(x)) \le C_d \left(1 + x + \omega(x)\right). \tag{II.6.40}$$

Considering $x \to 0$, respectively $x \to +\infty$, and an allometric form for ω, we obtain that (II.6.40) implies that $-\delta \le \kappa_1$, respectively that $2\kappa_2 + \delta \le \max(1, \kappa_2)$. First if $\kappa_2 > 1$, then we necessarily have $1 < -\delta = \kappa_1 = \kappa_2$. Else, we verify that we have $-\delta \le \kappa_1 \le \kappa_2 \le \dfrac{1-\delta}{2} \le 1$. Then, the first point of Assumption II.1.13 gives

$$\forall x > 0, \quad \max(\phi(R_{\max})C_\gamma - C_\alpha, C_\alpha)x^\alpha(1 + \omega'(x)) \le C_g \left(1 + x + \omega(x)\right). \tag{II.6.41}$$

Considering $x \to 0$ and $x \to +\infty$, we verify that (II.6.41) implies that $0 \le \alpha \le \max(1, \kappa_2)$. In particular, we obtain that Assumption II.1.13 cannot be verified if $\delta > 0$ (because in that case, we would have $0 > -\delta \ge 1 - \alpha \ge 0$). Also, if we suppose by contradiction that $\alpha > 1$ and $\kappa_2 > 1$, then considering (II.6.41) when $x \to +\infty$, we would have $\alpha + \kappa_2 - 1 \le \max(1, \kappa_2) = \kappa_2$ so $\alpha \le 1$ which is a contradiction. Hence, we always have $\alpha \in [0, 1]$. Finally, we consider the second point of Assumption II.1.13. First if $-1 \le \delta \le 0$, then ω is non-decreasing, and Lipschitz continuous on $(1, +\infty)$ by the previous work, so the second point of Assumption II.1.13 is equivalent to (II.1.12), which entails $\beta \le 1$. Else if $\delta < -1$, we verify that the left-hand side in the second point of Assumption II.1.13 is of order $x^{\beta - 2\delta - 2}$ in a neighborhood of $+\infty$, and the right-hand side is of order $x^{-\delta}$, hence we necessarily have $\beta \le 2 + \delta$. As $\max(\beta, \delta) \ge \alpha - 1$, Assumption II.1.13 cannot hold true if $\delta < \alpha - 3$. The converse implications in Lemma II.6.1 are straightforward verifications. □

Remark: Note in particular that under the allometric setting, if we want to pick a weight function that has an allometric form with $0 \le \kappa_1 \le \kappa_2$ and verifies Assumptions II.1.4, II.1.5 and Assumption II.1.13, we restrict ourselves to the case $\alpha \in [0, 1]$. If we want to consider for example the case $\beta + 1 = \delta + 1 = \alpha = 3/4$ supported by the Metabolic Theory of Ecology [SDF08], we can pick $\kappa_1 = 1/4$ and $\kappa_2 = 5/8$ according to Lemma II.6.2 (and this is the best choice in the sense that we cannot choose another weight function that has an allometric form and dominates this particular weight near 0 or $+\infty$). Obviously, we investigated here only a precise form of the weight function ω, other choices may be possible to be less restrictive in the case $\beta + 1 = \delta + 1 = \alpha = 3/4$, or to study for example the case $\alpha < 0$. We leave this for future work.

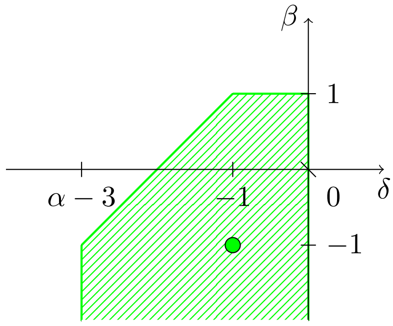


Figure II.2: Visual representation of the constraints on δ and β in Lemma II.6.2, with $\gamma = \alpha \in [0, 1]$ (we took $\alpha = 0$ on this figure). The admissible coefficients (δ, β) are those in the green hatched area, and verify one of the two following conditions: (i) $(\alpha - 3 \leq \delta < -1$ and $\beta \leq 2 + \delta)$; (ii) $(-1 \leq \delta \leq 0$ and $\beta \leq 1)$. Remark that the green area always contains the particular case $\beta = \delta = \alpha - 1 = \gamma - 1$ highlighted by the Metabolic Theory of Ecology, represented by a green dot on the figure.

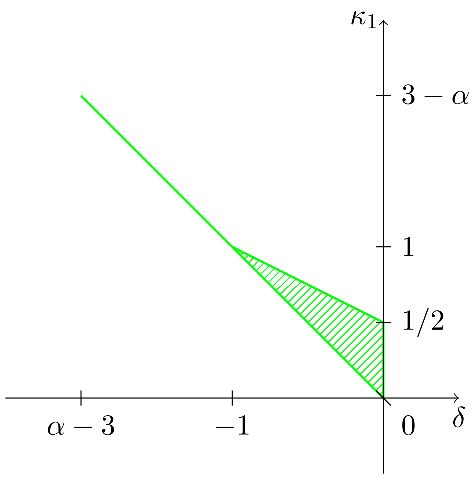


Figure II.3: Visual representation of the constraints on κ_1 in Lemma II.6.2, depending on the value of δ (we took $\alpha = 1$ on this figure). This represents the fact that if $\alpha - 3 \leq \delta < -1$, then $\kappa_1 = -\delta$, and if $-1 \leq \delta \leq 0$, then $-\delta \leq \kappa_1 \leq (1 - \delta)/2$. Note that κ_2 verifies the same conditions in Lemma II.6.2, so that this graph is also valid to visualize the constraints on κ_2. To sum up, if we want to pick an admissible triplet $(\delta, \kappa_1, \kappa_2)$, we first choose (δ, κ_1) on the green line or in the green hatched area. Then, the remaining possible values for κ_2 are such that the two following conditions hold true: $\kappa_2 \geq \kappa_1$, and (δ, κ_2) is also on the green line or in the green hatched area.

Lemma II.6.3. *Under the allometric setting, suppose that the weight function ω has an allometric form with $0 \le \kappa_1 \le \kappa_2$. Then,*

(*i*) *Assumption II.3.1 is equivalent to*

$$\kappa_1 \le \alpha \le \kappa_2 \text{ and } \beta \le \kappa_2,$$

(*ii*) *Assumption II.5.8 is equivalent to*

$$\exists \eta \in (0,1), \quad \kappa_1 \le \alpha \le \max(\kappa_2, 1-\eta) \text{ and } \beta \le \max(\kappa_2, 1-\eta).$$

Proof. This is an immediate verification (in particular, if ω has an allometric form, the fact that $1/\omega$ is bounded near $+\infty$ is automatically verified). □

Remark: If we want to be able to apply Theorem II.3.1 under the allometric setting, we thus have to gather all the restrictive assumptions of Lemmas II.6.1, II.6.2 and II.6.3. For example, if we work with the less restrictive Assumption II.5.8, and fix $\beta = \delta = \alpha - 1$ (this is motivated by Theorem I.2.1 in Chapter I), this gives the following constraints on κ_1 and κ_2,

- $0 \le 1 - \alpha \le \kappa_1 \le \kappa_2 \le 1 - \dfrac{\alpha}{2} \le 1$,
- $\exists \eta \in (0,1)$, $\alpha \in [\kappa_1, \max(\kappa_2, 1-\eta)]$, which implies with the previous point that $\alpha \in [1/2, 1)$.

In particular, ω is dominated by $1+\mathrm{Id}$, both near 0 and $+\infty$ (see again Figure II.1). Hence, from Corollary II.5.3, the best we can obtain in that case is an expression for any $\langle \mu_t, \varphi \rangle$, where φ' is bounded (which is equivalent to $\varphi \in \mathfrak{B}_{1+\mathrm{Id}}(\mathbb{R}_+^*)$). In particular, any accumulation point μ_t^* integrates Id, and we can recover the total energy in the population. More generally, μ_t^* integrates $x \mapsto x^k$ for every $k \in [0,1]$. We will illustrate numerically the convergence result of Theorem II.3.1 under the allometric setting in Section III.4.2.

Finally, we draw links between the general setting and specific frameworks already existing in the literature in Example 2 and Example 3. Importantly, note that we are not going to be fully explicit and the reader have to adapt our methods to the described frameworks to recover a result similar to Theorem II.3.1. For example, in our model, during a birth event, the allocation of masses between parent and offspring is very specific, whereas in classical growth-fragmentation models [DHKR15, CCF16], the mass of the offspring is determined via a general division kernel. We verify that we can adapt our notations and methods of proof to compare our model to the differents cases we investigate.

Example 2 (Age-structure setting): We introduce functional parameters similar to those of the age-structured model in [Tra08]. There is no interaction with resources in this specific case, thus the function g is of the form $\psi - \ell$, and we do not define ϕ. Also, the age-structure imposes naturally that we work with $x_0 = 0$, which does not modify fundamentally the previous model, we simply define our functions on $\mathbb{R}^+$, and it makes sense to see individuals appear with age 0. The author of [Tra08] proposes constraints on functional parameters that we can rewrite as

- $\ell \equiv 0$,
- $\exists \bar{b}, \forall x \ge 0, \quad 0 \le b(x) \le \bar{b}$,

- $\exists \overline{\psi} > 0, \forall x \geq 0, \quad 0 < \psi(x) \leq \overline{\psi}(1+x)$,
- $\exists \overline{d} > 0, \forall x \geq 0, \quad d(x) \leq \overline{d} \quad \text{and} \quad \forall y > 0, \quad \int_y^{+\infty} \frac{d}{\psi}(x)\mathrm{d}x = +\infty.$

Note in particular that the death rate is bounded. For biological motivations, we could aim to introduce an increasing and unbounded death rate, modelling the fact that individuals are more likely to die as their age increases. For example, if we set

- $\ell \equiv 0$,
- $\forall x > 0, \quad b(x) = 1$,
- $\forall x > 0, \quad \psi(x) = 1$,
- $\forall x > 0, \quad d(x) = \sqrt{x}$,

then with the convention $d/\ell = +\infty$, we recover immediately Assumptions II.1.4 and II.1.5. We verify that we can choose $\omega : x \geq 0 \mapsto x^{1/4}$ to verify Assumption II.1.13. Finally, without interaction with resources, Proposition II.1.11 has to be modified. We verify that with the assumptions $\ell \equiv 0$, $\psi \equiv 1$ and $\mathbb{E}(E_0) < +\infty$, we can prove with the same techniques as in Proposition II.1.16 that for all $T \geq 0$, $\mathbb{E}(\sup_{t\in[0,T]} E_t) < +\infty$. This suffices to recover all our results for an age-structured model similar to [Tra08], but with an unbounded death rate. Furthermore, Theorem II.3.1 is stronger than the tightness result of Proposition 3.1. in [Tra08]. Let $T \geq 0$, $(\mu_t^*)_{t\in[0,T]}$ an accumulation point of the sequence highlighted in Theorem II.3.1, and $\varphi \in \mathcal{C}^1(\mathbb{R}_+^*)$. In [Tra08], for $t \in [0,T]$, the author obtained an expression for quantities of the form $\langle \mu_t^*, \varphi \rangle$, if φ and φ' are bounded. By Lemma II.5.3, we obtain an expression of $\langle \mu_t^*, \varphi \rangle$, for any φ such that only $\varphi/(1+\mathrm{Id})$ is bounded. In particular, we directly obtain an expression for $\langle \mu_t^*, \mathrm{Id} \rangle$, which represents the sum of ages of all individuals in the population (analogous to the total biomass of the population in the case of a mass-structured model), without using a monotone convergence argument.

Example 3 (Chemostat setting): For the sake of completeness, we finally mention the mass-structured model including interaction with a resource R in [CF15]. In this paper, individual masses remain in a compact $[0,M]$ over time, thus we define our functions on this segment instead of $\mathbb{R}_+^*$, and we fix $x_0 \in (0,M)$. We give details about this chemostat setting here, because it is the initial model on which we based our construction of Section II.1. The authors of [CF15] propose constraints on functional parameters that we can rewrite as

- $\ell \equiv 0$,
- b non-decreasing and continuous on $[0,M]$ with $b(x) \xrightarrow[x\to x_0+]{} 0$ (see Section 2.1 in [CF15] for the general shape of b),
- concerning the resource consumption, a Gompertz model for the x dynamics and a Monod law for the R dynamics, *i.e.*

$$\exists \eta > 0, \forall x \in [0,M], \forall R \geq 0, \quad f(x,R) = \frac{R}{\eta + R} \log\left(\frac{M}{x}\right) x,$$

- $d \equiv D$, with some constant $D > 0$.

We presented here a precise form for b, ϕ and ψ, used for numerical simulations in [CF15], but we can be a bit more general in the choice of these three functions (see Section 2.1 in [CCF16]). With the convention $d/\ell = +\infty$, we recover again Assumption II.1.4. Concerning Assumption II.1.5, recall that it is meant to ensure $\tau_{\exp} = +\infty$ almost surely, together with Assumption II.1.4. In the chemostat setting, individual energies are compactly supported and the death rate is constant equal to D over time, so that this fact is immediately verified (any individual dies in finite time). Finally, as individual energies remain in $[0, M]$ over time, we only need to verify Assumption II.1.13 on this segment. Suppose that $\omega \in \mathcal{C}^1([0, M], \mathbb{R}_+^*)$, then every point of Assumption II.1.13 is verified on the segment $[0, M]$ by continuity of the considered functions. Hence, we can pick any non-decreasing $\omega \in \mathcal{C}^1([0, M], \mathbb{R}_+^*)$ in that case. In other words, as rates are bounded and individual energies are compactly supported in the chemostat setting, we verify that we can adapt our work with any weight function ω to recover the results of [CF15]. Our aim in this chapter was to go beyond the previous setting, by providing an adapted framework to relax both the assumption of *a priori* compactly supported masses, and the bounds on the growth, birth and death rates, and in particular to study Example 1 with the choice of allometric functional parameters.

Chapter III– Link between stochastic and deterministic models, and a numerical illustration with allometric parameters

In this chapter, we fix $T \geq 0$ and we work under the renormalized setting, so that we can freely use every result of Chapter II, and in particular Corollary II.1.7, Corollary II.1.17 and Theorem II.3.1, and except in Section III.5, we fix the energy at birth x_0 once and for all (see again Section II.1.1 for details on the birth mechanism). The first aim of Chapter III in Section III.1 is to present additional results on the accumulation points $(\mu_t^*, R_t^*)_{t\in[0,T]}$ highlighted in Theorem II.3.1. In the following, up to the extraction of a converging subsequence, we suppose that $\left((\mu_t^K, R_t^K)_{t\in[0,T]} \right)_{K\in\mathbb{N}^*}$ converges in law towards $(\mu_t^*, R_t^*)_{t\in[0,T]}$ in $\mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w}) \times [0, R_{\max}])$. Furthermore, we suppose for the sake of simplicity that the initial condition of the limiting process (μ_0^*, R_0^*) is deterministic. Under this assumption, $(\mu_t^*, R_t^*)_{t\in[0,T]}$ is itself deterministic and characterized by Equations (II.3.24) and (II.3.25). In similar contexts, if in addition the initial condition μ_0^* admits a density u_0 with respect to Lebesgue measure on $\mathbb{R}_+^*$, then μ_t^* also admits a density u_t for every $t \in [0,T]$ (see Proposition 5.4. in [FM04] or Section 3.3. in [Tra08]). We only give a partial justification of the previous result in our setting in Proposition III.1.3, with restrictive assumptions. In the rest of Chapter III, we assume that μ_t^* is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}_+^*$ for every $t \in [0,T]$. This gives rise to function solutions to the weak formulation of the PDE associated to $(\mu_t^*, R_t^*)_{t\in[0,T]}$ (whereas Theorem II.3.1 provides only measure solutions to this PDE). With the notations of Section II.1.1 in Chapter II and under additional regularity conditions, we show in Proposition III.1.5 that the density u_t verifies, for every $t \in (0,T]$ and $x \in \mathbb{R}_+^* \setminus \{x_0\}$,

$$\partial_t u_t(x) + \partial_x \Big(g(x, R_t^*) u_t(x) \Big) = b(x+x_0) u_t(x+x_0) - (b(x) + d(x)) u_t(x),$$

with

$$\frac{\mathrm{d}R_t^*}{\mathrm{d}t} = \varsigma(R_t^*) - \chi \int_{\mathbb{R}_+^*} f(x, R_t^*) u_t(x) \mathrm{d}x$$

and the boundary condition

$$\int_{\mathbb{R}_+^*} b(y) u_t(y) \mathrm{d}y = \Big(u_t(x_0+) - u_t(x_0-) \Big) g(x_0, R_t^*).$$

In Section III.1.2, we search for specific solutions to the previous PDE system, associated to an eigenvalue problem. After this preliminary work, valid under general conditions, we focus in all the rest of this chapter on the allometric example described in Example 1 of Section II.6. In particular, under this allometric setting and the assumption that there exists a non-trivial equilibrium for the previous PDE, we obtain in Corollary III.1.9 that the value of the resource at equilibrium R_{eq} is unique. We draw a link with Chapter I, by remarking that this equilibrium is compatible with only one set of allometric relationships between (I.2.6) and (I.2.7) (see Theorem I.2.1), and the only compatible one is precisely (I.2.6), *i.e.* the one supported by the Metabolic Theory of Ecology. We further present in Section III.1.3, under the allometric setting, an attempt to reduce the problem of solving a PDE system for the limit $(\mu_t^*, R_t^*)_{t\in[0,T]}$, to solving a coupled system of ODE, by studying quantities of the form $(\langle \mu_t^*, x^k \rangle)_{t\in[0,T]}$ for $k \in [0,1]$.

The rest of Chapter III is dedicated to numerical illustrations of the theoretical results of Chapter II, and a discussion about the difficulties encountered to carry out the simulations. They are implemented with `Python`, under the allometric setting where the functional parameters described in Section II.1.1 are power functions (see Example 1), and for the renewal of the resource, we place ourselves in a chemostat setting. We specify our simulation parameters for the rest of this chapter in Section III.2. Then in Section III.3, we discuss the usual way to simulate the stochastic process $(\mu_t^K, R_t^K)_{t\geq 0}$ for $K \geq 1$, which model a population interacting with a varying resource, via a Gillespie algorithm [Gil76]. We will denote this stochastic model as the *individual-based model*, or simply IBM in the following. In Section III.4, our main goal is to illustrate numerically the tightness result of Theorem II.3.1. In fact, our simulations lead to the convergence of the IBM towards a unique limit, which motivated the discussion of Section II.5.1. First, in Section III.4.1, we work under the specific context described in Section III.1 where we study a density u_t solution to a coupled PDE system with the resource R_t^*. We will denote this deterministic model as the *PDE model* in the following. We provide our algorithm for the deterministic simulations of $(u_t, R_t^*)_{t\geq 0}$ given a fixed initial condition (u_0, R_0^*). Then in Section III.4.2, we compare simulations of the IBM to simulations of the PDE model for increasing values of K. The parameter K can be interpreted as a scaling parameter representing the initial size of the population. In the case of a chemostat, we can also interpret K as a scaling parameter of the total volume of the vessel where bacteria interact with nutrients (see Section 5.1 in [CF15]). We observe numerically the convergence of the IBM towards the PDE model when $K \longrightarrow +\infty$. Finally, in Section III.5, we explore possible biological behaviors simulated by our model, by making the parameter x_0 vary.

III.1 Link between the IBM, the PDE model and ODEs

We suppose that the initial condition of the limiting process (μ_0^*, R_0^*) is deterministic, so we study a deterministic process $(\mu_t^*, R_t^*)_{t\in[0,T]}$ determined through Equations (II.3.24) and (II.3.25) (this process exists thanks to Theorem II.3.1), which are the weak formulation of a PDE system. In Section III.1.1, for $T \geq 0$ and under restrictive assumptions, we prove in Proposition III.1.3 that μ_t^* admits a density u_t with respect to Lebesgue measure on $\mathbb{R}_+^*$ for $t \in [0,T]$. This result is only partial in the sense that our additional assumptions are not verified under the allometric setting for example. Thus, in the rest of this section, we will assume that μ_t^* is absolutely continuous with respect to Lebesgue measure for $t \in [0,T]$. Under additional regularity assumptions, we prove that this entails the existence of function solutions to the PDE in Proposition III.1.5. Then in Section III.1.2, we formulate a similar PDE system with fixed resources $R \geq 0$, and we search for specific *decorrelated* solutions to this new PDE system of the form $u_t(x) : (t,x) \in [0,T] \times \mathbb{R}_+^* \mapsto u_R(x)e^{\Lambda_R t}$ with some $\Lambda_R \in \mathbb{R}$ and u_R a positive function. We are particularly interested in non-trivial equilibria for the PDE system (*i.e.* values R_{eq} for which there exists a decorrelated solution u_{eq} associated to $\Lambda_{R_{\text{eq}}} = 0$ to the PDE system with fixed resources R_{eq}), but note that we shall not prove the existence of decorrelated solutions nor equilibria associated to the PDE system. Nevertheless, this gives rise to an eigenvalue problem, and we give necessary conditions verified by (u_R, Λ_R, R) for the existence of such solutions. Our main result is Corollary III.1.9, which identify the unique possible value of the resource R_{eq} at a non-trivial equilibrium in a precise allometric setting, if we assume the existence of such a non-trivial equilibrium. Then, we deduce that the only choice of allometric relationships compatible with this non-trivial equilibrium between (I.2.6) and (I.2.7) (see Theorem I.2.1) is the first one, supported by the Metabolic Theory of Ecology. Finally, we further present in Section III.1.3 an attempt to reduce the problem of solving the PDE system to solving a coupled system of ODEs, for the study of 'allometric' quantities of the form $\langle \mu_t^*, x^k \rangle$ with $k \in [0,1]$ for $t \in [0,T]$.

III.1.1 PDE associated to the accumulation points of Theorem 1

We will investigate in the following the case where there exists function solutions to the weakly formulated PDE system (II.3.24)-(II.3.25). We want to study the context where for all $t \in [0,T]$, μ_t^* admits a density u_t with respect to the Lebesgue measure on $\mathbb{R}_+^*$. First, we give a partial result that proves the existence of such a density under the restrictive Assumption III.1.2, following a classical procedure of [FM04]. We begin with some definitions, and remark that we can define (μ_t^*, R_t^*) for any $t \geq 0$, simply by saying that it coincides on any $[0,T]$ with $T \geq 0$ with the unique solution to (II.3.24)-(II.3.25) (recall that we work under the renormalized setting of Chapter II). As in Section II.5.1, for $t_0 \geq 0$ and $z > 0$, we consider the following equation with unknown function $G_{t_0,z}$ and terminal condition at t_0, associated to the resource R_t^*:

$$\begin{cases} G'_{t_0,z}(t) = g(G_{t_0,z}(t), R_t^*), \\ G_{t_0,z}(t_0) = z. \end{cases} \tag{III.1.1}$$

With the same arguments as in the proof of Proposition II.1.3, for every $z > 0$ and $t_0 \geq 0$, there exists a neighborhood $V_{t_0,z} \subseteq \mathbb{R}^+$ of t_0, such that there exists a unique local positive solution $G_{t_0,z}(.)$ to the previous equation, defined on $V_{t_0,z}$, and we can assume that this neighborhood is maximal. It is then possible to define $\mathfrak{V} := \{(t_0,z,t) \in \mathbb{R}^+ \times \mathbb{R}_+^* \times \mathbb{R}^+,\ t \in V_{t_0,z}\}$, which is an open subset of $\mathbb{R}^+ \times \mathbb{R}_+^* \times \mathbb{R}^+$, on which the flow $(t_0,z,t) \mapsto G_{t_0,z}(t)$ is well-defined and $\mathcal{C}^{1,1,2}(\mathfrak{V})$ by the same arguments as in the proof of Proposition II.1.3. For $(t_0,z,t) \in \mathfrak{V}$, we can write

$$G_{t_0,z}(t) = z + \int_{t_0}^{t} g(G_{t_0,z}(s), R_s^*) \mathrm{d}s. \tag{III.1.2}$$

Contrary to classical results for similar models where the regularity of the flow is valid on the whole space $\mathbb{R}^+ \times \mathbb{R}_+^* \times \mathbb{R}^+$ (see Proposition 2.1 in [Tra08] or Proposition 2.1 in [CF15]), we only have a result on $\mathfrak{V}$, because of the possibly unbounded growth rate (individual energy following this deterministic flow may explode in finite time). For $(t_0,t) \in \mathbb{R}^+ \times \mathbb{R}^+$, we define the open set $\mathfrak{O}_{t_0,t} := \{z > 0,\ (t_0,z,t) \in \mathfrak{V}\}$; and for $(z,t) \in \mathbb{R}_+^* \times \mathbb{R}^+$, we also define the open set $\mathfrak{U}_{z,t} := \{t_0 \geq 0,\ (t_0,z,t) \in \mathfrak{V}\}$.

Lemma III.1.1. *For every $(t_0, t) \in \mathbb{R}^+ \times \mathbb{R}^+$, $z \mapsto G_{t_0,z}(t)$ is a $\mathcal{C}^1$-diffeomorphism from $\mathfrak{D}_{t_0,t}$ to its image. This image is denoted as $\mathfrak{D}^{-1}_{t_0,t}$ in the following.*

Proof. Thanks to the regularity of our functional parameters, g and R^* are $\mathcal{C}^1$ functions, so locally Lipschitz continuous, hence classical arguments entails this regularity with respect to the terminal condition z (see Théorème II.10. in Chapter X of [ZQ96]). □

However, an important new ingredient of our model compared to existing literature is that for $(t_0, z) \in \mathbb{R}^+ \times \mathbb{R}^*_+$, the sign of $g(G_{t_0,z}(t), R^*_t)$ in (III.1.1) may change over time, so that $G_{t_0,z}(.)$ would not be monotonous on $V_{t_0,z}$, hence would not be a $\mathcal{C}^1$-diffeomorphism from $V_{t_0,z}$ to its image. This is why we introduce the following additional assumption.

Assumption III.1.2.

(i) *For every $(t_0, z) \in \mathbb{R}^+ \times \mathbb{R}^*_+$, $t \mapsto G_{t_0,z}(t)$ is a $\mathcal{C}^1$-diffeomorphism from $V_{t_0,z}$ to its image.*

(ii) *For every $(z, t) \in \mathbb{R}^*_+ \times \mathbb{R}^+$, $t_0 \mapsto G_{t_0,z}(t)$ is a $\mathcal{C}^1$-diffeomorphism from $\mathfrak{U}_{z,t}$ to its image. This image is denoted as $\mathfrak{U}^{-1}_{z,t}$ in the following.*

Remark: We acknowledge that Assumption III.1.2 is restrictive and may not be verified in a general context where $(R^*_t)_{t\in[0,T]}$ is not monotonous over time (which will be the case under the allometric setting, at least numerically, of the third row of Figure III.4). Essentially, by considering (III.1.2), Assumption III.1.2 is the same as saying that for all $t \geq 0$, $g(., R^*_t) > 0$ (or for all $t \geq 0$, $g(., R^*_t) < 0$). This assumption is for instance verified if the resource R^* is constant equal to $R_{\max}$ over time thanks to Assumption II.1.1, or if we choose g to be always positive (as in [CF15]), but absolutely not guaranteed if the resource vary over time or if g is not of constant sign uniformly on x and R. Still, even if we did not succeed in proving it without Assumption III.1.2, we aim to provide in Proposition III.1.3 a context where for all $t \in [0, T]$, μ^*_t is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^*_+$. The proof of this result in a general context without Assumption III.1.2 is left open to conjecture, and we give a possible line of research after the proof of Proposition III.1.3.

Proposition III.1.3. *We work under the assumptions of Theorem II.3.1 with a deterministic initial condition (μ^*_0, R^*_0). We assume that there exists a density u_0 with $\mu^*_0(\mathrm{d}x) = u_0(x)\mathrm{d}x$, and that Assumption III.1.2 holds true. Then, for all $t \in [0, T]$, there exists an integrable density $u_t : \mathbb{R}^*_+ \mapsto \mathbb{R}^+$ such that*

$$\mu^*_t(\mathrm{d}x) = u_t(x)\mathrm{d}x.$$

Proof. We use Radon-Nikodym theorem, *i.e.* it suffices to show that for every negligible Borel set $B \subseteq \mathbb{R}^*_+$ (*i.e.* such that the Lebesgue measure of B is 0) and $t \geq 0$, then $\langle \mu^*_t, \mathbb{1}_B \rangle = 0$. First, we consider a positive function $\varphi \in \mathcal{C}^\infty_c(\mathbb{R}^*_+)$. As in Section II.5.1, we can associate to φ the function

$$y^t_s(z) : (s, z, t) \in \mathfrak{V} \mapsto \varphi(G_{s,z}(t)),$$

and then extend this function by $y^t_s(z) = 0$ if $(s, z, t) \notin \mathfrak{V}$. By the same argument as in Section II.5.1 (using the fact that for every (s, z), $V_{s,z}$ is the maximal neighborhood on which the solution $t \mapsto G_{s,z}(t)$ is well-defined, so this solution escapes any compact near the boundary of $\mathfrak{V}$ and especially the compact support $\mathrm{supp}(\varphi)$ of the function φ), this defines a global $\mathcal{C}^{1,1,2}$ function $y : (s,z,t) \mapsto y_s^t(z)$ on $\mathbb{R}^+ \times \mathbb{R}_+^* \times \mathbb{R}^+$ verifying the transport equation in Proposition II.5.6, replacing R_s with R_s^*. By convention, even if $G_{s,z}(t)$ is not well-defined if $(s,z,t) \notin \mathfrak{V}$, we will write for every $(s,z,t) \in \mathbb{R}^+ \times \mathbb{R}_+^* \times \mathbb{R}^+$ that

$$y_s^t(z) = \varphi(G_{s,z}(t))\mathbb{1}_{\{x\in\mathfrak{D}_{s,t}\}} = \varphi(G_{s,z}(t))\mathbb{1}_{\{s\in\mathfrak{U}_{z,t}\}}.$$

We fix $t \geq 0$ in the following. Then, from Equation (II.3.25) applied to y^t, the fact that it is a positive function (so we can neglect negative terms in the following upper bound) and that it verifies Proposition II.5.6, we obtain

$$\begin{aligned}\langle \mu_t^*, \varphi\rangle = \langle \mu_t^*, y_t^t\rangle \leq \langle \mu_0^*, y_0^t\rangle + \int_0^t\int_{\mathbb{R}_+^*} b(x)y_s^t(x_0)\mu_s^*(\mathrm{d}x)\mathrm{d}s + \int_0^t\int_{\mathbb{R}_+^*} b(x)y_s^t(x-x_0)\mu_s^*(\mathrm{d}x)\mathrm{d}s \\ = \int_{\mathbb{R}_+^*}\varphi\left(G_{0,x}(t)\right)\mathbb{1}_{\{x\in\mathfrak{D}_{0,t}\}}u_0(x)\mathrm{d}x + \int_0^t \varphi(G_{s,x_0}(t))\mathbb{1}_{\{s\in\mathfrak{U}_{x_0,t}\}}\int_{\mathbb{R}_+^*} b(x)\mu_s^*(\mathrm{d}x)\mathrm{d}s \\ + \int_0^t\int_{x_0}^{+\infty} b(x)\varphi(G_{s,x-x_0}(t))\mathbb{1}_{\{s\in\mathfrak{U}_{x-x_0,t}\}}\mu_s^*(\mathrm{d}x)\mathrm{d}s. \quad \text{(III.1.3)}\end{aligned}$$

By Lemma III.1.1, we can define $u \in \mathfrak{D}_{0,t}^{-1} \mapsto G_{0,u}^{-1}(t)$ the inverse function of $x \in \mathfrak{D}_{0,t} \mapsto G_{0,x}(t)$, which is also a $\mathcal{C}^1$-diffeomorphism from $\mathfrak{D}_{0,t}^{-1}$ to $\mathfrak{D}_{0,t}$. We then perform the change of variables $u = G_{0,x}(t)$ in the left-most integral of the right-hand side in (III.1.3) to obtain

$$\int_{\mathbb{R}_+^*}\varphi\left(G_{0,x}(t)\right)\mathbb{1}_{\{x\in\mathfrak{D}_{0,t}\}}u_0(x)\mathrm{d}x = \int_{\mathbb{R}_+^*}\varphi\left(u\right)\left[u_0(G_{t,u}^{-1}(0))\partial_u G_{t,u}^{-1}(0)\mathbb{1}_{\left\{u\in\mathfrak{D}_{0,t}^{-1}\right\}}\right]\mathrm{d}u.$$

Similarly, thanks to Assumption III.1.2, we can define $u \in \mathfrak{U}_{x_0,t}^{-1} \mapsto G_{u,x_0}^{-1}(t)$ the inverse function of $s \in \mathfrak{U}_{x_0,t} \mapsto G_{s,x_0}(t)$, which is a $\mathcal{C}^1$-diffeomorphism from $\mathfrak{U}_{x_0,t}^{-1}$ to $\mathfrak{U}_{x_0,t}$. From the change of variables $u = G_{s,x_0}(t)$ in the second integral of the right-hand side of (III.1.3), we obtain

$$\int_0^t \varphi(G_{s,x_0}(t))\mathbb{1}_{\{s\in\mathfrak{U}_{x_0,t}\}}\int_{\mathbb{R}_+^*} b(x)\mu_s^*(\mathrm{d}x)\mathrm{d}s = \int_{\mathbb{R}_+^*}\varphi(u)\left[\mathbb{1}_{\{u\in\mathfrak{U}_{x_0,t}^{-1}\}}\partial_u G_{u,x_0}^{-1}(t)\int_{\mathbb{R}_+^*} b(x)\mu_{G_{u,x_0}^{-1}(t)}^*(\mathrm{d}x)\right]\mathrm{d}u.$$

More generally, by Assumption III.1.2, we can define $u \in \mathfrak{U}_{x-x_0,t}^{-1} \mapsto G_{u,x-x_0}^{-1}(t)$ the inverse function of $s \in \mathfrak{U}_{x-x_0,t} \mapsto G_{s,x-x_0}(t)$ for every $x > x_0$, and this is a $\mathcal{C}^1$-diffeomorphism from $\mathfrak{U}_{x-x_0,t}^{-1}$ to $\mathfrak{U}_{x-x_0,t}$. We perform the change of variables $(u,y) = (G_{s,x-x_0}(s), t)$ in the right-most integral of the right-hand side of (III.1.3) to obtain

$$\begin{aligned}\int_0^t\int_{x_0}^{+\infty} b(x)\varphi(G_{s,x-x_0}(t))\mathbb{1}_{\{s\in\mathfrak{U}_{x-x_0,t}\}}\mu_s^*(\mathrm{d}x)\mathrm{d}s \\ = \int_{\mathbb{R}_+^*}\varphi(u)\int_{x_0}^{+\infty}\left[\mathbb{1}_{\{u\in\mathfrak{U}_{y-x_0,t}^{-1}\}}\partial_u G_{u,y-x_0}^{-1}(t)b(y)\mu_{G_{u,y-x_0}^{-1}(t)}^*(\mathrm{d}y)\right]\mathrm{d}u.\end{aligned}$$

Hence, by (III.1.3), we finally have $\langle \mu_t^*, \varphi\rangle \leq \int_{\mathbb{R}_+^*}\varphi(u)\Psi(t,u)\mathrm{d}u$, with

$$\begin{aligned}\Psi(t,u) := u_0(G_{t,u}^{-1}(0))\partial_u G_{t,u}^{-1}(0)\mathbb{1}_{\left\{u\in\mathfrak{D}_{0,t}^{-1}\right\}} + \mathbb{1}_{\{u\in\mathfrak{U}_{x_0,t}^{-1}\}}\partial_u G_{u,x_0}^{-1}(t)\int_{\mathbb{R}_+^*} b(x)\mu_{G_{u,x_0}^{-1}(t)}^*(\mathrm{d}x) \\ + \int_{x_0}^{+\infty}\mathbb{1}_{\{u\in\mathfrak{U}_{y-x_0,t}^{-1}\}}\partial_u G_{u,y-x_0}^{-1}(t)b(y)\mu_{G_{u,y-x_0}^{-1}(t)}^*(\mathrm{d}y).\end{aligned}$$

Let us show that $u \mapsto \Psi(t,u)$ is integrable for every $t \geq 0$. Let $(\tilde{\varphi}_n)_{n\in\mathbb{N}}$ be an increasing sequence of $\mathcal{C}_c^\infty(\mathbb{R}_+^*)$ positive functions that converges pointwise towards the constant function equal to 1 on $\mathbb{R}_+^*$. For any $n \in \mathbb{N}$, we associate as before a function $y_{t,n}$ to $\tilde{\varphi}_n$, and recall that as we just made several changes of variables to obtain the expression of Ψ, with (III.1.3) we have for all $t \in [0,T]$,

$$\int_{\mathbb{R}_+^*} \tilde{\varphi}_n(u)\Psi(t,u)\mathrm{d}u = \langle \mu_0^*, y_0^{t,n}\rangle + \int_0^t \int_{\mathbb{R}_+^*} b(x) y_s^{t,n}(x_0)\mu_s^*(\mathrm{d}x)\mathrm{d}s + \int_0^t \int_{\mathbb{R}_+^*} b(x) y_s^{t,n}(x-x_0)\mu_s^*(\mathrm{d}x)\mathrm{d}s.$$

By monotone convergence, from Assumption II.1.13, the fact that every $\tilde{\varphi}_n$ is lower than or equal to 1, and the previous equality, there exists a constant $C > 0$ such that for every $t \in [0,T]$,

$$\int_{\mathbb{R}_+^*} |\Psi(t,u)|\,\mathrm{d}u = \lim_{n\to+\infty} \int_{\mathbb{R}_+^*} \tilde{\varphi}_n(u)\,|\Psi(t,u)|\,\mathrm{d}u \leq C\Big(\langle \mu_0^*, 1\rangle + TC_b \sup_{t\in[0,T]} \langle \mu_t^*, 1+\mathrm{Id}+\omega\rangle\Big),$$

which is finite by Lemma II.5.3, so that $u \mapsto \Psi(t,u)$ is integrable for every $t \geq 0$. Let us get back to our initial aim of proving that $\langle \mu_t^*, \mathbb{1}_B\rangle = 0$. We consider a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of $\mathcal{C}_c^\infty(\mathbb{R}_+^*)$ functions that converges pointwise towards $\mathbb{1}_B$. Without loss of generality, we can assume that the functions φ_n are positive and uniformly bounded on $\mathbb{R}_+^*$, and that the sequence $(\varphi_n)_{n\in\mathbb{N}}$ is non-increasing with φ_0 integrable. We can then apply the previous work and obtain

$$\langle \mu_t^*, \mathbb{1}_B\rangle = \lim_{n\to+\infty} \langle \mu_t^*, \varphi_n\rangle \leq \lim_{n\to+\infty} \int_{\mathbb{R}_+^*} \varphi_n(y)\Psi(t,y)\mathrm{d}y = \int_{\mathbb{R}_+^*} \mathbb{1}_B(y)\Psi(t,y)\mathrm{d}y = 0.$$

We use monotone convergence for the first equality, then (III.1.3), followed by dominated convergence because Ψ is integrable, and finally the fact that B is a negligible event for the last equality. This concludes the proof. □

Remark: It might be possible to adapt the previous proof if we can split $[0,T]$ into a finite number of intervals $(I_n)_{1\leq n\leq M}$ with $M \geq 1$, such that for every $1 \leq n \leq M$, $s \in I_n \mapsto G_{s,z}(t)$ is a $\mathcal{C}^1$-diffeomorphism on its image (equivalently, if we can split $[0,T]$ into a finite number of intervals such that on each of these intervals, $g(., R_t^*)$ is either > 0 or < 0). This may allow us to work without Assumption III.1.2, but the technical point would be to prove that there is indeed a finite number of such intervals, and we leave this for future work. Also, the previous line of research will not work if $g(x_0, R_t^*) = 0$ on an interval I, because in that case, μ_t will have a Dirac mass at x_0 for $t \in I$.

In the rest of this section, we thus make the following assumption.

Assumption III.1.4. *For all $t \in [0,T]$, there exists an integrable density $u_t : \mathbb{R}_+^* \mapsto \mathbb{R}^+$ such that*

$$\mu_t^*(\mathrm{d}x) = u_t(x)\mathrm{d}x.$$

Intuitively, the specific birth dynamics of Section II.1.2, where offspring always have the same energy x_0, may create an accumulation of the mass of the measure μ_t^* at x_0 for $t > 0$, and so u_t will be discontinuous at x_0. This is different from the classical case depicted in the literature where the density function is usually assumed to be continuous at every point $x > 0$ (see Proposition 3.6. in [Tra08]). We precise our intuition in the following proposition.

Proposition III.1.5. *Let $T \geq 0$. Under the assumptions of Theorem II.3.1, under Assumption III.1.4, assume that:*

- *for all $t \in [0,T]$, the function $x \in \mathbb{R}_+^* \setminus \{x_0\} \mapsto u_t(x)$ is $\mathcal{C}^1$,*
- *for all $t \in [0,T]$, the limits $u_t(x_0+)$ and $u_t(x_0-)$ exist and are finite,*
- *for all $x \in \mathbb{R}_+^*$, the function $t \in [0,T] \mapsto u_t(x)$ is $\mathcal{C}^1$,*
- *there exists a locally integrable function F on $\mathbb{R}_+^*$, such that for all $t \in [0,T]$, for all $x > 0$, $|\partial_t u_t(x)| \leq F(x)$.*

Then, the density $(t,x) \in [0,T] \times \mathbb{R}_+^* \mapsto u_t(x)$ and the quantity of resources $(R_t^*)_{t\in[0,T]}$ verify the following PDE system. For every $t \in (0,T]$ and $x \in \mathbb{R}_+^* \setminus \{x_0\}$,

$$\partial_t u_t(x) + \partial_x\Big(g(x,R_t^*)u_t(x)\Big) = b(x+x_0)u_t(x+x_0) - (b(x)+d(x))u_t(x) \tag{III.1.4}$$

and

$$\frac{\mathrm{d}R_t^*}{\mathrm{d}t} = \varsigma(R_t^*) - \chi\int_{\mathbb{R}_+^*} f(x,R_t^*)u_t(x)\mathrm{d}x, \tag{III.1.5}$$

so that in particular $t \mapsto R_t^*$ is $\mathcal{C}^1$ on $[0,T]$. We also have the boundary condition

$$\int_{\mathbb{R}_+^*} b(y)u_t(y)\mathrm{d}y = \Big(u_t(x_0+) - u_t(x_0-)\Big)g(x_0,R_t^*), \tag{III.1.6}$$

and the initial condition u_0 at time $t=0$.

Proof. Considering Equation (II.3.24), which provides an expression of R_t^* with an integrated form for $t \in [0,T]$, and thanks to our regularity assumptions on functional parameters ς, f and u_t, we obtain that R^* is $\mathcal{C}^1$ on $[0,T]$, and Equation (III.1.5) follows by definition of ρ. To show (III.1.4) and (III.1.6), we first remark that from the uniform domination assumption on $\partial_t u_t$ by a locally integrable function F on $\mathbb{R}_+^*$, for every test function $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+^*)$ and $t \in [0,T]$, we have

$$\partial_t\left(\int_{\mathbb{R}_+^*} \varphi(x)u_t(x)\mathrm{d}x\right) = \int_{\mathbb{R}_+^*} \varphi(x)\partial_t u_t(x)\mathrm{d}x. \tag{III.1.7}$$

Let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+^*)$, then from Equation (II.3.25) and by assumption on the functional parameters b, d, g and u_t, $t \in [0,T] \mapsto \langle \mu_t^*, \varphi\rangle$ is differentiable, and for every $t \in [0,T]$, we first have

$$\begin{aligned}\partial_t\left(\int_{\mathbb{R}_+^*} \varphi(x)u_t(x)\mathrm{d}x\right) = \partial_t\left(\langle \mu_t^*, \varphi\rangle\right) = \int_{\mathbb{R}_+^*}\Big[g(x,R_t^*)\varphi'(x) + b(x)\Big(\varphi(x_0)+\varphi(x-x_0)-\varphi(x)\Big)&\\ - d(x)\varphi(x)\Big]u_t(x)\mathrm{d}x&\\ = \int_{\mathbb{R}_+^*} g(x,R_t^*)\varphi'(x)u_t(x)\mathrm{d}x + \int_{\mathbb{R}_+^*}\Big[b(x)\varphi(x_0)u_t(x)&\\ + b(x+x_0)u_t(x+x_0)\varphi(x) - (b(x)+d(x))\varphi(x)u_t(x)\Big]\mathrm{d}x,&\end{aligned} \tag{III.1.8}$$

where we used a change of variables $x - x_0 \longleftarrow x$ and the fact that $b \equiv 0$ on $(0, x_0]$ for the term $b(x+x_0)u_t(x+x_0)\varphi(x)$. If we consider a function φ such that $\varphi(x_0) = 0$, then we integrate by parts and use the fact that φ has compact support to obtain

$$\begin{aligned}\int_{\mathbb{R}_+^*} g(x, R_t^*)\varphi'(x)u_t(x)\mathrm{d}x &= \int_0^{x_0} g(x, R_t^*)\varphi'(x)u_t(x)\mathrm{d}x + \int_{x_0}^{+\infty} g(x, R_t^*)\varphi'(x)u_t(x)\mathrm{d}x \\ &= -\int_{\mathbb{R}_+^*} \varphi(x)\partial_x\Big(g(x, R_t^*)u_t(x)\Big)\mathrm{d}x,\end{aligned}$$

where we first splitted the integrals to obtain vanishing terms in the integration by parts, and then recombine them together to obtain a global integral on $\mathbb{R}_+^*$. Hence, we finally obtain, for all test function φ such that $\varphi(x_0) = 0$,

$$\int_{\mathbb{R}_+^*} \varphi(x)\Big[\partial_t u_t(x) + \partial_x\Big(g(x, R_t^*)u_t(x)\Big) + (b(x)+d(x))u_t(x) - b(x+x_0)u_t(x+x_0)\Big]\mathrm{d}x = 0,$$

which entails (III.1.4) for all $x \in \mathbb{R}_+^* \setminus \{x_0\}$, since the integrand above is continuous for every $x \neq x_0$. Finally, if we consider now $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+^*)$ such that $\varphi(x_0) \neq 0$, we obtain in the same manner

$$\begin{aligned}\int_{\mathbb{R}_+^*} g(x, R_t^*)\varphi'(x)u_t(x)\mathrm{d}x &= \int_0^{x_0} g(x, R_t^*)\varphi'(x)u_t(x)\mathrm{d}x + \int_{x_0}^{+\infty} g(x, R_t^*)\varphi'(x)u_t(x)\mathrm{d}x \\ &= \varphi(x_0)g(x_0, R_t^*)\Big(u_t(x_0-) - u_t(x_0+)\Big) \\ &\qquad - \int_{\mathbb{R}_+^*} \varphi(x)\partial_x\Big(g(x, R_t^*)u_t(x)\Big)\mathrm{d}x,\end{aligned}$$

so from (III.1.4), (III.1.7) and (III.1.8), we obtain (III.1.6), which ends the proof. □

Remark: In Equation (III.1.4), we can relate the terms to the individuals dynamics depicted in Section II.1.1. For $x \neq x_0$, the infinitesimal variation of $u_t(x)$ over a small step time is due to three factors: the energy gain and loss dynamics lying in the transport term $\partial_x\Big(g(x, R_t^*)u_t(x)\Big)$; the possible creation of offspring at x from individuals with energy $x+x_0$, expressed with the term $b(x+x_0)u_t(x+x_0)$; and the possible birth or death events occuring from individuals with energy x, expressed with the term $-(b(x)+d(x))u_t(x)$. Finally, the boundary condition (III.1.6) expresses the fact that every birth event creates offspring with energy x_0, instantaneously driven out from this point by the continuous evolution of energies with speed $g(x_0, R_t^*)$.

The resolution of the system (III.1.4), (III.1.5), (III.1.6) with initial condition u_0 can be implemented with a finite differences scheme presented in Section III.4.1. In this section, we will discuss numerous problems arising with the numerical approximation of the PDE, among which dealing with the boundary condition (III.1.6), the fact that the growth rate g can be either non-negative or negative depending on the amount of resources over time, and the fact that our energy discretization has to approximate well the term $b(x+x_0)u_t(x+x_0)$.

III.1.2 Eigenvalue problem and equilibrium for the PDE model with fixed resources

In this section, we fix an initial condition u_0, and we introduce a new PDE system similar to (III.1.4), (III.1.5), (III.1.6), but with fixed resources R over time. Our goal is to draw a link between the results of Chapter I and Chapter II.

Definition III.1.6 (PDE with fixed resources). *In the following, for every $R \geq 0$, the function $(t, x) \in \mathbb{R}^+ \times \mathbb{R}_+^* \mapsto u_t(x)$ is said to be a solution to the PDE with fixed resources R if it verifies the assumptions of Proposition III.1.5 and*

$$\forall t > 0, \forall x > 0, \quad \partial_t u_t(x) + \partial_x \Big(g(x, R) u_t(x) \Big) = b(x + x_0) u_t(x + x_0) - (b(x) + d(x)) u_t(x),$$

with the boundary condition

$$\forall t > 0, \quad \int_{\mathbb{R}_+^*} b(y) u_t(y) \mathrm{d}y = \Big(u_t(x_0+) - u_t(x_0-) \Big) g(x_0, R).$$

It is possible to adapt the techniques used in Chapter II to obtain a similar tightness result as Theorem II.3.1, but with the model of Chapter I with fixed resources. Even if we shall not give a proof of this fact here, because it would need to repeat all the steps of the proof of Theorem II.3.1, the PDE with fixed resources R naturally arises in the large-population limit for the model of Chapter I, as it was the case for the PDE system (III.1.4), (III.1.5), (III.1.6) for the model of Chapter II. We fix $R \geq 0$ in the following, and search for specific solutions to the PDE with fixed resources R and initial condition u_0.

Definition III.1.7 (Decorrelated solutions). *A solution to the PDE with fixed resources R is called a decorrelated solution, if there exists $\Lambda_R \in \mathbb{R}$ and a positive function $u_R \in \mathcal{C}^1(\mathbb{R}_+^* \setminus \{x_0\})$ such that $u_R(x_0+)$ and $u_R(x_0-)$ exist and are finite, with*

$$\forall t > 0, \forall x > 0, \quad u_t(x) := u_R(x) e^{\Lambda_R t}.$$

We write (u_R, Λ_R) for such a solution. It is called a non-trivial equilibrium if $\Lambda_R = 0$, and we denote it as u_{eq} in that case.

The search for such solutions is motivated for example by the work of Campillo, Champagnat and Fritsch (see Section 4.1 in [CCF16]) and more generally gives rise to an eigenvalue problem associated to the PDE with fixed resources.

Conjecture III.1.1. *Under additional assumptions on the functional parameters of the model, for every R high enough (to guarantee that the system does not go extinct), there exists decorrelated solutions to the PDE with fixed resources R. This is equivalent to say that there exists solutions (u_R, Λ_R, R) to the eigenvalue problem*

$$\begin{cases} \partial_x \left(g(., R) u_R \right) + (\Lambda_R + b + d) u_R = b(. + x_0) u_R(. + x_0), \\ \displaystyle\int_{\mathbb{R}_+^*} b(y) u_R(y) \mathrm{d}y = \Big(u_R(x_0+) - u_R(x_0-) \Big) g(x_0, R), \end{cases}$$

with $u_R > 0$ and $\displaystyle\int_{\mathbb{R}_+^*} u_R(x) \mathrm{d}x = 1$.

A proof of Conjecture III.1.1 could be based on the study of the expectation of the stochastic process $(\mu_t)_{t\geq 0}$ defined in Chapter I with fixed resources R, conditioned to non-extinction, and the general criteria for the existence of quasi-stationary distributions developed in [CV23]. It might also be possible to adapt arguments from [DJG10]. We leave this for future work.

Conjecture III.1.2. *Under additional assumptions on the functional parameters of the model, there exists R_{eq} such that the PDE with fixed resources R_{eq} admits a non-trivial equilibrium u_{eq} (i.e. there exists a solution to the eigenvalue problem of Conjecture III.1.1 with $\Lambda_{R_{\text{eq}}} = 0$).*

In Section III.4.2, we will verify Conjecture III.1.2 numerically under the allometric setting described in Example 1 of Section II.6 with $\gamma = \alpha = \delta + 1$. Even if proving the existence of decorrelated solutions or non-trivial equilibria is left for future work, we can still give necessary conditions for a solution (u_R, Λ_R, R) to the previous eigenvalue problem to exist.

Lemma III.1.8. *We assume that there exists (u_R, Λ_R) a decorrelated solution to the PDE with fixed resources $R \geq 0$. We also assume that there exists an integrable function H on $\mathbb{R}_+^*$, such that*

$$\forall x > 0, \quad |(1 + x + \omega(x))u_R(x)| \leq H(x). \tag{III.1.9}$$

Then, for every $f \in \mathcal{C}^1(\mathbb{R}_+^*) \cap \mathfrak{B}_{1+\text{Id}+\omega}(\mathbb{R}_+^*)$,

$$\begin{aligned}\Lambda_R \int_0^{+\infty} f(x)u_R(x)\mathrm{d}x = &\int_0^{+\infty} f'(x)g(x,R)u_R(x)\mathrm{d}x \\ &+ \int_0^{+\infty} \Big(f(x_0) + f(x - x_0) - f(x)\Big)b(x)u_R(x)\mathrm{d}x \\ &- \int_0^{+\infty} f(x)d(x)u_R(x)\mathrm{d}x.\end{aligned}$$

Proof. Let us fix $f \in \mathcal{C}^1(\mathbb{R}_+^*) \cap \mathfrak{B}_{1+\text{Id}+\omega}(\mathbb{R}_+^*)$ and write $u : (t,x) \in [0,T] \times \mathbb{R}_+^* \mapsto u_R(x)e^{\Lambda_R t}$. First, we check that u verifies the assumptions of Proposition III.1.5, and thanks to the domination (III.1.9), all the following integrals are well-defined and finite. The solution u is decorrelated, so we immediately obtain

$$\int_0^{+\infty} f(x)\partial_t u_t(x)\mathrm{d}x = \Lambda_R \int_0^{+\infty} f(x)u_t(x)\mathrm{d}x.$$

Then, we use (III.1.4) to obtain

$$\begin{aligned}\int_0^{+\infty} f(x)\partial_t u_t(x)\mathrm{d}x = -\int_0^{+\infty} f(x)\partial_x\Big(g(x,R_t)u_t(x)\Big)\mathrm{d}x - \int_0^{+\infty} f(x)(b(x)+d(x))u_t(x)\mathrm{d}x \\ + \int_0^{+\infty} f(x)b(x+x_0)u_t(x+x_0)\mathrm{d}x,\end{aligned}$$

Performing an integration by parts for the first integral in the right-hand side, then use the boundary condition (III.1.6) for the additional term that appears, and finally perform a change of variables $y \leftarrow x + x_0$ in the last integral (recall that $b \equiv 0$ on $(0, x_0)$) leads to the wanted result (we also simplify everywhere by $e^{\Lambda_R t}$). □

Remark: If Assumption III.1.4 is verified for $u : (t, x) \in [0, T] \times \mathbb{R}_+^* \mapsto u_R(x)e^{\Lambda_R t}$ associated to the decorrelated solution (u_R, Λ_R) highlighted in Lemma III.1.8, then the integrability condition (III.1.9) follows from similar arguments as in the proof of Lemma II.5.3, but adapted to the model of Chapter I with constant resources.

Lemma III.1.8 provides an infinite number of necessary conditions that has to be verified by (u_R, Λ_R) for a decorrelated solution to the PDE with fixed resources R to exist. For $t \geq 0$, $k \in [0, 1]$ and such a decorrelated solution, we define

$$M_k := \int_0^{+\infty} x^k u_R(x)\mathrm{d}x.$$

Under the assumptions of Lemma III.1.8, these quantities are finite and positive as $u_R > 0$. For example, if we take $f \equiv 1$, we obtain

$$\Lambda_R M_0 = \int_0^{+\infty} \Big[b(x) - d(x)\Big] u_R(x)\mathrm{d}x.$$

If we take $f \equiv \mathrm{Id}$, we obtain

$$\Lambda_R M_1 = \int_0^{+\infty} \Big[g(x, R) - xd(x)\Big] u_R(x)\mathrm{d}x \tag{III.1.10}$$

Remark that we can also isolate the term Λ_R, and this provides relationships between integral quantities depending on the density u_R, and especially the M_k for $k \in [0, 1]$, which is particularly adapted under the allometric setting (because power functions naturally arise in this setting). For example, with the notations of Example 1 and if γ, α and $\delta + 1$ are in $[0, 1]$, (III.1.10) translates into

$$\Lambda_R M_1 = \phi(R)C_\gamma M_\gamma - C_\alpha M_\alpha - C_\delta M_{\delta+1}. \tag{III.1.11}$$

Corollary III.1.9. *Under the allometric setting described in Example 1 of Section II.6 with $\alpha \in [0, 1]$ and $\gamma = \alpha = \delta + 1$, assume that there exists u_{eq} a non-trivial equilibrium for the PDE with fixed resources R_{eq}, such that there exists an integrable function H on $\mathbb{R}_+^*$ with (III.1.9). Then, we necessarily have*

$$\phi(R_{\mathrm{eq}}) = \frac{C_\delta + C_\alpha}{C_\gamma}. \tag{III.1.12}$$

In particular, if $\phi : R \geq 0 \mapsto \dfrac{R}{\kappa + R}$ with some $\kappa > 0$, and $c := \dfrac{C_\delta + C_\alpha}{C_\gamma} < 1$, then we obtain

$$R_{\mathrm{eq}} = \frac{\kappa c}{1 - c}.$$

Proof. We use Lemma III.1.8 with $f \equiv \mathrm{Id}$ and $\Lambda_R = 0$ because we search for a non-trivial equilibrium. The result follows from (III.1.11), the fact that $\gamma = \alpha = \delta + 1$ and $M_\alpha > 0$. □

Hence, under the allometric setting with the precise parameters specified in Corollary III.1.9, we can identify the unique possible value of the resource at the equilibrium, if it exists. Finally, we have the following result if $\phi(R) < 1$ for $R \geq 0$ (this assumption is really not harmful, it is usually the case for the choice of the function ϕ [YI92, CF15, DDA22]).

Lemma III.1.10. *We work under the allometric setting described in Example 1 of Section II.6 with $\alpha \in [0,1]$, $\gamma = \alpha = \delta + 1$ and $\phi(R) < 1$ for $R \geq 0$. Assume in addition that $C_\delta \geq C_\gamma - C_\alpha$. Then, for every $R \geq 0$, the PDE with fixed resources R admits no non-trivial equilibrium u_{eq} such that there exists an integrable function H on $\mathbb{R}_+^*$ with (III.1.9).*

Proof. Let us suppose by contradiction that there exists an amount of resources R_{eq} associated with a non-trivial equilibrium u_{eq} verifying (III.1.9). Then by Corollary III.1.9 and by assumption on $(C_\delta, C_\gamma, C_\alpha)$ and ϕ, we must have

$$1 > \phi(R_{\text{eq}}) = \frac{C_\delta + C_\alpha}{C_\gamma} \geq 1,$$

which is a contradiction and this ends the proof. □

Lemma III.1.10 allows us to draw a final link between the considerations of this section and Chapter I.

Corollary III.1.11. *We work under the allometric setting described in Example 1 of Section II.6 with $\alpha \in (0,1]$ and $\phi(R) < 1$ for $R \geq 0$, and assume that the allometric coefficients verify the condition (I.2.7) highlighted in Theorem I.2.1. Then, for every $R \geq 0$, the PDE with fixed resources R admits no non-trivial equilibrium u_{eq} verifying the integrability condition (III.1.9).*

Proof. If the allometric parameters verify the condition (I.2.7), then $C_\delta \geq C_\gamma - C_\alpha$ and we conclude by Lemma III.1.10. □

Remark: The main result of Chapter I is Theorem I.2.1, which highlights two possible sets of allometric coefficients (I.2.6) and (I.2.7) to guarantee that there exists an amount of resource R for which the population branching process is supercritical for every $x_0 > 0$. In Chapter I, we cannot distinguish between (I.2.6) and (I.2.7) in terms of supercriticality, even if we conjecture a difference in the expectation of the offspring distribution in Conjecture I.2.3. In this section of Chapter III, for every $R \geq 0$, we study the existence of a non-trivial equilibrium for the PDE with fixed resources R, that can be linked to the stochastic model of Chapter I by the same arguments as the tightness result of Theorem II.3.1. Corollary III.1.11 states that if we choose the set of allometric parameters (I.2.7), then there is no possible non-trivial equilibrium for the PDE with fixed resources. On the contrary, we verify numerically that if we choose the allometric set (I.2.6) (which is precisely the set highlighted by the Metabolic Theory of Ecology), then the system seems to converge to a non-trivial equilibrium (see Figure III.4). Even if some theoretical ideas need to be developed, especially to prove the existence of this equilibrium, this result allows us to distinguish between the two sets of allometric parameters (I.2.6) and (I.2.7). Along with the result of Theorem I.2.1, this could be a way to justify the allometric relationships usually depicted in the literature [BMT93, MM19] as the only set allowing both survival of the population in the context of abundant resources, and existence of a critical amount of resources for which the system reaches a non-trivial equilibrium .

III.1.3 An exploratory attempt of reduction to an ODE system

In this section, we consider again a general solution $(u_t(.), R_t^*)_{t\in[0,T]}$ to the PDE system (III.1.4), (III.1.5), (III.1.6) verifying the assumptions of Proposition III.1.5. In particular, we define for every $t \in [0,T]$ and $k \in \mathbb{R}$ the following quantities, possibly finite or infinite,

$$M_k(t) := \int_0^{+\infty} x^k u_t(x) \mathrm{d}x.$$

We present an unsuccessful attempt to approximate numerically quantities of the form $M_k(t)$ over time. We wanted to deduce from the PDE system a new system of coupled ODEs, under the allometric setting with $\alpha \in [0,1]$ and $\gamma = \alpha = \delta + 1 = \beta + 1$. Our main motivation was to recover a simpler model still structured by allometric relationships, but for quantities of the form $M_k(.)$ that have a clear biological interpretation (for $t \geq 0$, we can interpret $M_0(t)$, respectively $M_1(t)$, as the population size, respectively the total energy in the population, at time t). Also, solving a coupled system of ODEs usually reduces the complexity of algorithms and computing time. A lighter numerical scheme paves the way to a numerical comparison between different allometric parameter sets, and could lead to a study of allometric relationships with an evolutionary point of view as in [FCO17]. We begin with a preliminary result, at the root of this exploratory work.

Lemma III.1.12. *Let $(u_t(.), R_t^*)_{t\in[0,T]}$ a solution to the PDE system* (III.1.4)*,* (III.1.5)*,* (III.1.6) *verifying the assumptions of Proposition III.1.5. We also assume that there exists an integrable function H on $\mathbb{R}_+^*$, such that*

$$\forall x > 0, \forall t \in [0,T], \quad |(1 + x + \omega(x))\partial_t u_t(x)| \leq H(x).$$

Then, for every $f \in \mathcal{C}^1(\mathbb{R}_+^*) \cap \mathfrak{B}_{1+\mathrm{Id}+\omega}(\mathbb{R}_+^*)$,

$$\begin{aligned}\partial_t \left(\int_0^{+\infty} f(x) u_t(x) \mathrm{d}x \right) = & \int_0^{+\infty} f'(x) g(x, R_t^*) u_t(x) \mathrm{d}x \\ & + \int_0^{+\infty} \Big(f(x_0) + f(x - x_0) - f(x) \Big) b(x) u_t(x) \mathrm{d}x \\ & - \int_0^{+\infty} f(x) d(x) u_t(x) \mathrm{d}x.\end{aligned}$$

Proof. Let $f \in \mathcal{C}^1(\mathbb{R}_+^*) \cap \mathfrak{B}_{1+\mathrm{Id}+\omega}(\mathbb{R}_+^*)$. First, by the uniform domination by H, by assumption on f and Lemma II.5.3, and under the regularity assumptions of Proposition III.1.5, the function $t \in [0,T] \mapsto \int_{\mathbb{R}_+^*} f(x) u_t(x) \mathrm{d}x$ is differentiable and

$$\forall t \in [0,T], \quad \partial_t \left(\int_{\mathbb{R}_+^*} f(x) u_t(x) \mathrm{d}x \right) = \int_{\mathbb{R}_+^*} f(x) \partial_t u_t(x) \mathrm{d}x.$$

We can develop the right-hand side using (III.1.4) and conclude in the same way as in the proof of Lemma III.1.8. □

Remark: By Lemma III.1.12, we obtain for all $k \in [0,1]$ and $t \in [0,T]$,

$$\begin{aligned}\partial_t M_k(t) = & \int_0^{+\infty} k x^{k-1} g(x, R_t) u_t(x) \mathrm{d}x + \int_0^{+\infty} \Big(x_0^k + (x - x_0)^k - x^k \Big) b(x) u_t(x) \mathrm{d}x \\ & - \int_0^{+\infty} x^k d(x) u_t(x) \mathrm{d}x.\end{aligned}$$

Under the allometric setting, the functions b, d and $g(., R)$ are power functions, so our aim is to obtain ODEs linking the functions $M_k(.)$. We choose $x_0 << 1$ and use the following approximations:

- $\forall x > 0, \quad \mathbb{1}_{x \leq x_0} \approx 1$, so that $b(x) \approx C_\beta x^\beta$;
- $\forall x > 0, \forall k \in \mathbb{R}, \quad (x - x_0)^k - x^k = x^k \left(\left(1 - \frac{x_0}{x}\right)^k - 1 \right) \approx -k x_0 x^{k-1}$.

Obviously, these approximations are reasonable only for high values of $x >> x_0$, but we choose to explore numerically what happens if we assume that this holds true for all $x > 0$. By the previous approximations and under the allometric setting with $\gamma = \alpha = \delta + 1 = \beta + 1$, this gives for all $k \in [0, 1]$, for all $t \in [0, T]$,

$$\partial_t M_k(t) = \left(k\left(C_\gamma \phi(R_t) - C_\alpha\right) - C_\delta\right) M_{\alpha+k-1}(t) + x_0^k C_\beta M_{\alpha-1}(t) - k x_0 C_\beta M_{\alpha+k-2}(t). \tag{III.1.13}$$

Unfortunately, the approximation (III.1.13) cannot provide a closed system of ODEs that only involves a finite number of $M^k(.)$, because $\partial_t M_k(.)$ depends on both moments $M_{\alpha+k-1}(.)$ and $M_{\alpha+k-2}(.)$. A line of research would be to verify numerically that for $k \in \mathbb{R}$ and $t \geq 0$ high enough, $kM_{\alpha+k-1}(t)$ and $kM_{\alpha+k-2}(t)$ are well-defined and negligible compared to $x_0^k M_{\alpha-1}(t)$ (we rely on the intuition that individual energies will remain close to x_0 so that high order moments will become negligible compared to low order ones), which will allow us to neglect terms in (III.1.13) to construct a closed system of ODEs. We leave this for future work. Instead, we choose to work under the following additional assumption:

$$\forall k \in \{\alpha - 1, \alpha, 1\}, \forall t \in [0, T], \quad M_{\alpha-1+k}(t) M_0(t) = M_{\alpha-1}(t) M_k(t) < +\infty. \tag{III.1.14}$$

Equation (III.1.14) is motivated by preliminary numerical simulations of the PDE, for which it is verified at least for high values of t, where the PDE seems to reach an equilibrium. We finally apply (III.1.13) to $k = 0$, 1 and α, uses (III.1.14) and also rewrite (III.1.5) under the allometric setting to obtain the following closed system of ODEs. It involves the four functions M_0, M_1, M_α and R^* over time, with the usual notation $\dot{F}(t) := \frac{\mathrm{d}F(t)}{\mathrm{d}t}$ for $F \in \mathcal{C}^1(\mathbb{R}_+^*)$:

$$\begin{cases} \dot{M}_0(t) &= (C_\beta - C_\delta) \dfrac{M_\alpha(t) M_0(t)}{M_1(t)}, \\ \dot{M}_1(t) &= \left(C_\gamma \phi(R_t) - C_\alpha - C_\delta\right) M_\alpha(t), \\ \dot{M}_\alpha(t) &= \left(\alpha\left(C_\gamma \phi(R_t^*) - C_\alpha\right) - C_\delta\right) \dfrac{M_\alpha(t)^2}{M_1(t)} + x_0^\alpha C_\beta M_{\alpha-1}(t) - \alpha x_0 C_\beta \dfrac{M_\alpha(t)^2}{M_0(t)}, \\ \dot{R}_t^* &= \varsigma(R_t^*) - C_\gamma \phi(R_t^*) M_\alpha(t). \end{cases}$$

Our numerical simulations lead to the following results: the population size $M_0(t)$ goes to $+\infty$, with a total energy $M_1(t)$ going to 0 when $t \to +\infty$, and the amount of resources stabilizes at $R_{\text{ode}} > R_{\text{eq}}$, where R_{eq} verifies (III.1.12) (the population size increases, but the consumption of resources stabilizes because individuals become smaller and smaller). In addition to the biologically questionable behavior of the population, the wrong equilibrium for the resource with respect to Corollary III.1.9 leads us to conclude that our approximations provide an inaccurate numerical scheme. We leave the question of a successful reduction of the PDE system (III.1.4), (III.1.5), (III.1.6) to an ODE system open for future work. Still, we observe that solving this problem is not easy, which raises an interesting questioning about the usual choice of ODE systems to model a chemostat or more generally interacting populations. In the literature, we find mainly ODEs involving only population sizes (see Section 3 in [FBC21] for a unifying model), whereas we observe in our case that other moments associated to the population (total energy M_1, moment M_α) are closely linked to the population size M_0. This is mainly due to our allometric assumptions, but allometric relationships seems to be a widespread phenomenon in the study of ecosystems [Pet86, BMT93, MM19], Hence, we think that deriving successfully ODE systems associated to our PDE model (and thus associated to our stochastic model by Theorem II.3.1) could be a fruitful line of research, interacting with evolutionary biology.

III.2 Simulation parameters

In this section, we fix our simulation parameters for the rest of this chapter. Recall that between random jumps, individual energies and the amount of resources evolves deterministically according to (II.2.20) and (II.2.21), and the stochastic dynamics are described in Section II.1.1. For $x > 0$ and $R \geq 0$, we set:

1. $\ell(x) := C_\alpha x^\alpha$,
2. $b(x) := \mathbb{1}_{x>x_0} C_\beta x^\beta$,
3. $f(x, R) := \dfrac{R}{\kappa + R} C_\gamma x^\gamma$ (*i.e.* $\phi(R) = \dfrac{R}{\kappa + R}$ and $\psi(x) = C_\gamma x^\gamma$),
4. $d(x) := C_\delta x^\delta$,
5. $\varsigma(R) := D(R_{\text{in}} - R)$,

with $\alpha \in (0,1)$, $\beta, \gamma, \delta \in \mathbb{R}$, $C_\alpha, C_\beta, C_\gamma, C_\delta, \kappa, D \in \mathbb{R}_+^*$ and $R_{\text{in}} \in [0, R_{\max}]$. Also, we will specify a conversion efficiency coefficient $\chi > 0$. We choose a deterministic initial condition for the resources $R_0 \in [0, R_{\max}]$, and K random initial individual energies, chosen independently and according to an initial distribution with compact support denoted as $[x_{\min}, x_{\max}]$, and absolutely continuous with respect to Lebesgue measure, with a density u_0 shown on Figure III.1, which is given by

$$\forall x > 0, \quad u_0(x) := C \left(\frac{(x - x_{\min})(x_{\max} - x)}{(x_{\max} - x_{\min})^2} \right)^5 \mathbb{1}_{x \in [x_{\min}, x_{\max}]},$$

with a constant $C > 0$ being such that $\displaystyle\int_{\mathbb{R}_+^*} u_0(x)\mathrm{d}x = 1$.

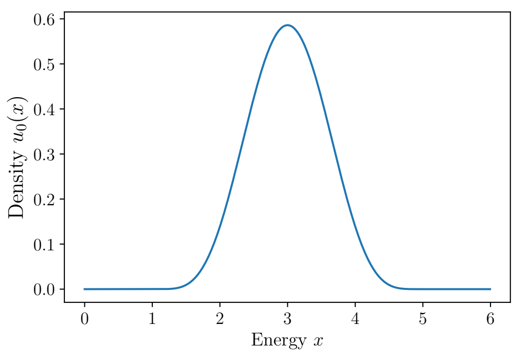


Figure III.1: Initial density $u_0(.)$

We simulate the IBM and the PDE model during a time $T > 0$. Under the allometric setting with $\alpha \in (0,1)$, for $t \in [0,T]$, we can upper bound the maximal energy an individual can reach by the solution of

$$\begin{cases} f'(t) = (\phi(R_{\max})C_\gamma - C_\alpha)f^\alpha(t), \\ f(0) = x_{\max} \end{cases}$$

taken at time T, where $R_{\max} > 0$ is defined in II.1.1. This upper bound is precisely given by

$$M_T := \left(x_{\max}^{1-\alpha} + (1-\alpha)(\phi(R_{\max})C_\gamma - C_\alpha)T\right)^{\frac{1}{1-\alpha}}. \tag{III.2.15}$$

Hence, we work on the fixed energy window $[0, M_T]$ in the following. More precisely, unless specified otherwise, we fix the set of parameters on Table III.1, which corresponds to the set supported by the Metabolic Theory of Ecology [BGA+04, MM19].

Parameter	Value
α	0.75
γ	0.75
β	-0.25
δ	-0.25
C_α	1
C_γ	2
C_β	0.1
C_δ	0.05
κ	5
D	0.275

Parameter	Value
R_{in}	2
$R_{\max}$	2
χ	200
K	1000
x_0	1
R_0	1
$x_{\min}$	1
$x_{\max}$	5
T	200
M_T	208688

Table III.1: Simulation parameters.

With Conjecture III.1.2 in mind, in all our numerical simulations under the allometric setting with the previous simulation parameters, we can use the following criterion to qualitatively evaluate if our numerical schemes behave badly.

Definition III.2.1 (Invalidation criterion of a numerical scheme for the simulation of the PDE). *Under the allometric setting described in Example 1 of Section II.6 with $\alpha \in [0,1]$ and $\gamma = \alpha = \delta + 1$, a numerical scheme for the resolution of the PDE system (III.1.4), (III.1.5), (III.1.6) is said to be inaccurate, if we observe that as $t \to +\infty$,*

$$R_t \nrightarrow R_{\text{eq}},$$

where R_{eq} verifies (III.1.12).

In Section III.4.2, we will verify that the resource seems to stabilise at the expected equilibrium on the third row of Figure III.4, where the dotted line is constant equal to R_{eq}. Hence, we do not reject the numerical scheme used in Section III.4.2, but we will insist in this section on the fact that obtaining accurate simulations is the result of a complex calibration of our discretization grid.

III.3 Exact simulation of individual-based models

Let us fix $T \geq 0$, and consider the renormalized process $(\mu_t^K, R_t^K)_{t\in[0,T]}$ described in Section II.2. Recall that this process is constructed iteratively, by defining a sequence of jump times through Poisson point measures, and between these jump times, the process is deterministic. Although these mathematical objects are mathematically well-defined, it is very natural to wonder how to simulate this process numerically, and the goal of this section is to provide some guidelines. First in Section III.3.1, we recall the structure of the classical rejection sampling algorithm [Gil76] in the case of bounded rates. Then in Section III.3.2, we provide another algorithm called the *individual-clock* algorithm and compare its performances to the rejection sampling algorithm. This discussion is motivated by the fact that in our setting, the death rate d is unbounded. Hence, on the one hand, the rejection sampling algorithm becomes theoretically inaccurate, which is why we propose the theoretically exact individual-clock algorithm. On the other hand, it still takes significantly less computing time to implement a rejection sampling algorithm (which is itself already a long procedure compared to the simulations of PDEs). The main aim of this section is thus to justify numerically the use of a rejection sampling algorithm to simulate the IBM in this chapter.

III.3.1 The rejection sampling algorithm

First, let us recall a classical way to simulate the IBM via rejection sampling with bounded birth and death rates, introduced by Gillespie ([Gil76], see also Section 2.2 in [CF15]). In this section only, we assume that there exists constants $\bar{b} > 0$ and $\bar{d} > 0$ such that $b(x) \leq \bar{b}$ and $d(x) \leq \bar{d}$ for all $x > 0$. We recall the structure of one step of the simulation algorithm of the IBM in that situation.

1. Start at time $t \geq 0$ from an initial condition $(\Xi_t, R_t) \in \left(\mathbb{R}_+^*\right)^N \times [0, R_{\max}]$, with $\Xi_t := \left(\xi_t^j\right)_{1 \leq j \leq N}$ and some $N \in \mathbb{N}$. If $N = 0$, it means that there are no individuals in the population anymore, and we simply actualize the amount of resources R_t, following Equation (II.1.3) with $\mu_t = 0$ (so it is a deterministic ordinary differential equation). Else, we continue the procedure with the following steps.

2. Create a variable $\tau := N(\bar{b} + \bar{d})$, and draw an independent random variable Δt with exponential law with parameter τ.

3. Actualize the individual energies and the amount of resources to obtain them at time $t + \Delta t$, following the flow X described in Section II.1.2, *i.e.* $(\Xi_t, R_t) \longleftarrow \left(\left(X_{\Delta t}^j(\Xi_t, R_t)\right)_{1 \leq j \leq N}, X_{\Delta t}^{\Re}(\Xi_t, R_t)\right)$, and also actualize $t \longleftarrow t + \Delta t$.

4. Draw an independent uniform random variable i in $\{1, ..., N\}$, and an independent uniform random variable U in $[0, 1]$.

5. If $U \leq \dfrac{b(\xi_t^i)}{\bar{b} + \bar{d}}$, then there is a birth for individual i. We actualize $\xi_t^i \longleftarrow \xi_t^i - x_0$, $\Xi_t \longleftarrow (\xi_t^1, ..., \xi_t^N, \xi_t^{N+1})$ with $\xi_t^{N+1} := x_0$ and $N \longleftarrow N + 1$.

6. Else if $U \leq \dfrac{b(\xi_t^i) + d(\xi_t^i)}{\bar{b} + \bar{d}}$, then individual i dies. We actualize
$$\Xi_t \longleftarrow \left(\xi_t^1, ... \xi_t^{i-1}, \xi_t^{i+1}, ... \xi_t^N\right) \in \left(\mathbb{R}_+^*\right)^{N-1}$$
and $N \longleftarrow N - 1$.

7. Else, this is a rejection event and there is no jump.

At the end of Step 6, 7 or 8, we simply start again at time $t + \Delta t$, with a new initial condition. Remark that the actualization of individual energies and resources during a step is not necessarily exact. If we cannot solve theoretically the system of coupled equations (II.1.4)-(II.1.5), we use a numerical scheme like Euler method to approximate the deterministic flow between jumps. One main weakness of this algorithm is the possible high number of rejections (Step 7) occurring if the deterministic and global upper bounds $\overline{b}$ and/or $\overline{d}$ are too high compared to the random individual birth and death rates in the population. In addition, remark that at Step 2, the random variable Δt (*i.e.* the time increment at this step of the rejection sampling algorithm) increases with the population size N. Hence, when the initial number of individuals becomes significant, IBM simulations take a very long time. This sheds new light on the result of Theorem II.3.1. When the parameter K becomes significant, and if there exists a unique solution $(\mu_t^*, R_t^*)_t$ to (II.3.24)-(II.3.25), we can reasonably approximate the process $(\mu_t^K, R_t^K)_t$ with the trajectories of $(\mu_t^*, R_t^*)_t$ (this will be illustrated on Figure III.4), which reduces substantially the computing time. We will describe our algorithm for the simulation of the PDE model associated to $(\mu_t^*, R_t^*)_t$ in Section III.4.1.

III.3.2 Simulation of the IBM with unbounded birth and death rates

If b and/or d is not bounded, we cannot give an almost sure upper bound for quantities of the form $b(\xi_t^i)$ and/or $d(\xi_t^i)$ for $t \geq 0$. In particular, in our setting and from the construction of the process $(\mu_t, R_t)_t$ in Section II.1.2, there is a positive probability that an individual energy ξ_t^i becomes arbitrarily close to 0 and with the simulation parameters of Table III.1, $d(\xi_t^i)$ would then explode. This makes the rejection sampling algorithm of Section III.3.1 theoretically inaccurate. In this section, we present an exact algorithm, denoted as the *individual clock algorithm*, and compare its trajectories to those obtained with the previous rejection sampling method.

One step of the individual clock algorithm:

1. Start at time $t \geq 0$ from an initial condition $(\Xi_t, R_t) \in \left(\mathbb{R}_+^*\right)^N \times [0, R_{\max}]$, with $\Xi_t := \left(\xi_t^j\right)_{1 \leq j \leq N}$ and some $N \in \mathbb{N}$. If $N = 0$, it means that there are no individuals in the population anymore, and we simply actualize the amount of resources R_t, following Equation (II.1.3) with $\mu_t = 0$ (so it is a deterministic ordinary differential equation). Else, we continue the procedure with the following steps.

2. Create N independent random variables $(E_j)_{1 \leq j \leq N}$ with exponential laws with parameter 1.

3. We compute $\Delta t > 0$ such that
$$\Delta t := \inf\{s \in [0, t_{\exp}(\Xi_t, R_t)), \exists 1 \leq j \leq N,\ I_j^s \geq E_j\},$$
with $I_j^s := B_j^s + D_j^s$ and these quantities are defined for every $s \in [0, t_{\exp}(\Xi_t, R_t))$ by
$$(B_j^s, D_j^s) := \left(\int_t^{t+s} b\left(X_\tau^j(\Xi_t, R_t)\right) \mathrm{d}\tau, \int_t^{t+s} d\left(X_\tau^j(\Xi_t, R_t)\right) \mathrm{d}\tau \right).$$
Note that we have almost surely $\Delta t < t_{\exp}(\Xi_t, R_t)$, thanks to Corollary II.1.7 (there is almost surely a random jump before one individual energy vanishes or explodes in finite time). Then, we do
$$(\Xi_t, R_t) \longleftarrow \left(\left(X_{\Delta t}^j(\Xi_t, R_t) \right)_{1 \leq j \leq N}, X_{\Delta t}^{\mathfrak{R}}(\Xi_t, R_t) \right) \quad \text{and} \quad t \longleftarrow t + \Delta t.$$

Thus, we actualize the individual energies and the amount of resources, following the flow X described in Section II.1.2.

4. Write $\mathfrak{J}$ for the set of indices j for which $I_j^{\Delta t} \geq E_j$, define $i := \text{argmin}_{j\in\mathfrak{J}} I_j^{\Delta t}$, and draw an independent uniform random variable U in $[0,1]$.

5. If $U \leq \dfrac{B_i^{\Delta t}}{B_i^{\Delta t} + D_i^{\Delta t}}$, then there is a birth for individual i. We actualize $\xi_t^i \longleftarrow \xi_t^i - x_0$, $\Xi_t \longleftarrow (\xi_t^1, ..., \xi_t^N, \xi_t^{N+1})$ with $\xi_t^{N+1} := x_0$ and $N \longleftarrow N+1$.

6. Else, individual i dies. We actualize $\Xi_t \longleftarrow \left(\xi_t^1, ...\xi_t^{i-1}, \xi_t^{i+1}, ...\xi_t^N\right) \in \left(\mathbb{R}_+^*\right)^{N-1}$ and $N \longleftarrow N-1$.

Note that in addition to the possible numerical approximation of the deterministic flow between jumps, we may also approximate numerically the integrals (B_j^s, D_j^s) at Step 3. We associate an exponential clock to each individual in the population to determine the next jump time, whereas the rejection sampling algorithm requires only one exponential variable. Hence, unfortunately, even if it does not require bounds on the birth and death rates, the individual clock algorithm is even worse than the rejection sampling algorithm in terms of computing time, especially when we consider large population sizes. This is why we ask ourselves if a rejection sampling algorithm with artificial bounds $\overline{b}$ and $\overline{d}$ could approximate the exact simulations of the individual clock algorithm (*i.e.* we use a rejection sampling algorithm with b and d replaced by $b \wedge \overline{b}$ and $d \wedge \overline{d}$). Our goal is to justify numerically the use of a rejection sampling algorithm for the IBM in the next sections. Under the allometric setting with the set of parameters of Table III.1, as $\beta < 0$ and $b(x) = 0$ for $x \leq x_0$, the birth rate is already bounded. Thus, we focus on the death rate, but our method can easily be adapted to also study the influence of an artificial bound on the birth rate if $\beta > 0$.

Our strategy is the following: we fix an artificial upper bound $\overline{d}$ and define the ratio $R_d := \overline{d}/d(x_0)$. Then, we simulate $n := 100$ times the IBM on the time window $[0, 200]$, with the rejection sampling algorithm, and at every jump time, we verify if the instantaneous death rate of an individual is higher than $\overline{d}$ (in that case, it means that the rejection sampling algorithm is making an approximation at this step). We finally compute the proportion P_d of such problematic events among all the jumps that occurred in the population on the time window $[0, 200]$. Finally, we plot the proportion P_d as a function of R_d for different values of $\overline{d}$. On Figure III.2, we realize that as R_d increases, the proportion P_d decreases very fast. When the time horizon T increases, the proportion of problematic events possibly occuring also increases, but this is still not harmful if the ratio R_d is chosen large enough. In practice, in the following, before using the rejection sampling algorithm to simulate the IBM, we will first adjust the artificial bound $\overline{d}$ depending on the parameters of simulation we want to use and the time window $[0, T]$. Recall also that if $\overline{d}$ is too high, then the computing time of the rejection sampling algorithm can be extremely long, so we pick the smallest value of $\overline{d}$ such that the proportion P_d is less than 10^{-6} for the set of parameters and the value of T we consider. For instance, with the parameters of Table III.1, we can choose $R_d = 2.10^4$ to verify this criterion. The individual clock algorithm could certainly be optimized, and we leave for future work a more precise comparison of the rejection sampling algorithm and exact algorithms for the simulation of the IBM.

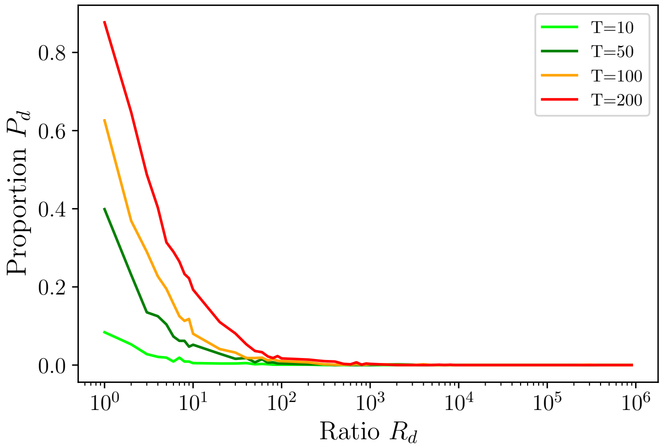


Figure III.2: Proportion P_d of inaccurate jumps for the rejection sampling algorithm, as a function of the ratio $R_d := \overline{d}/d(x_0)$, plotted on a log-scale. For 30 values of R_d going from 1 to 10^6, we simulated $n := 100$ times the IBM on the time window $[0, 200]$, and plotted the proportion P_d for intermediate values of $T \in [0, 200]$.

III.4 Numerical convergence of the IBM towards the PDE model when $K \to +\infty$

In this section, we present simulations of the PDE system (III.1.4), (III.1.5), (III.1.6) and compare them to the IBM simulated with the rejection sampling algorithm of Section III.3, when the initial population size K goes to $+\infty$. This illustrates the tightness result of Theorem II.3.1, in the context where the measure μ_t^* is absolutely continuous with respect to Lebesgue measure for every $t \in [0, T]$ and the associated density is a function solution to the system (III.1.4), (III.1.5), (III.1.6). This is a very useful result if we want to reduce the computing time associated to simulations of our model for the study of large populations. In Section III.4.1, we adapt the method used by Fritsch (see Section 6 in [CF15]) to our allometric setting and specific birth dynamics. We further discuss the technical difficulties encountered for the calibration of parameters of this algorithm. In Section III.4.2, we present the results of our simulations.

III.4.1 Algorithm for the simulation of the PDE system (III.1.4), (III.1.5), (III.1.6)

The transport term (III.1.4) associated to the boudary condition (III.1.6) accounts for a conservative system for the energy transfer due to birth events (which is coherent with the fact that our birth dynamics in Section II.1.1 are conservative). The only creation of energy in the whole system is the income of resources in (III.1.5), and the loss of energy comes from the washout of resources term in ς, individual deaths in the population and the coefficient of conversion efficiency χ. To respect the conservative aspect of (III.1.4) and (III.1.6), we use a finite differences scheme of order 1 to approximate the transport term. To simulate the temporal evolution of the system, we simply use a classical Euler scheme. There is a numerical approximation due to these low order schemes, especially in the context of unbounded growth, birth and death rates. Nevertheless, in this section, our goal is not to go deep in the optimization of our algorithm, nor to give a mathematical proof of the convergence of our numerical scheme. We rather present the difficulties we encountered in the calibration of the energy discretization grid. Indeed, we proposed a specific choice of energy discretization adapted to our birth dynamics, inspired by usual strategies depicted in the literature [BDG19, RT24].

III.4.1.1 Description of the algorithm

First, we consider the initial condition u_0 with compact support of Section III.2, and recall that with our simulation parameters, we work on the compact energy window $[0, M_T]$. To implement a finite differences scheme, we discretize both time interval $[0, T]$ and energy interval $[0, M_T]$.

- For the energy discretization grid, we write $(y_i)_{i\in\{0,\ldots,N\}}$ with some $N \in \mathbb{N}^*$, $y_0 = 0$ and $y_N := M_T$, and for $i \in \{1, ..., N\}$, we define $\Delta y_i := y_i - y_{i-1}$. Also, for every $i \in \{0, ..., N\}$, we define

$$i^{x_0} := \underset{j\in\{0,\ldots,N\}}{\operatorname{argmin}} |y_j - y_i - x_0|, \tag{III.4.16}$$

  so that $y_{i^{x_0}}$ is the point of the discretization that is the closest to $y_i + x_0$ in absolute value. We will use these indices to approximate the term $b(x + x_0)u_t(x + x_0)$ in (III.1.4) in our finite differences scheme.

- For the time discretization grid, we fix $M \in \mathbb{N}^*$, define $\Delta t := T/M$, and choose a regular grid $(t_k)_{k\in\{0,\ldots M\}}$ with $t_k := k\Delta t$ for $k \in \{0, ...M\}$.

For every $k \in \{0, ...M\}$ and $i \in \{0, ...N\}$, we define inductively the quantities $u_{k,i}$, $u^{x_0}_{k,i}$ and r_k as approximations of the quantities $u_{t_k}(y_i)$, $u_{t_k}(y_i + x_0)$ and $R^*_{t_k}$. As we have to verify the boundary condition (III.1.6), we distinguish the index i_0 such that $y_{i_0} = x_0$ (we will choose our discretization grid to ensure that such an index exists). First, $u_t(.)$ is the density associated to μ^*_t, whose support is included in $\mathbb{R}^*_+$, and we choose our energy window and initial condition u_0 so that $u_t(M_T) = 0$ for every $t \in [0, T]$, so we impose the boundary conditions

$$\forall k \in \{0, ..., M\}, \quad u_{k,0} = u_{k,N} = 0.$$

Then, for the initial condition at time 0, we set $u_{0,i} := u_0(y_i)$ and $u^{x_0}_{0,i} := u_0(y_i + x_0)$ for every $i \in \{1, ..., N\}$, and $r_0 := R^*_0$. Finally, we will follow the classical procedure of an upwind scheme, except for the fact that for $t \in [0, T]$ the sign of the transport term $g(x, R^*_t)$ in (III.1.4) may vary depending on the energy x. We thus have to include in the procedure the possibility for any $u_{k,i}$ with $i \in \{1, ..., N - 1\}$ to be actualized because energy comes from both $u_{k,i-1}$ and $u_{k,i+1}$. Hence, for every $k \in \{0, ..., M - 1\}$ and $i \in \{1, ..., N - 1\}\setminus\{i_0\}$, we set, following (III.1.4), (III.1.5) and the boundary condition (III.1.6),

$$\begin{cases} \dfrac{u_{k+1,i} - u_{k,i}}{\Delta t} &= -\dfrac{1}{\Delta y_i}\Bigg[g(u_{k,i}, r_k)u_{k,i} - g(u_{k,i-1}, r_k)u_{k,i-1}\mathbb{1}_{\{g(u_{k,i-1},r_k)>0\}} \\ &\qquad\qquad + g(u_{k,i+1}, r_k)u_{k,i+1}\mathbb{1}_{\{g(u_{k,i+1},r_k)<0\}}\Bigg] \\ &\qquad\qquad - \Big(b(u_{k,i}) + d(u_{k,i})\Big)u_{k,i} + b(u^{x_0}_{k,i})u^{x_0}_{k,i}, \\ \dfrac{r_{k+1} - r_k}{\Delta t} &= \varsigma(r_k) - \chi \sum_{i=1}^{N} f(y_i, r_k)\Delta y_i, \\ u^{x_0}_{k+1,i} &= u_{k+1,i^{x_0}} \\ u_{k+1,i_0} &= u_{k+1,i_0-1} + \dfrac{\Delta y_{i_0}}{g(x_0, r_{k+1})} \sum_{j=1}^{N} b(y_j)u_{k+1,j}\Delta y_j. \end{cases}$$

III.4.1.2 Calibration of the discretization grids

First, note that as M_T goes to $+\infty$ when $T \to +\infty$ (see (III.2.15)), depending on our choice of energy discretization, the computing time of our algorithm can explode when T increases. Remark also that the fact that $u_t(.)$ is compactly supported for $t \in [0, T]$ is not necessarily guaranteed within the general framework of Section II.1.1 (here it is due to the fact that we work under the allometric setting with $0 \le \alpha \le 1$). Additional approximations in our numerical scheme could arise if the density $u_t(.)$ was not compactly supported over time. Importantly, we do not want to truncate arbitrarily our energy window to work in a prescribed compact independent of T. Indeed, the individual growth rate is unbounded (of the form $x \mapsto x^\alpha$ with $\alpha = 3/4$ with the simulation parameters of Section III.2), so that individuals with high energies contribute significantly to resource consumption. Hence, even if for $t \in [0, T]$, the density $x \mapsto u_t(x)$ decreases rapidly (see Figure III.5), we want to be able to approximate well these densities for high values of x to obtain the correct equilibrium for resources (*i.e.* to respect the criterion highlighted in Definition III.2.1).

We begin with the description of the energy discretization. In the literature, the energy discretization to simulate similar equations with a transport term can be chosen to fit growth dynamics between two time steps, in order to limit numerical diffusion and dissipation (see the choice of a geometric discretization in Section 3.2 of [BDG19], or in Section 3.2 of [RT24] in the case of a linear growth). Choosing an energy grid that fit growth dynamics and is valid for all time $t \in [0, T]$ is not possible here because the growth rate g also depends on the amount of resources R_t that fluctuates over time. We could imagine a grid that itself evolves over time, but this would increase computing time and complexify the algorithm. We did not investigate deeper in this direction and choose another energy discretization adapted to the reference value of energy x_0 (it is the reference unit for energy transfer due to births) to approximate well the term $b(x + x_0)u_t(x + x_0)$ in (III.1.4). We divide $[0, M_T]$ into several energy windows, and on each of these windows, we fix a constant energy step. In addition to the parameters of Table III.1, we show on Table III.2 the precise energy steps we choose with our simulation parameters.
On Figure III.3, we do not represent the precise energy discretization of Table III.2, but rather a typical shape to explain our approach.

- For small values of energy comparable to x_0, we want to be very precise for both birth and growth dynamics between two time steps, so we choose very small energy steps

Energy window	$[0, 10]$	$[10, 10^2]$	$[10^2, 10^3]$	$[10^3, \times 10^4]$	$[10^4, 6.5 \times 10^4]$	$[6.5 \times 10^4, M_T]$
Energy step h	5×10^{-4}	1	10	10^2	2×10^3	2×10^4
x_0/h	2000	1	0.1	0.01	2×10^{-3}	2×10^{-4}

Table III.2: Energy discretization for the finite differences scheme

of the form $\frac{p}{q}x_0$ with $p \in \mathbb{N}$ and a large (compared to x_0) $q \in \mathbb{N}^*$. On Table III.2, we pick $q := 2000$.

- For intermediate values of energy x, we progressively increase the energy steps since the approximation of the $b(x+x_0)u_t(x+x_0)$ term has less impact on the simulation. This is due to the following feedback after our first simulations: apparently, the simulated density u_t takes very low and decreasing values for energies higher than $10x_0$. The energy steps are of the form kx_0 with $k \in \mathbb{N}^*$, with $k \leq 10^2$. On Table III.2, we pick $k = 1$, 10 and then 10^2.

- For high values of energy, we choose energy steps of the form Dx_0 with $D >> 1$ (on Table III.2, we pick $D = 2000$ and then 20000). The approximation of the term $b(x + x_0)u_t(x + x_0)$ seems now to be negligible. Furthermore, increasing the energy steps reduces the computing time and storage needed for the simulations. This is a compromise between efficiency and acurracy of our algorithm.

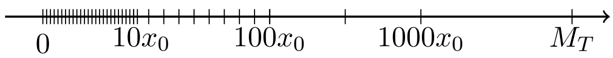


Figure III.3: Typical shape of our choice of grid with reference unit equal to x_0, adapted to birth dynamics.

Note that in the lowest energy window (*i.e.* on $[0, 10]$), our refined energy discretization allows us to verify $y_{i^{x_0}} = y_i + x_0$. In other terms, if G is the discretization grid in the lowest energy window, we have $y_i \in G \Rightarrow y_i + x_0 \in G$. The scheme respects the birth mechanism in the low-energy regime, where the density u_t is concentrated, which is similar to the choice of a geometric grid in the case of equal mitosis in [BDG19, RT24].

Let us now focus on time discretization. We enforce the following Courant-Friedrich-Lévy (CFL) condition for the choice of the time step Δt:

$$\forall i \in \{1, ..., N\}, \quad \max\left(g(y_i, R_{\max}), \ell(y_i)\right) \frac{\Delta t}{\Delta y_i} \leq 1. \tag{III.4.17}$$

By doing so, we limit the numerical diffusivity associated to the transport term between two time steps. For $i \in \{1, ...N-2\}$, an individual with energy contained in the cell $[y_i, y_{i+1}]$ at time t grows during Δt up to the cell $[y_{i+1}, y_{i+2}]$, or decrease down to the cell $[y_{i-1}, y_i]$, or stays in the cell $[y_i, y_{i+1}]$, but cannot "skip" cells during Δt. Finally, we chose the highest possible time step Δt that verifies the CFL condition, in order to reduce computing time. Under the allometric setting, with the grid depicted in Table III.2, this gives $\Delta t := 8 \times 10^{-5}$.

We insist on the fact that the previously described procedure is absolutely not optimized in terms of computing time or theoretical accuracy of the discretization. The precise calibration of the model has been done by hand, to find an acceptable compromise between computing time and precision of the numerical scheme (verified thanks to the criterion of Definition III.2.1). This is only a first exploratory attempt that could be automatized. This raises an original questioning about calibration of time and space steps of finite differences schemes for PDEs similar as (III.1.4), that we keep open for future work.

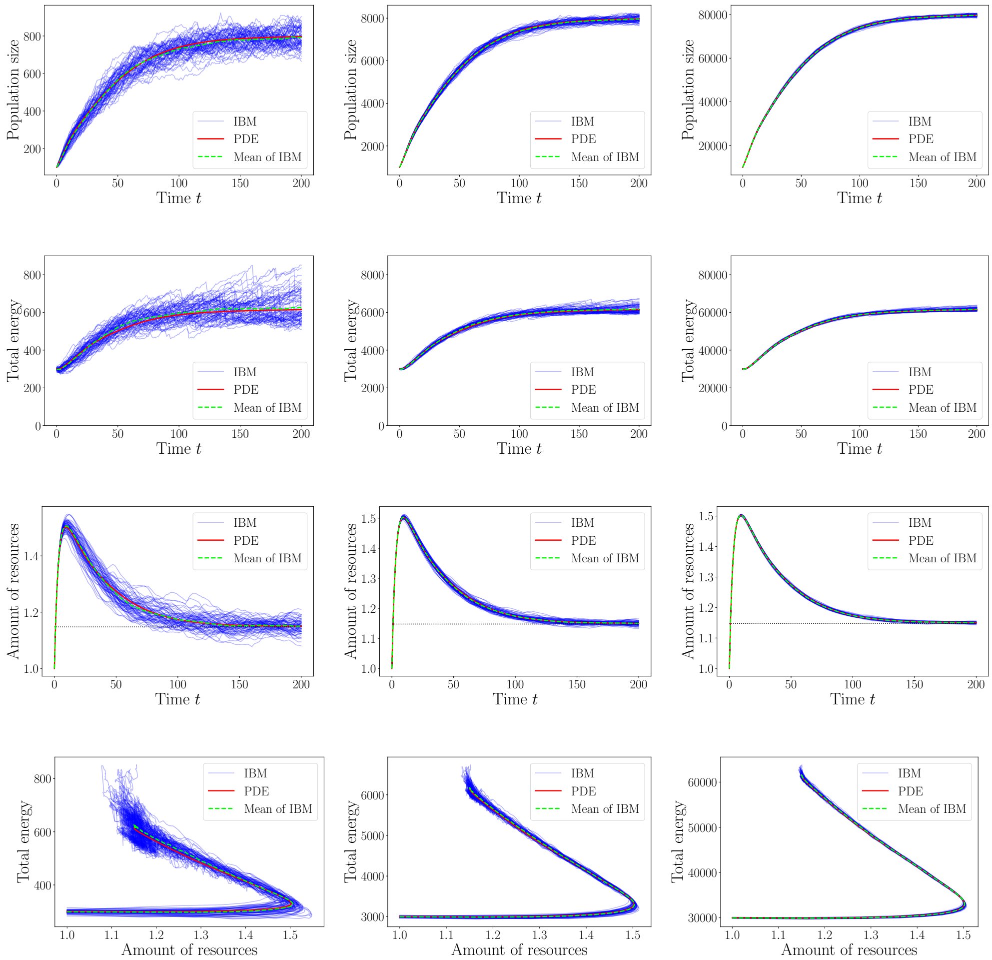
III.4.2 Comparison of IBM and PDE model

In this section, we compare simulations of the IBM using the rejection sampling algorithm described in Section III.3.1 with parameters $\overline{d} = 2.10^4$ and $\overline{b} = 0.1$ obtained from the discussion of Section III.3.2, and simulations of the PDE system (III.1.4), (III.1.5), (III.1.6) with the algorithm described in Section III.4.1. We use the simulation parameters of Table III.1, except for the value of K, which is going to increase to illustrate the tightness result of Theorem 1. As in [CF15], we compare three different regimes.

1. Small population size, with $K = 100$;
2. Medium population size, with $K = 1000$;
3. Large population size, with $K = 10000$.

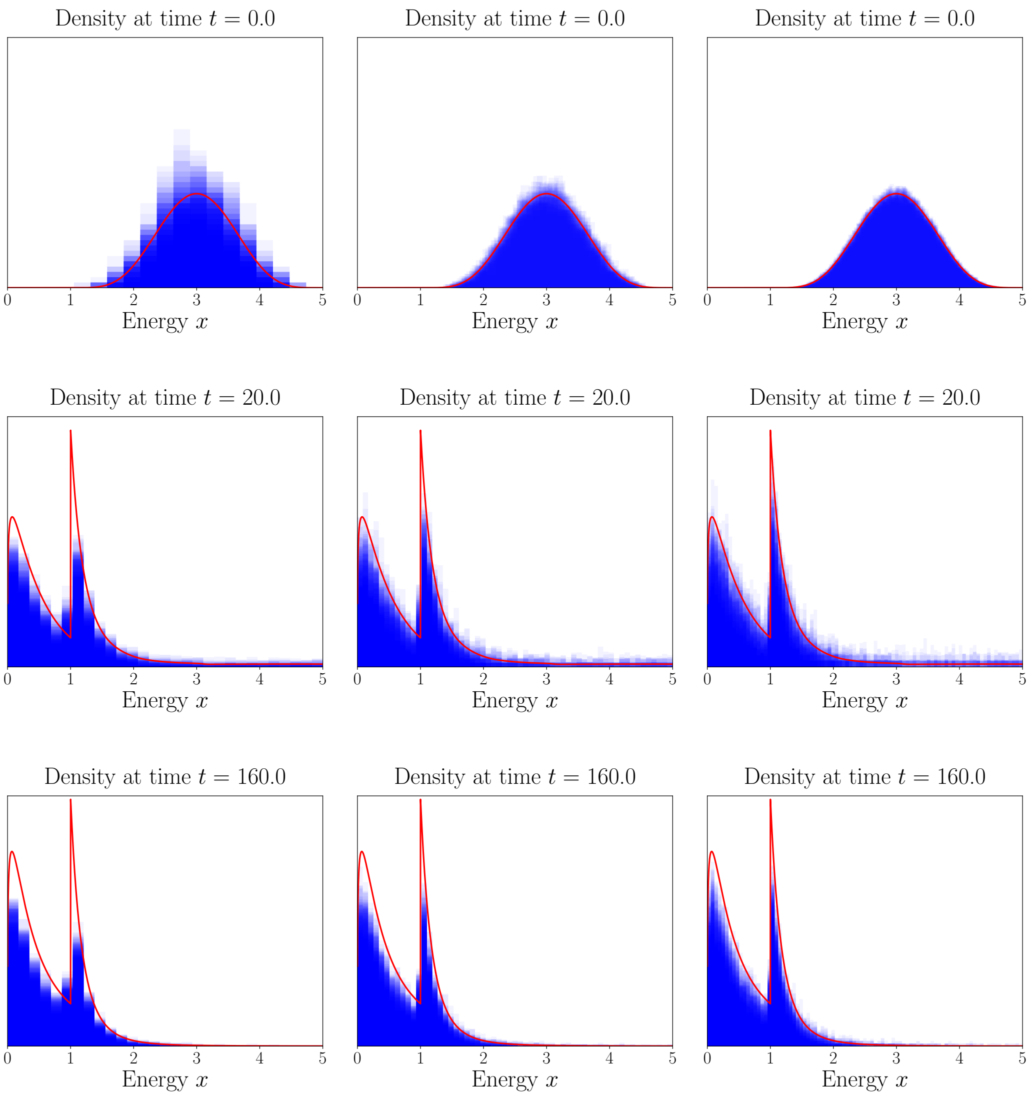
For each of these regimes, we start from the same initial condition u_0 depicted on Figure III.1, which is distributed on $[x_{\min}, x_{\max}]$, and simulate 100 independent runs of the IBM. The convergence of the IBM towards the PDE model is illustrated on Figure III.4, where we present the evolution of the population size, the total energy of the population, and the amount of resources over time for $t \in [0, 200]$. The fact that the limit is apparently unique motivates the discussion of Section II.5.1. We also represent a phase portrait energy/resource on this time window. Remark that we illustrate the convergence of $\langle \mu_t^K, 1 \rangle$ (population size) and $\langle \mu_t^K, \mathrm{Id} \rangle$ (total energy), where $1 + \mathrm{Id}$ is not dominated by ω, and this is motivated by Conjecture II.5.2. These simulation results are very similar to those obtained in Section 6 in [CF15], but the main difference is the deviations of the IBM from the PDE in terms of total energy, that we observe on the second line of Figure III.4. As time increases, it seems that the variability of the IBM trajectories around the PDE also increases, even if this phenomenon has less impact as K goes to $+\infty$ by our tightness result. This variability comes precisely from the main new contribution of our work compared to existing literature, which is the fact that individual energies are not bounded and have a positive probability to increase very fast (recall the typical shapes of individual trajectories without competition for resource in Section I.5.1). In Section III.5, we will further make vary the value of x_0 and observe the impact of this variation on the total energy deviations of the IBM from the PDE. Considering the third row on Figure III.4, we verify that the amount of resources R_t converges towards the unique possible value of the resource at equilibirum R_{eq} identified in Corollary III.1.9. Hence, all the presented simulations are not invalidated with respect to our criterion in Definition III.2.1.

Then on Figure III.5, at times $t = 0$, 20 and 160, we show a numerical approximation of the renormalized energy distribution $\tilde{u}_t : x > 0 \mapsto \dfrac{u_t(x)}{\int_0^{+\infty} u_t(y)\mathrm{d}y}$, where $(t,x) \mapsto u_t(x)$ is solution to the PDE system (III.1.4), (III.1.5), (III.1.6) with initial condition u_0. On this curve, we superimpose renormalized histograms of the empirical energy distribution in the population for 100 independent IBM simulations, taken at the same times t to illustrate again the convergence result of Theorem II.3.1. We observe numerically that for every $t > 0$, the density $x \mapsto u_t(x)$ is discontinuous at x_0, as expressed in the boundary condition of Proposition III.1.5. Furthermore, the density is rapidly (we can observe this phenomenon from $t = 5$) concentrated on a precise energy window and has a bimodal shape, with a peak near x_0 and another one near 0, which seems natural with our birth rule in mind. The density decreases very fast to 0 after x_0 and seems to stabilize after time $t = 150$. Finally, our main observation is that the system seems to reach a non-trivial equilibrium, different from $(R_{\text{in}}, 0)$ (extinction of the population), which was at the origin of Conjecture III.1.2. A more precise illustration of the uniqueness of this non-trivial equilibrium would need to make the initial condition u_0 vary, and we leave this for future work.



(a) $K = 100$ (b) $K = 1000$ (c) $K = 10000$

Figure III.4: From the first row to the third row, time evolutions of N_t^K, E_t^K and R_t^K (in blue), respectively $N_t^* := \langle \mu_t^*, 1 \rangle$, $E_t^* := \langle \mu_t^*, \mathrm{Id} \rangle$ and R_t^* (in red), representing the population size, the total energy of the population and the amount of resources, and associated respectively with the trajectories of 100 independent IBM simulations (in blue) and the numerical resolution of the PDE system (III.1.4), (III.1.5), (III.1.6) with initial condition u_0 (in red). The fourth row presents the energy/resource phase portrait. The green dotted curve is the mean value of the stochastic simulations in blue. These graphs are presented for small (left), medium (middle) and large (right) initial population sizes. On the third row, the dotted black line locates the value R_{eq} assumed to be an equilibrium for the amount of resource (we recover the value highlighted in Corollary III.1.9).



(a) $K = 100$ (b) $K = 1000$ (c) $K = 10000$

Figure III.5: Energy distribution on the energy window $[0, 5]$ at time $t = 0$ (above), 20 (middle) and 160 (bottom) for small (left), medium (middle) and large (right) initial population sizes. The red curve represents the renormalized energy distribution $\tilde{u}_t(.)$ for the corresponding value of t. The blue histogram represent the empirical energy distribution of individuals for 100 independent runs of IBM. The number of bins for the histograms is adapted to population sizes on each subfigure.

III.5 Simulations under the allometric setting with varying x_0

In this section, we investigate numerically what happens under the allometric setting, when we make the parameter x_0 vary, for a fixed initial population size $K = 1000$. Our biological motivation is the following: on the one hand, the Metabolic Theory of Ecology assumes very strong constraints on allometric relationships, as in (2.3.2), across the broad range of living organisms [Pet86, BGA+04, SDF08] (*i.e.* (2.3.2) is valid for every $x_0 > 0$)

and these are called *interspecific* allometries; on the other hand, recent papers argue for varying allometric relationships depending on the typical size of the species we consider [DOM+10, MM19, WLK24], which are called *intraspecific* allometries. In Chapter I and II, our main results (Theorems I.2.1 and II.3.1) do not depend on a precise value for x_0, hence relate to the study of *interspecific* allometries. In the following, we work with the simulation parameters of Table III.1, except that we compare three different values of x_0:

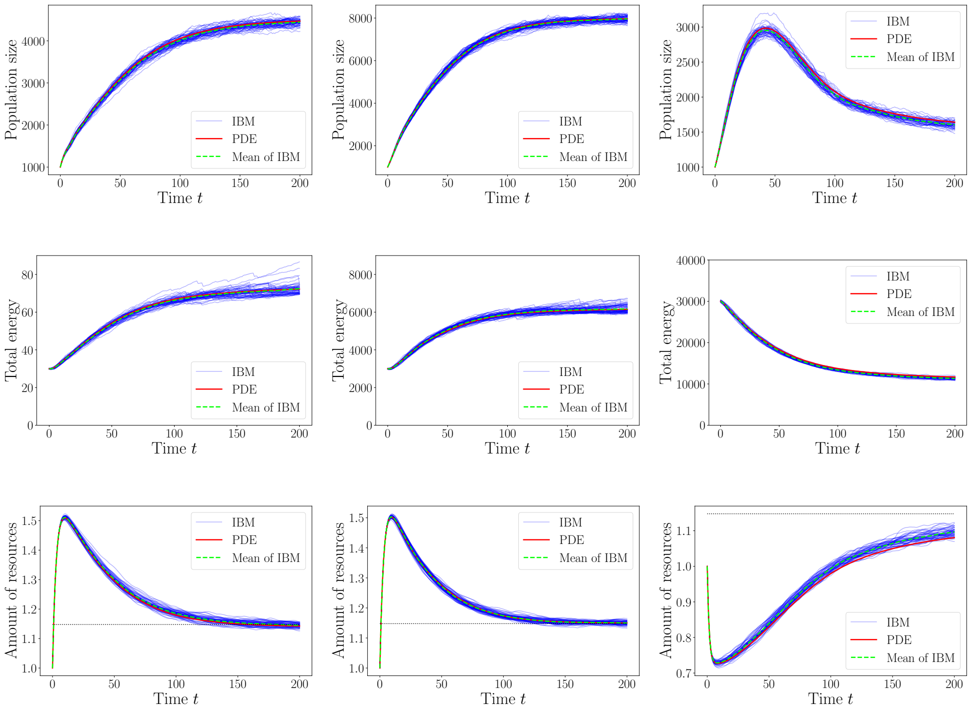
1. Small energy at birth, with $x_0 = 10^{-2}$;
2. Medium energy at birth, with $x_0 = 1$;
3. Large energy at birth, with $x_0 = 100$.

For each of these regimes, we keep the same shape for the initial density depicted on Figure III.1, with rescaled values for $x_{\min}$ and $x_{\max}$ (we simply pick $x_{\min} = x_0 = x_{\max}/5$). For the simulation of the PDE, we also adapt the discretization grid in energy presented on Table III.2, by rescaling the energy windows depending on the new maximal value M_T and keeping the same values for the ratio x_0/h on each energy window. We observe on Figure III.6 the different evolutions of population size, total energy of the population and amount of resources over time for each value of x_0. First, for both small and large values of x_0, we observe that the mean of the IBM trajectories does not coincide with the solution of the PDE, and this deviation is stronger than for the medium case $x_0 = 1$.

For the small value of x_0 (on the left of Figure III.6), the average shape of the curves remains the same, but it seems that a fraction of the stochastic simulations of the IBM deviate from the average behavior as T increases (in particular for the energy behavior on the middle row). This raises a questioning about the large deviations behavior of the stochastic process described in Section II.1.2 when $x_0 \to 0$, that we left open for future work. A possible biological intepretation is that as x_0 decreases, our model describes the behavior of living species with birth dynamics comparable to that of plants [TBDR12]. These species spread very small seeds (*i.e.* with small x_0) comparatively to the typical size of an individual of the population, and thus are less impacted in their growth by the birth dynamics, which translates into large deviations in energy (*i.e.* very large individuals appear with trajectories comparable to Figure I.3f).

Then, for the large value of x_0 (on the right of Figure III.6), the average shape of the curves becomes different and we observe a transitional regime before the stabilization of the system. At first, a lot of offspring appears and the energies of the parents become close to 0 (this explains the first increase of the population size), and then these parents die before reaching energy x_0 again (this is the large x_0 regime, so the probability to reach energy x_0 from a small energy is small). To obtain a similar behavior than in the small and medium regimes for x_0, one would certainly need to rescale properly the income of resources. In fact, it would be an interesting line of research to compute numerically the scaling of R_{in} in terms of x_0 to obtain the same (rescaled) simulations for both large, medium and large x_0. This would provide an allometric relationship between a species and its prey (here the resource R), and pave the way to the conception of food web models with allometric relationships between trophic levels. We leave this for future work.

Finally, we propose a final possible line of research. Our birth mechanism may not be adapted to model living species if x_0 is too large, because of the fact that if an individual with an energy of the form $x_0 + \eta$ with some $\eta > 0$ gives birth to offspring, then this individual reaches energy η. This can hardly be related to the behavior of known species



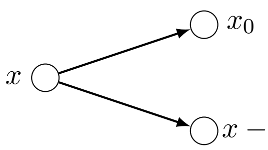
(a) $x_0 = 10^{-2}$ (b) $x_0 = 1$ (c) $x_0 = 100$

Figure III.6: From the first row to the third row, time evolutions of N_t^K, E_t^K and R_t^K (in blue), respectively $N_t^* := \langle \mu_t^*, 1 \rangle$, $E_t^* := \langle \mu_t^*, \mathrm{Id} \rangle$ and R_t^* (in red), representing the population size, the total energy of the population and the amount of resources, and associated respectively with the trajectories of 100 independent IBM simulations (in blue) and the numerical resolution of the PDE system (III.1.4), (III.1.5), (III.1.6) with initial condition u_0 (in red). The green dotted curve is the mean value of the stochastic simulations in blue. These graphs are presented for small (left), medium (middle) and large (right) energy at birth x_0. On the third row, the dotted black line locates the value R_{eq} assumed to be an equilibrium for the amount of resource.

when $\eta << x_0$ (in that case, parents appear with a negligible energy compared to that of their offspring) [Kar06]. A possible line of research would be to introduce an additional parameter $x_r > 0$ characterizing the energy dedicated to reproduction within a species, to modify the birth rate in Section II.1.1 into

$$\forall x > 0, \quad b_{x_0,x_r}(x) := \tilde{b}(x) \mathbb{1}_{\{x > x_0 + x_r\}},$$

and if the parent has energy x at the time of birth, distribute energies between parent and offspring as shown in the diagram below:



This enforces the fact that parents always have an energy higher than that of their offspring at the time of birth. We leave the precise study of these modified birth dynamics for future work.

A– Appendix of Chapter I

A.1 Construction of the population process

A.1.1 Construction of the measure-valued process μ

We give a formal construction of the population process μ, using the construction of the individual process ξ in Section I.4.1. In the following, we fix $x_0 > 0$, $R \geq 0$, and we work under the general setting of Section I.1.1.1. We define μ_t for $t \in [0, \overline{\Theta}[$, where $\overline{\Theta} < +\infty$ if there is an accumulation of jump times (and in that case, $\overline{\Theta}$ is the supremum of the jump times), and $\overline{\Theta} = +\infty$ otherwise. Then, we prove in Proposition A.1.1, that under Assumptions I.1.1 and I.1.2, then $\overline{\Theta} = +\infty$ almost surely for every $\mu_0 \in \mathcal{M}_P$. Recall from Section I.1.1.3 that we adopt the Ulam-Harris-Neveu notation to index individuals in the population. We define

$$\mathcal{U} := \bigcup_{n\in\mathbb{N}} (\mathbb{N}^*)^{n+1}.$$

Over time, every individual will have an index of the form $u := u_0...u_n$ with some $n \geq 0$, and some positive integers $u_0, ..., u_n$, *i.e.* some $u \in \mathcal{U}$. The *generation* of u is $|u| := n$. For every $u := u_0...u_n$ and $k \geq 1$, we define $uk := u_0...u_nk$.

At time $t = 0$, we pick a random variable $\mu_0 \in \mathcal{M}_P$. It means that there exists a random variable $\mathcal{C}_0 \in \mathbb{N}$ and a random vector $(\xi_0^1, ..., \xi_0^{\mathcal{C}_0}) \in (\mathbb{R}_+^*)^{\mathcal{C}_0}$ such that

$$\mu_0 = \sum_{u\in V_0} \delta_{\xi_0^u},$$

with $V_0 := \{1, ..., \mathcal{C}_0\}$. Then, we define a family of processes $(\xi^u)_{u\in\mathcal{U}}$, independent conditionally to μ_0, such that for every $u \in \mathcal{U}$, ξ^u is distributed as the process ξ of Section I.4.1, started from ξ_0^u if $u \in V_0$, and started from x_0 if $u \in \mathcal{U} \setminus V_0$ (which corresponds to the fixed amount of energy transferred to offspring). We use the construction of Section I.4.1 with independent exponential random variables $(E_i^u)_{u\in\mathcal{U},i\geq 1}$ and independent uniform random variables $(U_i^u)_{u\in\mathcal{U},i\geq 1}$. We assume that these sequences are independent from each other, and both independent from μ_0. For $u \in \mathcal{U}$, we write $(J_k^u)_{k\geq 1}$ for the jump times of ξ^u, the time of death is T_d^u and the time when the process possibly reaches $\flat$ is $T_0^u \wedge T_\infty^u$. Also, for $u \in \mathcal{U}$ and $\tau \geq 0$, we define the shifted process $\xi^{u,\tau}$ on $[\tau, +\infty[$ with $\xi_t^{u,\tau} := \xi_{t-\tau}^u$ for every $t \geq \tau$. Over time, we will describe μ_t as a point measure given by

$$\mu_t := \sum_{u\in V_t} \delta_{\xi_t^{u,\tau_u}}, \tag{A.1.1}$$

which means that at time t, alive individuals have indices u in some $V_t \subseteq \mathcal{U}$. Recall that by definition of alive individuals, V_t does not contain individuals absorbed in $\{\partial, \flat\}$. Individual u is born at time τ_u and follows the process ξ^{u,τ_u}. In particular, we set $\tau_u = 0$

for $u \in V_0$. At time $t = 0$, the initial condition of the population process is $\mu_0 := \sum_{u \in V_0} \delta_{\xi_0^{u,0}}$.

We now define the sequence $(\Theta_n)_{n\in\mathbb{N}}$ of successive times of jump of the population process. First, we set $\Theta_0 := 0$, and then suppose that our process is described until some time Θ_n with $n \geq 0$. If $\Theta_n = +\infty$, the process is already well-defined for every $t \geq 0$ and we set $\Theta_{n+1} = +\infty$. Else, at time Θ_n, the process is of the form (A.1.1) with some finite $V_{\Theta_n} \subseteq \mathcal{U}$. The next time of jump for the whole population is then

$$\Theta_{n+1} := \inf_{u \in V_{\Theta_n}} \{(J_k^u \wedge T_d^u \wedge T_0^u \wedge T_\infty^u) + \tau_u, \quad k \in \mathbb{N}^*, J_k^u + \tau_u \geq \Theta_n\},$$

with the convention $\inf(\varnothing) = +\infty$. For $s \in [\Theta_n, \Theta_{n+1}[$, we set

$$\mu_s = \sum_{u \in V_{\Theta_n}} \delta_{\xi_s^{u,\tau_u}}.$$

If $\Theta_{n+1} = +\infty$, we have defined the process μ_t for every $t \geq 0$. Else, almost surely, the infimum in the definition of Θ_{n+1} is reached at a single element $u \in V_{\Theta_n}$. First, if Θ_{n+1} is of the form $J_k^u + \tau_u$ for some $k \in \mathbb{N}^*, u \in V_{\Theta_n}$ and $J_k^u \neq T_d^u$, it means that this jump is the k-th birth for the individual u. In that case, we set

$$\mu_{\Theta_{n+1}} = \mu_{\Theta_{n+1}-} + \delta_{\xi_{\Theta_{n+1}}^{uk,\Theta_{n+1}}}$$

and $\tau_{uk} := \Theta_{n+1}$, $V_{\Theta_{n+1}} = V_{\Theta_n} \cup \{uk\}$.
Then, if Θ_{n+1} is of the form $T_d^u + \tau_u$ for some $u \in V_{\Theta_n}$, it means that this jump is the death of individual u. In that case, we set

$$\mu_{\Theta_{n+1}} = \mu_{\Theta_{n+1}-} - \delta_{\xi_{\Theta_{n+1}-}^{u,\tau_u}}$$

and $V_{\Theta_{n+1}} = V_{\Theta_n} \setminus \{u\}$.
Finally, if Θ_{n+1} is of the form $(T_0^u \wedge T_\infty^u) + \tau_u$, it means that at this jump, the energy of the individual u reaches $\{0, +\infty\}$. In that case, we also remove the individual from $V_{\Theta_{n+1}}$:

$$\mu_{\Theta_{n+1}} = \mu_{\Theta_{n+1}-} - \delta_{\xi_{\Theta_{n+1}-}^{u,\tau_u}}$$

and $V_{\Theta_{n+1}} = V_{\Theta_n} \setminus \{u\}$. Finally, we define $\overline{\Theta} := \sup_{n\in\mathbb{N}} \Theta_n$. If for all $t \in [0, \overline{\Theta}[$, $u \notin V_t$, we set $\tau_u := +\infty$.

Remark that, until now, μ is well-defined only on $[0, \overline{\Theta}[$, with $\overline{\Theta}$ possibly finite or infinite. We work now under Assumptions I.1.1 and I.1.2. Under these assumptions, any individual process ξ^u is almost surely biologically relevant by Theorem I.1.1. Thus, we can construct the individual process ξ^u as in Section I.4.9.1, using a Poisson point measure $\mathcal{N}_u$ and an exponential random variable E_u for the death. It is equivalent in law to define another independent Poisson point measure $\mathcal{N}'_u$ with intensity $\mathrm{d}s \times \mathrm{d}h$, and to work with ξ^u defined for $t \geq 0$ by

$$\begin{aligned}\xi_t^u :=& A_{\xi_0^u}(t) + \int_0^t \int_{\mathbb{R}^+} \mathbb{1}_{\{h \leq b_{x_0}(\xi_{s-}^u)\}} \Big(A_{\xi_{s-}^u - x_0}(t-s) - A_{\xi_{s-}^u}(t-s)\Big) \mathcal{N}_u(\mathrm{d}s, \mathrm{d}h) \\ &+ \int_0^t \int_{\mathbb{R}^+} \mathbb{1}_{\{h \leq d(\xi_{s-}^u)\}} \Big(\partial - A_{\xi_{s-}^u}(t-s)\Big) \mathcal{N}'_u(\mathrm{d}s, \mathrm{d}h),\end{aligned}$$

with the convention $b_{x_0}(\partial) = d(\partial) = 0$. Then, from the construction of μ and as the individual processes $(\xi_t^u)_{t\geq 0}$ are independent from each other, we have for $t \in [0, \overline{\Theta}[$,

$$\begin{aligned}\mu_t = &\sum_{u\in V_0} \delta_{A_{\xi_0^u}(t)} \\ &+ \sum_{u\in\mathcal{U}} \Bigg[\int_0^t \int_{\mathbb{R}^+} \mathbb{1}_{\{u\in V_{s-}\}} \mathbb{1}_{\{h\leq b_{x_0}(\xi_{s-}^{u,\tau_u})\}} \\ &\left(\delta_{A_{x_0}(t-s)} + \delta_{A_{\xi_{s-}^{u,\tau_u}-x_0}(t-s)} - \delta_{A_{\xi_{s-}^{u,\tau_u}}(t-s)} \right) \mathcal{N}_u(\mathrm{d}s, \mathrm{d}h) \\ &- \int_0^t \int_{\mathbb{R}^+} \mathbb{1}_{\{u\in V_{s-}\}} \mathbb{1}_{\{h\leq d(\xi_{s-}^{u,\tau_u})\}} \delta_{A_{\xi_{s-}^{u,\tau_u}}(t-s)} \mathcal{N}'_u(\mathrm{d}s, \mathrm{d}h) \Bigg]. \end{aligned} \tag{A.1.2}$$

Notice that this definition is valid almost surely on $[0, \overline{\Theta}[$, because individual trajectories are biologically relevant. This is now very similar to the classical framework for the construction of measure-valued processes [FM04, CCF16, Mar16], but the birth rate b_{x_0} is not necessarily bounded on $\mathbb{R}_+^*$.

Proposition A.1.1. *Under the general setting of Section I.1.1.1, under Assumptions I.1.1 and I.1.2, for every random variable $\mu_0 \in \mathcal{M}_P$, we have*

$$\forall x_0 > 0, \forall R \geq 0, \quad \mathbb{Q}_{\mu_0,x_0,R}(\overline{\Theta} = +\infty) = 1.$$

Proof. If we write Λ for the law of μ_0, then

$$\mathbb{Q}_{\mu_0,x_0,R}(\overline{\Theta} < +\infty) = \int_{\mathcal{M}_P} \mathbb{Q}_{\mu,x_0,R}(\overline{\Theta} < +\infty)\mathrm{d}\Lambda(\mu).$$

Thus, it is sufficient to show that for any deterministic initial condition $\mu_0 := \sum_{u\in V_0} \xi_0^u \in \mathcal{M}_P$, for every $T > 0$, $\mathbb{Q}_{\mu_0,x_0,R}(\overline{\Theta} < T) = 0$. We fix such μ_0 and T in the following, then the initial energies $(\xi_0^u)_{u\in V_0}$ are in a compact depending on μ_0, and $\langle \mu_0, 1\rangle := \int_{\mathbb{R}_+^*} \mu_0(\mathrm{d}x) = \mathrm{Card}(V_0)$ is finite. For $M > 0$, we define the stopping time $\tau_M := \inf\{t \geq 0, \langle \mu_t, 1\rangle \geq M\}$. We have from (A.1.2) that for $t \leq \tau_M \wedge T$,

$$\langle \mu_t, 1\rangle \leq \langle \mu_0, 1\rangle + \sum_{u\in\mathcal{U}} \int_0^t \int_{\mathbb{R}^+} \mathbb{1}_{\{u\in V_{s-}\}} \mathbb{1}_{\{h\leq b_{x_0}(\xi_{s-}^{u,\tau_u})\}} \mathcal{N}_u(\mathrm{d}s, \mathrm{d}h). \tag{A.1.3}$$

The initial condition μ_0 is compactly supported, so individual trajectories are almost surely in a compact $\mathfrak{C}_{\mu_0,T}$ until time $t \leq T$. Also, b_{x_0} is bounded by a constant $\overline{b}_{\mu_0,T}$ on $\mathfrak{C}_{\mu_0,T}$, and $\mathrm{Card}(V_s) \leq M$ until time $t \leq \tau_M \wedge T$. Hence, the integrand in the previous upper bound is almost surely bounded, and classical arguments using (A.1.3) allow us to prove first that $\mathbb{E}\left(\sup_{t\leq\tau_M\wedge T}\langle \mu_t, 1\rangle\right) < C'_T$, where $C'_T < +\infty$ does not depend on M (see the proof of Corollary 4.3. in [**?**]). This entails that almost surely, τ_M goes to $+\infty$ when M goes to $+\infty$.

Finally, let us suppose by contradiction that $\overline{\Theta} < T$ with positive probability and work on this event. In particular, this accumulation of jump times means that there exists a random subsequence of $(\Theta_n)_{n\in\mathbb{N}}$ corresponding to an infinite number of birth times, and denoted as $(\mathcal{B}_n)_{n\in\mathbb{N}}$ in the following. Also, we fix $M > 0$ such that $\tau_M > T$, and remark that if

$$\exists n \in \mathbb{N}, \tau_M \leq \Theta_n, \tag{A.1.4}$$

then there is a contradiction because $\overline{\Theta} < T < \tau_M \leq \Theta_n \leq \overline{\Theta}$. Hence, to conclude, it suffices to show that on the event $\{\overline{\Theta} < T\}$, (A.1.4) is verified almost surely. We assume by contradiction that $\overline{\Theta} < T$, but (A.1.4) is not verified with positive probability. Then, we would have $\langle \mu_t, 1 \rangle < M$ for every $t \in [0, \overline{\Theta}[$. In that case, the subsequence $(\mathcal{B}_n)_{n \in \mathbb{N}}$ can be constructed in law as a subsequence of a Poisson point process with intensity $M\bar{b}_{\mu_0,T}$. The only accumulation point of $(\mathcal{B}_n)_{n \in \mathbb{N}}$ would then almost surely be $+\infty$, which contradicts the fact that for all $n \in \mathbb{N}$, $0 \leq \mathcal{B}_n \leq \overline{\Theta} < T$ and concludes the proof. □

Remark that in the inductive definition of μ, we assumed that the processes $(\xi^u)_{u \in \mathcal{U}}$ are independent. This is what we call a *branching property* for the population process, and will be illustrated in the next section.

A.1.2 Proof of Proposition I.1.6 and Proposition I.1.7 in Section I.1.1.4

First, we give the proof of Proposition I.1.6, based on Lemma A.1.2 below. Recall that for $n \in \mathbb{N}$,

$$G_n := \{u \in \mathcal{U}, |u| = n, \exists t \geq 0, u \in V_t\}$$

contains all the individuals of the n-th generation and $\Upsilon_n := \mathrm{Card}(G_n)$.

Lemma A.1.2. *Under Assumption I.1.5,*

$$\forall \mu_0 \in \mathcal{M}_P, \forall x_0 > 0, \forall R \geq 0, \quad \mathbb{Q}_{\mu_0,x_0,R}(\forall n \in \mathbb{N}, \Upsilon_n < +\infty) = 1.$$

Proof. We fix μ_0, x_0, R and use induction on n. For the 0-generation, because $\mathcal{C}_0 = \langle \mu_0, 1 \rangle$ is a random variable taking values in $\mathbb{N}$, we almost surely have

$$\mathrm{Card}(G_0) = \mathrm{Card}(V_0) < +\infty.$$

Now, let us suppose that our result holds for some $n \geq 0$, that is almost surely,

$$\mathrm{Card}(G_n) < +\infty. \tag{A.1.5}$$

By Assumption I.1.5, we have almost surely

$$\forall u \in G_n, \quad N^u < +\infty, \tag{A.1.6}$$

where N^u is the number of direct offspring of individual u defined in Section I.4.1. Combining (A.1.5) and (A.1.6) entails that almost surely,

$$\mathrm{Card}(G_{n+1}) = \sum_{u \in G_n} N^u < +\infty.$$

□

By Lemma A.1.2, Υ_1 is almost surely finite, and for all $n \geq 0$,

$$\Upsilon_{n+1} := \sum_{u \in G_n} N^u.$$

From the construction of Appendix A.1.1, the state at birth of every offspring after generation G_1 is the same, starting from x_0, and the resource remains constant equal to R, so the law of every N^u is ν_{x_0,R,x_0}. Also, all these individual trajectories are independent, so are the N^u. This is exactly the setting defining a Galton-Watson process with offspring distribution ν_{x_0,R,x_0}, which ends the proof of Proposition I.1.6. Now, we give the proof of Proposition I.1.7.

Lemma A.1.3. *Under Assumptions I.1.1 and I.1.2, for all $n \in \mathbb{N}$, there exists almost surely a random time $\sigma_n \in [0, +\infty]$ such that $\sigma_n = +\infty$, if and only if $G_n = \varnothing$; and if $\sigma_n < +\infty$, then*

$$(t \geq \sigma_n) \Leftrightarrow (\forall s \geq t, G_n \cap V_s = \varnothing).$$

In addition, $\overline{\sigma}_n := \sup_{0 \leq m \leq n} \sigma_m \xrightarrow[n \to +\infty]{} +\infty$.

Proof. First, under Assumptions I.1.1 and I.1.2, by Proposition A.1.1, we work on the almost sure event $\{\overline{\Theta} = +\infty\}$. By Corollary I.4.9 in Section I.4.5, Assumption I.1.2 implies Assumption I.1.5, so we use Lemma A.1.2 and work on the event $\{\forall n \in \mathbb{N}, \Upsilon_n < +\infty\}$. For $n \in \mathbb{N}$, we define $\sigma_n := \sup\{\tau_u + T_d^u, u \in G_n\}$, with the convention $\sup(\varnothing) = +\infty$. Under Assumptions I.1.1 and I.1.2, thanks to Theorem I.1.1, for every $t \geq 0$, $u \in V_t$, $\tau_u + T_d^u < +\infty$ almost surely. Also, for every $n \in \mathbb{N}$, $\Upsilon_n = \mathrm{Card}(G_n) < +\infty$. Hence, $\sigma_n = +\infty$, if and only if $G_n = \varnothing$. Also, it is clear by definition that if $\sigma_n < +\infty$, then

$$(t \geq \sigma_n) \Leftrightarrow (\forall s \geq t, G_n \cap V_s = \varnothing).$$

The sequence $(\overline{\sigma}_n)_{n \in \mathbb{N}}$ is increasing, so converges to some $\tilde{\Theta} \leq +\infty$. If by contradiction, $\tilde{\Theta} < +\infty$, we would have

$$\forall t \geq \tilde{\Theta}, \forall n \in \mathbb{N}, G_n \cap V_t = \varnothing,$$

so no random jumps can occur after time $\tilde{\Theta}$, which contradicts the fact that $\overline{\Theta} = +\infty$. □

We take now $x_0 > 0$, and we work under Assumptions I.1.1 and I.1.2, so by Proposition A.1.1, we have $\overline{\Theta} = +\infty$ almost surely and we work on this event. By Corollary I.4.9 in Section I.4.5, Proposition I.1.6 and [AN04, Theorem 1, p.7], for all $R \geq 0$, for all $\mu_0 \in \mathcal{M}_{P,x_0,R}$ (so $\mathfrak{B} = \{G_1 \neq \varnothing\}$ occurs with positive probability), we have

$$m_{x_0,R}(x_0) > 1 \Leftrightarrow \mathbb{Q}_{\mu_0,x_0,R}(\forall n \in \mathbb{N}, G_n \neq \varnothing) > 0.$$

Thus, to prove Proposition I.1.7, it suffices to show that on the event $\{\overline{\Theta} = +\infty\}$,

$$\{\forall t \geq 0, V_t \neq \varnothing\} = \{\forall n \in \mathbb{N}, G_n \neq \varnothing\}.$$

We begin with the inclusion $\subseteq$. If there exists $n \in \mathbb{N}$ such that $G_n = \varnothing$, then we can define $m := \min\{n \in \mathbb{N}, G_n = \varnothing\}$. Remark that $\mu_0 \in \mathcal{M}_{P,x_0,R}$ implies that $\mu_0 \neq 0$, so $m \geq 1$. Then, by Lemma A.1.3, $\sigma_{m-1} < +\infty$ and $\sigma_n = +\infty$ for every $n \geq m$. Hence, $V_t = \varnothing$ for every $t \in [\sigma_{m-1}, +\infty[$. Then for the inclusion $\supseteq$, suppose that: $\forall n \in \mathbb{N}, G_n \neq \varnothing$. With Lemma A.1.3, we have $\overline{\sigma}_n \xrightarrow[n \to +\infty]{} +\infty$. Let $t \geq 0$, then there exists $n \in \mathbb{N}$ with $\overline{\sigma}_n > t$. By definition, there exists $s \in [t, \overline{\sigma}_n[$ and $m \leq n$ such that $G_m \cap V_s \neq \varnothing$, so necessarily $V_t \neq \varnothing$. This reasoning means exactly

$$\{\forall t \geq 0, V_t \neq \varnothing\} \supseteq \{\forall n \in \mathbb{N}, G_n \neq \varnothing\}.$$

A.2 Additional content for Section I.4.9

A.2.1 Proof of Proposition I.4.25

Let G be a measurable function. We have, as an immediate consequence of the telescopic expression of X given by (I.4.34),

$$G(X_t) = G(A_{\xi_0}(t)) + \int_0^t \int_{\mathbb{R}^+} \mathbb{1}_{\{h \leq b_{x_0}(X_{s-})\}} \Big(G(A_{X_{s-}-x_0}(t-s)) - G(A_{X_{s-}}(t-s))\Big) \mathcal{N}(\mathrm{d}s, \mathrm{d}h). \quad \text{(A.2.7)}$$

For any fixed $t \geq 0$, we apply (A.2.7) to $x \mapsto F(t, x)$:

$$F(t, X_t) = F(t, A_{\xi_0}(t)) + \int_0^t \int_{\mathbb{R}^+} \mathbb{1}_{\{h \leq b_{x_0}(X_{s-})\}} \Big(F(t, A_{X_{s-}-x_0}(t-s)) - F(t, A_{X_{s-}}(t-s)) \Big) \mathcal{N}(\mathrm{d}s, \mathrm{d}h). \quad \text{(A.2.8)}$$

Take $z > 0$ and $0 \leq s \leq t$. Since F and $u \in [s, t] \mapsto A_z(u - s)$ are respectively $\mathcal{C}^{1,1}$ and $\mathcal{C}^1$, the fundamental theorem of calculus gives:

$$F(t, A_z(t-s)) = F(s, z) + \int_s^t \Big(\partial_1 F(u, A_z(u-s)) + \partial_2 F(u, A_z(u-s)) g(A_z(u-s), R) \Big) \mathrm{d}u.$$

We get

$$F(t, X_t) = F(0, \xi_0) + F_0 + \int_0^t \int_{\mathbb{R}^+} \mathbb{1}_{\{h \leq b_{x_0}(X_{s-})\}} \Big(F(s, X_{s-} - x_0) - F(s, X_{s-}) \Big) \mathcal{N}(\mathrm{d}s, \mathrm{d}h) + F_1,$$

with

$$\begin{aligned}
F_0 :=& \int_0^t \Big(\partial_1 F(u, A_{\xi_0}(u)) + \partial_2 F(u, A_{\xi_0}(u)) g(A_{\xi_0}(u), R) \Big) \mathrm{d}u, \\
F_1 :=& \int_0^t \int_{\mathbb{R}^+} \mathbb{1}_{\{h \leq b_{x_0}(X_{s-})\}} \int_s^t \Bigg[\Big(\partial_1 F(u, A_{X_{s-}-x_0}(u-s)) \\
& \qquad + \partial_2 F(u, A_{X_{s-}-x_0}(u-s)) g(A_{X_{s-}-x_0}(u-s), R) \Big) \\
& - \Big(\partial_1 F(u, A_{X_{s-}}(u-s)) + \partial_2 F(u, A_{X_{s-}}(u-s)) g(A_{X_{s-}}(u-s), R) \Big) \Bigg] \mathrm{d}u \, \mathcal{N}(\mathrm{d}s, \mathrm{d}h).
\end{aligned}$$

Recall that the first two integrals in the definition of F_1 refer to the Poisson point measure $\mathcal{N}$, so are only a formal writing of an almost surely finite random sum, meaning that we can swap the integrals and obtain

$$\begin{aligned}
F_1 =& \int_0^t \int_0^u \int_{\mathbb{R}^+} \mathbb{1}_{\{h \leq b_{x_0}(X_{s-})\}} \\
& \Bigg[\Big(\partial_1 F(u, A_{X_{s-}-x_0}(u-s)) + \partial_2 F(u, A_{X_{s-}-x_0}(u-s)) g(A_{X_{s-}-x_0}(u-s), R) \Big) \\
& - \Big(\partial_1 F(u, A_{X_{s-}}(u-s)) + \partial_2 F(u, A_{X_{s-}}(u-s)) g(A_{X_{s-}}(u-s), R) \Big) \Bigg] \mathcal{N}(\mathrm{d}s, \mathrm{d}h) \, \mathrm{d}u.
\end{aligned}$$

If we look back at (A.2.8), we realize that

$$F_0 + F_1 = \int_0^t \Big(\partial_1 F(u, X_u) + \partial_2 F(u, X_u) g(X_u, R) \Big) \mathrm{d}u,$$

which, associated to the decomposition $\mathcal{N}(\mathrm{d}s, \mathrm{d}h) = \mathcal{N}_C(\mathrm{d}s, \mathrm{d}h) + \mathrm{d}s\mathrm{d}h$, leads to the expected expression of $F(t, X_t)$.

At this step, M_F is only a local martingale. Let $T > 0$, and $t \in [0, T]$. From a fixed initial condition $\xi_0 > 0$, as $A_{\xi_0}(.)$ is defined on $\mathbb{R}^+$ and $\mathcal{C}^1$, $X_t \leq A_{\xi_0}(t)$ is bounded by some constant C_T for $t \in [0,T]$. Also, $b_{x_0} \equiv 0$ on $]0, x_0]$ and is continuous on $]x_0, C_T]$, and F is $\mathcal{C}^{1,1}$. Hence, $b_{x_0}(X_t)$ and quantities of the form $F(t, X_t)$ are also bounded for $t \in [0,T]$. Finally, $M_{F,t}$ is L^2 for all $t \in [0,T]$, so M_F is a true martingale. To obtain the expression of $\langle M_F \rangle_t$, we give two different semi-martingale decompositions of $F^2(t, X_t)$. This is the technique used to prove Proposition 3.4. in [FM04]. We also refer to [RZ06]. First, we apply the previous result to F^2 and obtain:

$$\begin{aligned}
F^2(t, X_t) = F^2(0, \xi_0) &+ \int_0^t 2F(s, X_s)\Big(\partial_1 F(s, X_s) + \partial_2 F(s, X_s) g(X_s, R)\Big)\mathrm{d}s \\
&+ \int_0^t b_{x_0}(X_s)\Big(F^2(s, X_s - x_0) - F^2(s, X_s)\Big)\mathrm{d}s + M_{F^2,t}.
\end{aligned}$$

Then, we apply Itô's formula to $F(t, X_t)$ and the function $x \mapsto x^2$ to get:

$$\begin{aligned}
F^2(t, X_t) = &F^2(0, \xi_0) + \int_0^t 2F(s, X_s)\Big(\partial_1 F(s, X_s) + \partial_2 F(s, X_s) g(X_s, R)\Big)\mathrm{d}s \\
&+ \int_0^t 2F(s, X_s) b_{x_0}(X_s)\Big(F(s, X_s - x_0) - F(s, X_s)\Big)\mathrm{d}s \\
&+ \int_0^t 2F(s, X_s)\mathrm{d}M_{F,s} + \langle M \rangle_{F,t}.
\end{aligned}$$

By the Doob-Meyer theorem, we can identify the martingale parts and the finite variation parts of these two semi-martingale decompositions of $F^2(X_t)$, and deduce the expected expression of $\langle M \rangle_{F,t}$.

A.2.2 Proof of Lemma I.4.28

We simply use Lemma I.4.26 to affirm that

$$\begin{aligned}
F(Z_t) = F(Y_0) &+ \int_0^{\pi(t)} F'(Y_s)\frac{g(A_{\xi_0}(s), R)}{A_{\xi_0}(s)}\left(\frac{g(A_{\xi_0}(s)Y_s, R)}{g(A_{\xi_0}(s), R)} - Y_s\right)\mathrm{d}s \\
&+ \int_0^{\pi(t)} \int_{\mathbb{R}^+} \mathbb{1}_{\{h \le b_{x_0}(A_{\xi_0}(s)Y_{s-})\}}\left(F\left(Y_{s-} - \frac{x_0}{A_{\xi_0}(s)}\right) - F(Y_{s-})\right)\mathcal{N}(\mathrm{d}s, \mathrm{d}h).
\end{aligned}$$

Then, we perform the change of variables $u = \pi^{-1}(s)$ in the first integral, and also obtain by definition of $\widehat{\mathcal{N}}$:

$$\begin{aligned}
F(Z_t) = F(Z_0) &+ \int_0^t F'(Z_u)\pi'(u)\frac{g(A_{\xi_0}(\pi(u)), R)}{A_{\xi_0}(\pi(u))}\left(\frac{g(A_{\xi_0}(\pi(u))Z_u, R)}{g(A_{\xi_0}(\pi(u)), R)} - Z_u\right)\mathrm{d}u \\
&+ \int_0^t \int_{\mathbb{R}^+} \mathbb{1}_{\{k \le \pi'(u) b_{x_0}(A_{\xi_0}(\pi(u))Z_{u-})\}}\left(F\left(Z_{u-} - \frac{x_0}{A_{\xi_0}(\pi(u))}\right) - F(Z_{u-})\right)\widehat{\mathcal{N}}(\mathrm{d}u, \mathrm{d}k).
\end{aligned}$$

We conclude by definition of π', and splitting $\widehat{\mathcal{N}}(\mathrm{d}u, \mathrm{d}k) = \widehat{\mathcal{N}}_C(\mathrm{d}u, \mathrm{d}k) + \mathrm{d}u\mathrm{d}k$ thanks to Lemma I.4.27. By the same arguments as in the proof of Proposition I.4.25, the highlighted local martingale is a true L^2-martingale, and we compute $\langle \mathcal{M}_F \rangle_t$.

B– Appendix of Chapter II

B.1 Additional assumption for a biologically relevant process

In this section, we work under Assumptions II.1.4, II.1.5 and II.1.13, so that we avoid almost surely Situation 1 and Situation 2, by Corollary II.1.7 and Corollary II.1.17. In addition to the work of Sections II.1.3 and II.1.4, we present an additional stronger assumption, guaranteeing that every individual trajectory almost surely dies in finite time (see Proposition B.1.2). Even if it is not necessary for the convergence result in Theorem II.3.1 to hold true, we still want to model biologically relevant individual trajectories, in line with our Chapter 1. We define

$$T_{\text{inf}} := \inf\{J_n,\ \exists u \in V_{J_n}, \forall m > n,\ u \in V_{J_m}\},$$

with the convention $\inf(\varnothing) = +\infty$. Remark that T_{inf} is not a stopping time, because it is $(\mathcal{F}_t)_{t\geq 0}$ measurable. The process $(\mu_t, R_t)_t$ is said to be biologically relevant, if and only if $J_\infty \wedge \tau_{\exp} \wedge T_{\text{inf}} = +\infty$. We introduce the following assumption.

Assumption B.1.1 (**Individuals die in finite time**). *For all $x > 0$,*

$$\int_x^{+\infty} \frac{d(y)}{g(y, R_{\max})} \mathrm{d}y = +\infty.$$

Remark that Assumption B.1.1 immediately implies Assumption II.1.5. As in Section I.1.1.1, for any fixed $R \geq 0$, we write $(\xi, t) \mapsto A_{\xi,R}(t)$ for the deterministic flow associated to an individual energy starting from ξ at time 0, and following Equation (II.1.2) with fixed resources R instead of R_t. It is well-defined and positive on an interval $[0, t_{\max}(\xi, R))$, where $t_{\max}(\xi, R)$ is the deterministic time when it reaches 0 or $+\infty$ ($t_{\max}(\xi, R)$ is equal to $+\infty$ if this never happens).

Proposition B.1.2. *Under Assumptions II.1.4, II.1.13 and B.1.1, the process $(\mu_t, R_t)_{t\geq 0}$ is almost surely biologically relevant.*

Proof. First, thanks to Corollary II.1.7 and Corollary II.1.17, we have $J_\infty \wedge \tau_{\exp} = +\infty$ almost surely, and we work under this event. From the strong Markov property for Poisson point processes (see Example 10.4(a) in [DVJ07]), we have for every $n \in \mathbb{N}$ and $m > n$, conditionally to the event $\{J_n \neq +\infty\}$,

$$\mathbb{E}(J_m | \mathcal{F}_{J_n}) = \mathbb{E}_{(\mu_{J_n}, R_{J_n})}(J_{m-n}).$$

Hence, from the definiton of T_{inf}, it suffices to show that

$$\forall u \in V_0, \exists m \geq 1, u \notin V_{J_m}. \tag{B.1.1}$$

We follow the skeleton of proof of Proposition II.1.6, with additional technical steps. We suppose by contradiction that (B.1.1) is not verified. There exists $u \in V_0$, such that $u \in V_s$ for all $s \geq 0$, *i.e.* the individual indexed by u never dies. By construction of the process with Poisson point measures, we necessarily have

$$\int_0^{+\infty} d(\xi_s^u)\mathrm{d}s < +\infty. \tag{B.1.2}$$

Also, because $R \mapsto g(.,R)$ is non-decreasing and considering Equation (II.1.2), we immediately have, for all $u \in V_0$,

$$\forall\, 0 \leq t \leq s < t_{\exp}(\mu_0, R_0) \wedge t_{\max}(\xi_t^u, 0) \wedge t_{\max}(\xi_t^u, R_{\max}), \quad A_{\xi_t^u,0}(s) \leq \xi_s^u \leq A_{\xi_t^u,R_{\max}}(s). \tag{B.1.3}$$

Furthermore, we assess that one of the three following situations occurs:

(i) $\xi_s^u \xrightarrow[s \to +\infty]{} 0,$

(ii) $\xi_s^u \xrightarrow[s \to +\infty]{} +\infty,$

(iii) $\exists K \subseteq \mathbb{R}_+^*$ compact, $\displaystyle\int_0^{+\infty} \mathbb{1}_{\{\xi_s^u \in K\}}\mathrm{d}s = +\infty.$

Indeed, if (i) and (ii) are not verified, we show that (iii) holds true. First, we necessarily have $t_{\exp}(\mu_0, R_0) = +\infty$ in that case by definition. Furthermore, for any $u \in V_0$, there exists $0 < m < M$ such that

$$\forall t \geq 0, \exists s \geq t, \quad \xi_s^u \in [m, M].$$

Without loss of generality, we can pick m smaller or M higher, such that $\xi_0^u \in [m, M]$ and $m \leq x_0$. Thus, we can define three random sequences $(s_n)_{n\geq 0}$, $(t_n)_{n\geq 1}$ and $(u_n)_{n\geq 1}$, such that $s_0 = 0$, for all $n \geq 1$, $t_n := \inf\{t \geq s_{n-1}, \xi_t^u \notin [m/2, 2M]\}$, $u_n := \inf\{t \geq t_n, \xi_t^u \in [m/2, 2M]\}$ and $s_n := \inf\{t \geq u_n, \xi_t^u \in [m, M]\}$, with the convention $\inf(\varnothing) = +\infty$. We assess that $\theta_{u,[m/2,2M]} = +\infty$. Indeed, this is immediate if $t_n = +\infty$ for some $n \geq 1$. Else, by (B.1.3), and because the width of the jump during a birth event is constant equal to x_0 and the birth rate is equal to 0 on $(0, x_0]$, associated to the fact that $m/2 \leq x_0/2$, there exists a deterministic time $T > 0$ such that $s_n - u_n \geq T$ for all $n \geq 1$, and $\theta_{u,[m/2,2M]} \geq \sum_{n\geq 1}(s_n - u_n) = +\infty$.

The end of the proof is similar to the proof of Corollary II.1.7, replacing $b + d$ with d where it is needed. If (i) holds true, then (B.1.2) contradicts Assumption II.1.4; if (ii) holds true, then (B.1.2) contradicts Assumption B.1.1. Note that for both of these cases, we have, in addition to the proof of Proposition II.1.6, to take into account possible birth jumps for the individual u, and we verify that this does not change the result, again because the width of the jump is constant equal to x_0, and the birth rate is equal to 0 on $(0, x_0]$ (in particular, we replace ξ_0^u by $\xi_0^u - E(\xi_0^u/x_0)x_0$ in (II.1.7), where $E(x)$ is the usual integer part of x). Finally if (iii) occurs, we obtain

$$+\infty > \int_0^{+\infty} d(\xi_s^u)\mathrm{d}s \geq \int_0^{+\infty} \mathbb{1}_{\{\xi_s^u \in K\}} \inf_{x \in K} d(x)\mathrm{d}s = +\infty,$$

and this is again a contradiction. □

Discussion: We verify that the proofs of Corollary II.1.7 and Proposition B.1.2 are still valid even if the resource $(R_t)_t$ is constant over time. In that case, we recover the measure-valued process constructed in Chapter I, with no dynamics on resources. Under the setting of Chapter I, thanks to the branching property without varying resources, the event $\tau_{\text{exp}} \wedge J_\infty \wedge T_{\text{inf}} = +\infty$ is equivalent to the fact that a single individual trajectory is biologically relevant. We gave in Theorem I.1.1 of Chapter I a necessary and sufficient condition to verify this, which gathers Assumption II.1.4 and Assumption I.1.2 of Chapter I, that we recall below. We write $\mathfrak{R}_\infty = \{R \geq 0, \exists x > 0, \forall y \geq x, g(y,R) > 0\}$, and for such R, we write $\Omega_R := \{\xi > 0, \forall x \geq \xi, g(x,R) > 0\}$. Under Assumption II.1.1, $[R_{\max}, +\infty) \subseteq \mathfrak{R}_\infty$. Also, we use the operator $K_{x_0,R}$ from the first chapter.

Assumption B.1.3. *For all $x_0 > 0$, for all $R \in \mathfrak{R}_\infty$, for all $\xi_0 \in \Omega_R$,*

$$K^k_{x_0,R}\mathbf{1}(\xi_0) \xrightarrow[k\to+\infty]{} 0 \quad \text{and} \quad \int_{\xi_0}^{+\infty} \frac{(b+d)(x)}{g(x,R)} \mathrm{d}x = +\infty.$$

Proposition B.1.2 implies in particular that under Assumption II.1.4, then Assumption B.1.1 implies Assumption B.1.3. Let us show that the converse implication is false in general, consider for example, for $x > 0$ and $R \geq 0$, $b(x) = \mathbb{1}_{x>x_0} e^x$, $d(x) = 1/x^2$, $\psi(x) = 1$, $\phi(R) = 2\arctan(R)/\pi$ and $\ell(x) = 1/2$. Then, Assumption II.1.4 is verified, but not Assumption B.1.1. Moreover, Assumption B.1.3 holds true. To prove this fact, let us consider an individual trajectory $(\xi^u_t)_t$ with constant resources $R > \tan(\pi/3)$. We first use Borel-Cantelli lemma and the form of b to prove that $\mathbb{P}(\xi^u_t \xrightarrow[t\to+\infty]{} +\infty) = 0$. Then, we deduce from the choice of d, ψ, ϕ and ℓ that almost surely, $(\xi^u_t)_t$ dies in finite time. Finally, with Proposition 17 of Chapter 1, we conclude that Assumption B.1.3 holds true.

Assumption B.1.3 immediately implies Assumption II.1.5 by Lemma II.1.2 and the fact that $R \mapsto g(x,R)$ is non-decreasing for every $x > 0$. Thus, one can wonder if the less restrictive Assumption B.1.3 is sufficient to obtain $J_\infty \wedge \tau_{\text{exp}} \wedge T_{\text{inf}} = +\infty$ almost surely, even in the context of varying resources over time. We leave this for future work. We finish this discussion with the following proposition.

Proposition B.1.4. *There exists a choice of $R_{\max} > 0$ and functions b, ψ, ℓ, ϕ, ς, such that*

$$\Big(\forall (\mu_0, R_0) \in \mathcal{M}_P(\mathbb{R}_+^*) \times [0, R_{\max}]),\ \mathbb{P}(\tau_{\text{exp}} \wedge T_{\text{inf}} = +\infty) = 1\Big)$$
$$\Leftrightarrow (\textit{Assumptions II.1.4 and B.1.1}).$$

Proof. The converse implication is always true thanks to Propositions II.1.7 and B.1.2. We verify that we necessarily have to choose d verifying Assumptions II.1.4 and II.1.5, if we choose the following functional parameters, for $x > 0$, $R_{\max} = 1$: $b(x) = \mathbb{1}_{x>x_0} 1/x^2$, $\psi(x) = 2$, $\ell(x) = 1$ (b and $\psi - \ell$ are chosen so that there is a positive probability that no birth events occur for $t \geq 0$). Also, we choose ϕ such that $\phi(1/2) = 2/3$ and $\phi(R) = 4R/3$ for $R \leq 1/4$. Finally, we choose ς non-decreasing such that $\varsigma(1/2) \geq 4$ and $\varsigma(R) = R$ for $R \leq 1/4$, and we consider the initial conditions $\mu_0 = 3\delta_{x_0}$ and $R_0 = 1/4$, respectively $\mu_0 = 2\delta_{x_0}$ and $R_0 = 1/2$, to enforce Assumption II.1.4, respectively Assumption II.1.5. □

Remark: Proposition B.1.4 states that there exists a choice of functional parameters b, ψ, ℓ, ϕ, ς, such that we necessarily have to choose d verifying Assumption B.1.1 to obtain $\tau_{\text{exp}} \wedge T_{\text{inf}} = +\infty$ almost surely.

B.2 Topological background

B.2.1 Vague and w-weak topologies

In the following, we consider a positive and continuous function $w : \mathbb{R}_+^* \to \mathbb{R}_+^*$. We denote by $\mathcal{M}_w(\mathbb{R}_+^*)$ the set of positive measures on $(\mathbb{R}_+^*, \mathcal{B}(\mathbb{R}_+^*))$ that verify $\langle \mu, w\rangle < +\infty$. We also write $\mathcal{C}_c(\mathbb{R}_+^*)$ the space of continous functions with compact support, and $\mathcal{C}_w(\mathbb{R}_+^*)$ the space of continuous functions f such that f/w is bounded on $\mathbb{R}_+^*$. For $f \in \mathcal{C}_w(\mathbb{R}_+^*)$, we define $||f||_w := ||f/w||_\infty$, where $||.||_\infty$ is the usual uniform norm on the space of bounded continous functions $\mathcal{C}_b(\mathbb{R}_+^*)$. It is straightforward to verify that $||.||_w$ is a norm on $\mathcal{C}_w(\mathbb{R}_+^*)$ (it is a weighted norm with respect to $||.||_\infty$).

The vague, respectively w-weak, topology on $\mathcal{M}_w(\mathbb{R}_+^*)$ is the finest topology for which the applications $\mu \mapsto \langle \mu, f\rangle$ are continous, with f in $\mathcal{C}_c(\mathbb{R}_+^*)$, respectively in $\mathcal{C}_w(\mathbb{R}_+^*)$. We write $(\mathcal{M}_w(\mathbb{R}_+^*), v)$, respectively $(\mathcal{M}_w(\mathbb{R}_+^*), \mathrm{w})$, when we endow $\mathcal{M}_w(\mathbb{R}_+^*)$ with the vague topology, respectively the w-weak topology. Remark that if $w \equiv 1$, the w-weak topology coïncide with the usual weak topology, which is the finest topology for which the applications $\mu \mapsto \langle \mu, f\rangle$ are continous, with f continuous and bounded. Also, if $w \equiv 1$ the norm $||.||_w$ coincide with $||.||_\infty$. We show in the two following propositions that these topologies on $\mathcal{M}_w(\mathbb{R}_+^*)$ define Polish spaces.

Proposition B.2.1. *$(\mathcal{M}_w(\mathbb{R}_+^*), v)$ is metrizable by a distance d_v^w, such that $(\mathcal{M}_w(\mathbb{R}_+^*), d_v^w)$ is a Polish space.*

Proof. This result is well-known when $w \equiv 1$. In that case, we show that there exists $(f_n)_{n\in\mathbb{N}}$ a sequence of elements in $\mathcal{C}_c(\mathbb{R}_+^*)$, dense in this set for the uniform norm $||.||_\infty$. Then, the distance d_v^1 is given by, for every $\mu, \nu \in \mathcal{M}_1(\mathbb{R}_+^*)$,

$$d_v^1(\mu, \nu) := \sum_{n\in\mathbb{N}} \frac{|\langle \mu - \nu, f_n\rangle|}{2^n ||f_n||_\infty},$$

and we verify that $(\mathcal{M}_1(\mathbb{R}_+^*), d_v^1)$ is a Polish space (it can be seen as the dual space of $\mathcal{C}_c(\mathbb{R}_+^*)$ for the weak-* topology, hence is a locally compact metric space by the Banach-Alaoglu theorem). From this base case, we define the distance d_v^w, for every $\mu, \nu \in \mathcal{M}_w(\mathbb{R}_+^*)$,

$$d_v^w(\mu, \nu) := d_v^1(w * \mu, w * \nu),$$

where $w * \mu$ is the usual pushforward of μ by w (*i.e.* for every measurable function f, $\langle w * \mu, f\rangle := \langle \mu, wf\rangle$). □

Proposition B.2.2. *$(\mathcal{M}_w(\mathbb{R}_+^*), \mathrm{w})$ is metrizable by a distance d_P^w, such that $(\mathcal{M}_w(\mathbb{R}_+^*), d_P^w)$ is a Polish space.*

Proof. Again, this result is well-known when $\omega \equiv 1$. We define the Prokhorov distance, for every $\mu, \nu \in M_1(\mathbb{R}_+^*)$,

$$d_P^1(\mu, \nu) := \inf\{\varepsilon > 0, \forall A \in \mathcal{B}(\mathbb{R}_+^*),\ \mu(A) \leq \nu(A^\varepsilon) + \varepsilon,\ \nu(A) \leq \mu(A^\varepsilon) + \varepsilon\},$$

where $A^\varepsilon := \{y \in \mathbb{R}_+^*, \inf_{x\in A} |x - y| < \varepsilon\}$. We refer to Theorem 1.7 page 101 in [EK86] for the proof of the fact that $(\mathcal{M}_1(\mathbb{R}_+^*), d_P^1)$ is Polish. Finally, we define d_P^w for every $\mu, \nu \in \mathcal{M}_w(\mathbb{R}_+^*)$ by

$$d_P^w(\mu, \nu) := d_P^1(w * \mu, w * \nu).$$

□

Remark: In Propositions B.2.1 and B.2.2, d_v^w and d_P^w are distances because w is positive, so $w * \mu = w * \nu$ implies $\mu = \nu$. We also introduce the following distances.

Definition B.2.3. *The total variation distance on $\mathcal{M}_1(\mathbb{R}_+^*)$ is given for every $\mu, \nu \in \mathcal{M}_1(\mathbb{R}_+^*)$ by*

$$d_{\mathrm{TV}}(\mu,\nu) := \sup_{\substack{\phi\in\mathcal{D}\\ ||\phi||_\infty\leq 1}} |\langle \mu-\nu,\phi\rangle|,$$

with $\mathcal{D}$ any countable and dense subset of $\mathcal{C}_c^\infty(\mathbb{R}_+^*)$ for the topology of uniform convergence (which exists by Lemma II.4.1). Then, we define the w-total variation distance for every $\mu, \nu \in \mathcal{M}_w(\mathbb{R}_+^*)$ by

$$d_{\mathrm{TV}}^w(\mu,\nu) := d_{\mathrm{TV}}(w*\mu, w*\nu) = \sup_{\substack{\phi\in\mathcal{D}\\ ||\phi/w||_\infty\leq 1}} |\langle \mu-\nu,\phi\rangle|.$$

In the previous definition, we define the total variation distance considering only functions in $\mathcal{D}$, but a classical approximation argument gives

$$d_{\mathrm{TV}}(\mu,\nu) = \sup_{\substack{\phi\in\mathcal{C}_c^\infty(\mathbb{R}_+^*)\\ ||\phi||_\infty\leq 1}} |\langle \mu-\nu,\phi\rangle| = \sup_{\substack{\phi\in\mathcal{C}^0(\mathbb{R}_+^*)\\ ||\phi||_\infty\leq 1}} |\langle \mu-\nu,\phi\rangle|,$$

where $\mathcal{C}^0(\mathbb{R}_+^*)$ is the set of continuous functions on $\mathbb{R}_+^*$. It is well-known that the total variation distance is larger than or equal to the Prokhorov distance, so we easily extend it for all $\mu, \nu \in \mathcal{M}_w(\mathbb{R}_+^*)$ to

$$d_P^w(\mu,\nu) \leq d_{\mathrm{TV}}^w(\mu,\nu).$$

Definition B.2.4. *The Fortet-Mourier distance on $\mathcal{M}_1(\mathbb{R}_+^*)$ is given for every $\mu, \nu \in \mathcal{M}_1(\mathbb{R}_+^*)$ by*

$$d_{\mathrm{FM}}(\mu,\nu) := \sup_{\substack{\phi\in\mathcal{D}\\ ||\phi||_\infty\leq 1\\ ||\phi'||_\infty\leq 1}} |\langle \mu-\nu,\phi\rangle|,$$

with $\mathcal{D}$ any countable and dense subset of $\mathcal{C}_c^\infty(\mathbb{R}_+^*)$ for the topology of uniform convergence. Then, we define the w-Fortet-Mourier distance for every $\mu, \nu \in \mathcal{M}_w(\mathbb{R}_+^*)$ by

$$d_{\mathrm{FM}}^w(\mu,\nu) := \sup_{\substack{\phi\in\mathcal{D}\\ ||\phi/w||_\infty\leq 1\\ ||\phi'||_\infty\leq 1}} |\langle \mu-\nu,\phi\rangle|.$$

As for the total variation distance, a classical approximation argument gives

$$d_{\mathrm{FM}}^w(\mu,\nu) = \sup_{\substack{\phi\in\mathcal{C}_c^\infty(\mathbb{R}_+^*)\\ ||\phi/w||_\infty\leq 1\\ ||\phi'||_\infty\leq 1}} |\langle \mu-\nu,\phi\rangle| = \sup_{\substack{\phi\in\mathcal{C}^1(\mathbb{R}_+^*)\\ ||\phi/w||_\infty\leq 1\\ ||\phi'||_\infty\leq 1}} |\langle \mu-\nu,\phi\rangle|.$$

Lemma B.2.5. *Let w be a positive function on $\mathbb{R}_+^*$, and let $(\mu_K)_{K\in\mathbb{N}}$ and μ be elements of $\mathcal{M}_w(\mathbb{R}_+^*)$, then*

$$\left(d_{\mathrm{TV}}^w(\mu_K,\mu) \xrightarrow[K\to+\infty]{} 0\right) \Rightarrow \left(d_{\mathrm{FM}}^w(\mu_K,\mu) \xrightarrow[K\to+\infty]{} 0\right) \Rightarrow \left(d_P^w(\mu_K,\mu) \xrightarrow[K\to+\infty]{} 0\right).$$

Proof. The left-most implication is straightforward. The right-most implication is a classical result for $w \equiv 1$ and comes from a usual approximation argument by Lipschitz continuous and bounded functions and the Portmanteau theorem (see Theorem 8.3. in [R.M76]). We extend this approximation argument in $\mathcal{M}_w(\mathbb{R}_+^*)$ to any Lipschitz continuous functions dominated by w and the result follows. □

Remark: In fact, we could prove that the right-most implication in Lemma B.2.5 is an equivalence, and even provide an explicit comparison between d_{FM}^w and d_P^w (see Problem 2. p.150 in [EK86]), but do not need these precise results in our work.

B.2.2 Skorokhod spaces

In this section, we provide useful topological background about Skorokhod spaces, which mainly comes from Chapter 3 in [EK86].

Definition B.2.6. *Let (X, d) be a metric space and $T > 0$. We define $\mathbb{D}([0,T],(X,d))$ the Skorokhod space associated with X, which is the space of càdlàg functions from $[0,T]$ to X.*

We can endow this space with the topology of uniform convergence, metrizable with the distance d_u such that for every $(f,g) \in \mathbb{D}([0,T],(X,d))$

$$d_u(f,g) := \sup_{t\in[0,T]} d(f(t),g(t)).$$

However, this distance is not adapted to study càdlàg processes in the neighborhood of their discontinuities. Thus, we define another distance d_S, called the Skorokhod distance (Equation (5.2) p.117 in [EK86]). We give here the following characterization of convergence for this distance (Proposition 5.3 p.119 in [EK86]).

Proposition B.2.7. *Let (X,d), $T > 0$ be a metric space and d_S the associated Skorokhod distance on $\mathbb{D}([0,T],(X,d))$. Let $(f_n)_{n\in\mathbb{N}}$ and f be elements of $\mathbb{D}([0,T],(X,d))$. Then $d_S(f_n,f) \xrightarrow[n\to+\infty]{} 0$, if and only if there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of increasing continuous functions on $[0,T]$ with $\lambda_n(0) = 0$ and $\lambda_n(T) = T$, verifying*

$$\sup_{t\in[0,T]} d(f_n(t), f(\lambda_n(t))) \xrightarrow[n\to+\infty]{} 0,$$
$$\sup_{0\le s<t\le T} \log\left|\frac{\lambda_n(t)-\lambda_n(s)}{t-s}\right| \xrightarrow[n\to+\infty]{} 0.$$

The topology associated to the distance d_S is called the usual Skorokhod topology. By Theorem 5.6 p.121 in [EK86], if (X,d) is separable, respectively complete, then $\mathbb{D}([0,T],(X,d))$ endowed with the usual Skorokhod topology is separable, respectively complete.

Lemma B.2.8. *Let (X,d) be a metric space and $(f_n)_{n\in\mathbb{N}}$, respectively f, be elements of $\mathbb{D}([0,T],(X,d))$, respectively $\mathcal{C}([0,T],(X,d))$. Then,*

$$\left(d_S(f_n,f) \xrightarrow[n\to+\infty]{} 0\right) \Leftrightarrow \left(d_u(f_n,f) \xrightarrow[n\to+\infty]{} 0\right),$$

where d_S is the Skorokhod distance associated to d and d_u is the distance associated to the topology of uniform convergence on $\mathbb{D}([0,T],(X,d))$.

Proof. The converse implication is always true. For the direct implication, we use the sequence $(\lambda_n)_{n\in\mathbb{N}}$ of Proposition B.2.7 and assess that for every $n \in \mathbb{N}$,

$$d_u(f_n, f) \leq \sup_{t\in[0,T]} \{d(f_n(t), f(\lambda_n(t))\} + \sup_{t\in[0,T]} \{d(f(\lambda_n(t)), f(t))\}.$$

The first term of the right-hand side converges to 0 when $n \to +\infty$ by Proposition B.2.7. The same holds true for the second term because f is uniformly continuous on $[0, T]$ (Heine-Cantor theorem) and $(\lambda_n)_{n\in\mathbb{N}}$ converges uniformly towards the identity function on $[0, T]$ by Proposition B.2.7, which ends the proof. □

The following result will be useful in the study of measure-valued processes.

Lemma B.2.9. *Let $T > 0$, w a positive function on $\mathbb{R}_+^*$, d and d' two topologically equivalent distances on $\mathcal{M}_w(\mathbb{R}_+^*)$. Then the associated Skorokhod distances d_S and d'_S are topologically equivalent on $\mathbb{D}([0,T], \mathcal{M}_w(\mathbb{R}_+^*))$.*

Proof. Let $((\mu_t^n)_{t\in[0,T]})_{n\in\mathbb{N}}$ and $(\mu_t)_{t\in[0,T]}$ be elements of $\mathbb{D}([0,T], (\mathcal{M}_w(\mathbb{R}_+^*), d_S))$. From Proposition 6.5 p.125 in [EK86], we have $d_S(\mu_n, \mu) \xrightarrow[n\to+\infty]{} 0$, if and only if for all $t \in [0, T]$, for every sequence $(t_n)_{n\in\mathbb{N}}$ converging to t when n goes to $+\infty$,

- $d(\mu_{t_n}^n, \mu_t) \wedge d(\mu_{t_n}^n, \mu_{t-}) \xrightarrow[n\to+\infty]{} 0.$
- If $d(\mu_{t_n}^n, \mu_t) \xrightarrow[n\to+\infty]{} 0$, for all $s_n \geq t_n$ for $n \geq 0$ such that $s_n \xrightarrow[n\to+\infty]{} t$, then $d(\mu_{s_n}^n, \mu_t) \xrightarrow[n\to+\infty]{} 0.$
- If $d(\mu_{t_n}^n, \mu_{t-}) \xrightarrow[n\to+\infty]{} 0$, for all $0 \leq s_n \leq t_n$ for $n \geq 0$ such that $s_n \xrightarrow[n\to+\infty]{} t$, then $d(\mu_{s_n}^n, \mu_{t-}) \xrightarrow[n\to+\infty]{} 0.$

These conditions only depend on convergences involving the distance d, which is topologically equivalent to d', hence we can replace d with d' in all the items above, and obtain by Proposition 6.5 p.125 in [EK86] again that it is equivalent to $d'_S(\mu_n, \mu) \xrightarrow[n\to+\infty]{} 0$, which ends the proof. □

Finally, for any metric space (X, d), we write $\mathcal{B}(X, d)$ for the Borel σ-algebra on (X, d), *i.e.* the σ-algebra generated by open sets of (X, d).

Lemma B.2.10. *Let w be a positive function on $\mathbb{R}_+^*$, d_v^w and d_P^w the distances on $\mathcal{M}_w(\mathbb{R}_+^*)$ defined in Propositions B.2.1 and B.2.2. Then, we have*

(i) $\mathcal{B}(\mathcal{M}_w(\mathbb{R}_+^*), d_v^w) = \mathcal{B}(\mathcal{M}_w(\mathbb{R}_+^*), d_P^w)$.

(ii) *For every $T \geq 0$, $\mathcal{B}(\mathbb{D}([0,T], (\mathcal{M}_w(\mathbb{R}_+^*), d_v^w))) = \mathcal{B}(\mathbb{D}([0,T], (\mathcal{M}_w(\mathbb{R}_+^*), d_P^w)))$, where these spaces are endowed with the usual Skorokhod topology.*

As a consequence, being equal in law in the space $\mathbb{D}([0,T], (\mathcal{M}_w(\mathbb{R}_+^*), d_v^w))$ or in the space $\mathbb{D}([0,T], (\mathcal{M}_w(\mathbb{R}_+^*), d_P^w))$ are equivalent concepts.

Proof. First, we show (i). The w-weak topology on $\mathcal{M}_w(\mathbb{R}_+^*)$ is finer than the vague topology, so the inclusion $\subseteq$ is straightforward. For the converse inclusion, a base of open subsets for the w-weak topology are the $O_{f,x,\varepsilon} := \{\mu \in \mathcal{M}_w(\mathbb{R}_+^*), |\langle \mu, f\rangle - x| < \varepsilon\}$, with

$\varepsilon > 0$, $x \in \mathbb{R}$ and $f \in \mathcal{C}_w(\mathbb{R}_+^*)$. Let (f, x, ε) such parameters and $(f_i)_{i\in\mathbb{N}}$ a sequence of $\mathcal{C}_c^\infty(\mathbb{R}_+^*)$ that converges pointwise towards f. Then,

$$O_{f,x,\varepsilon} = \left\{\mu \in \mathcal{M}_w(\mathbb{R}_+^*), \exists p \in \mathbb{N}^*, \forall k \in \mathbb{N}^*, \exists i \in \mathbb{N}, \quad |\langle \mu, f_i\rangle - x| < \varepsilon - \frac{1}{p} + \frac{1}{k}\right\},$$

which shows that every open set for the w-weak topology is measurable for the vague topology, which concludes for the first point of the lemma.

Then, to show (ii), we use Proposition 7.1 p.127 in [EK86] to assess that

$$\mathcal{B}(\mathbb{D}([0,T], (\mathcal{M}_w(\mathbb{R}_+^*), d_v^w))) = \sigma\left(\pi_t^{-1}(\Gamma), \Gamma \in \mathcal{B}(\mathcal{M}_w(\mathbb{R}_+^*), d_v^w), t \in [0,T]\right),$$

where $\pi_t : \mu \in \mathbb{D}([0,T], \mathcal{M}_w(\mathbb{R}_+^*)) \mapsto \mu_t \in \mathcal{M}_w(\mathbb{R}_+^*)$ for every $t \in [0,T]$. By (i), we thus obtain

$$\begin{aligned}\mathcal{B}(\mathbb{D}([0,T], (\mathcal{M}_w(\mathbb{R}_+^*), d_v^w))) &= \sigma\left(\pi_t^{-1}(\Gamma), \Gamma \in \mathcal{B}(\mathcal{M}_w(\mathbb{R}_+^*), d_P^w), t \in [0,T]\right)\\ &= \mathcal{B}(\mathbb{D}([0,T], (\mathcal{M}_w(\mathbb{R}_+^*), d_P^w))).\end{aligned}$$

Remark that we are able to use Proposition 7.1 p.127 in [EK86], thanks to the separability of $\mathbb{D}([0,T], (\mathcal{M}_w(\mathbb{R}_+^*), d_v^w))$ and $\mathbb{D}([0,T], (\mathcal{M}_w(\mathbb{R}_+^*), d_P^w))$ endowed with the usual Skorokhod topology (see Propositions B.2.1 and B.2.2). □

Remark: However, note that it is not an equivalent concept to converge in law in $\mathbb{D}([0,T], (\mathcal{M}_w(\mathbb{R}_+^*), d_v^w))$ or in $\mathbb{D}([0,T], (\mathcal{M}_w(\mathbb{R}_+^*), d_P^w))$ (see Theorem II.4.2).

B.3 Proof of Theorems II.4.1 and II.4.2

B.3.1 Proof of Theorem II.4.1

We reorganize the structure of the proof of Theorem 2.1 in [Roe86]. A slight difference lies in the characterization of relatively compact sets in $(\mathcal{M}_\omega(\mathbb{R}_+^*), v)$ compared to the case $(\mathcal{M}_1(\mathbb{R}_+^*), v)$ treated in [Roe86]. Let $\varepsilon > 0$, our goal is to show that there exists C relatively compact in $\mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), v))$ such that for all $K \in \mathbb{N}$, $P^K(C) \geq 1 - \varepsilon$.

First, let us enumerate the elements of the countable set $D \cup \{\omega\}$ as $(f_k)_{k\in\mathbb{N}}$, with $f_0 = \omega$. For every $k \in \mathbb{N}$, by assumption, $(\pi_{f_k} * P^K)_{K\in\mathbb{N}}$ is a tight sequence of probability measures on $\mathbb{D}([0,T], \mathbb{R})$, so there exists a compact set $C_k \subseteq \mathbb{D}([0,T], \mathbb{R})$ such that for every $K \in \mathbb{N}$, $(\pi_{f_k} * P^K)(C_k) \geq 1 - \varepsilon/2^{k+1}$. We define

$$C := \bigcap_{k\in\mathbb{N}} \pi_{f_k}^{-1}(C_k),$$

which immediately verifies by construction, for every $K \in \mathbb{N}$,

$$P^K(C) = 1 - P^K\left(\bigcup_{k\in\mathbb{N}} \left(\pi_{f_k}^{-1}(C_k)\right)^c\right) \geq 1 - \sum_{k\in\mathbb{N}} \left(1 - (\pi_{f_k} * P^K)(C_k)\right) \geq 1 - \varepsilon.$$

Let us show that C is relatively compact in $\mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), v))$ to conclude. We begin with a preliminary definition.

Definition B.3.1. *For $\delta > 0$, we write Π_δ for the set of all finite partitions $0 = t_0 < ... < t_k = T$ of $[0,T]$, of any size k and verifying $\min_{1\leq i\leq k}(t_i - t_{i-1}) > \delta$. Let (X,d) be a metric space, then the δ-càdlàg modulus of continuity of any function $f \in \mathbb{D}([0,T],(X,d))$ is*

$$w_{\delta,d}(f) := \inf_{(t_i)_{0\leq i\leq k}\in\Pi_\delta} \sup_{1\leq i\leq k} \sup_{(s,t)\in[t_{i-1},t_i)} d(f(s),f(t)).$$

The space $\mathbb{D}([0,T],(\mathcal{M}_\omega(\mathbb{R}_+^*),v))$ is complete by Proposition B.2.1 and Theorem 5.6 p.121 in [EK86]. Hence, we can use Theorem 6.3 p.123 in [EK86] to assess that C is relatively compact, if and only if

$$\forall t \in [0,T]\cap\mathbb{Q}, \quad \{\mu_t, \mu \in C\} \text{ is relatively compact in } (\mathcal{M}_\omega(\mathbb{R}_+^*),v),$$

and

$$\lim_{\delta\to 0}\sup_{\mu\in C} w_{\delta,d_v}(\mu) = 0. \tag{B.3.4}$$

First, we have the inclusion $C \subseteq \pi_\omega^{-1}(C_0)$, which implies that there exists a constant $M > 0$, such that for every $t \in [0,T]$, for every $\mu \in C$, $\langle \mu_t, \omega\rangle \leq M$. For every $t \in [0,T]$, the subset $\{\mu_t, \mu \in C\}$ is thus relatively compact in $(\mathcal{M}_\omega(\mathbb{R}_+^*),v)$ (it is included in the set $\{\nu, \langle\nu,\omega\rangle \leq M\}$, which is compact by the Banach-Alaoglu theorem). Let us finally show (B.3.4). The set D is by assumption a dense countable subset of $\mathcal{C}_0(\mathbb{R}_+^*)$ for the topology of uniform convergence. The set $\mathcal{C}_0(\mathbb{R}_+^*)$ is composed of bounded continuous functions, and is an algebra that separates points, thus it is a separating class of functions for $(\mathcal{M}_\omega(\mathbb{R}_+^*),v)$ (Theorem 4.5 in [EK86]). In particular, $D = (f_k)_{k\in\mathbb{N}^*}$ is dense in a convergence determining set, so by Theorem 2.4. p.9 in [Kur81], to show (B.3.4), it suffices to show that for every $k \in \mathbb{N}^*$,

$$\lim_{\delta\to 0}\sup_{\mu\in C} w_{\delta,d}(\langle\mu, f_k\rangle) = 0,$$

where d is the usual distance on $\mathbb{R}$. For every $k \in \mathbb{N}$, C_k is compact in $\mathbb{D}([0,T],\mathbb{R})$, hence by Theorem 6.3 p.123 in [EK86] again,

$$\lim_{\delta\to 0}\sup_{\phi\in C_k} w_{\delta,d}(\phi) = 0.$$

For every $k \in \mathbb{N}$, we have the inclusion $C \subseteq \pi_{f_k}^{-1}(C_k)$, so finally

$$\lim_{\delta\to 0}\sup_{\mu\in C} w_{\delta,d}(\langle\mu, f_k\rangle) \leq \lim_{\delta\to 0}\sup_{\mu\in\pi_{f_k}^{-1}(C_k)} w_{\delta,d}(\langle\mu, f_k\rangle) \leq \lim_{\delta\to 0}\sup_{\phi\in C_k} w_{\delta,d}(\phi) = 0,$$

which entails (B.3.4) and ends the proof.

B.3.2 Proof of Theorem II.4.2

We follow the same structure of proof as in [MR93]. First, we show a deterministic result.

Lemma B.3.2. *Let w be a positive function on $\mathbb{R}_+^*$, $(\nu_n)_{n\in\mathbb{N}}$ and ν elements of $\mathcal{M}_w(\mathbb{R}_+^*)$. Then,*

$$d_P^w(\nu_n,\nu) \xrightarrow[n\to+\infty]{} 0, \textit{ if and only if } \left(d_v^w(\nu_n,\nu) \xrightarrow[n\to+\infty]{} 0 \textit{ and } |\langle\nu_n - \nu, w\rangle| \xrightarrow[n\to+\infty]{} 0.\right)$$

Proof. The direct implication is straightforward, we show the converse implication. Let $\varepsilon > 0$ and $f \in \mathcal{C}_w(\mathbb{R}_+^*)$, so there exists a constant $C > 0$ such that $|f| \leq Cw$. Also, we consider $(\zeta_p)_{p\in\mathbb{N}}$ an increasing sequence of positive functions in $\mathcal{C}_c^\infty(\mathbb{R}_+^*)$ that converges pointwise towards the constant function equal to 1 on $\mathbb{R}_+^*$. First, we write, for any $p \in \mathbb{N}$,

$$\begin{aligned}|\langle \nu_n, f\rangle - \langle \nu, f\rangle| &\leq |\langle \nu_n, f\zeta_p\rangle - \langle \nu, f\zeta_p\rangle| + |\langle \nu_n, f(1-\zeta_p)\rangle| + |\langle \nu, f(1-\zeta_p)\rangle| \\ &\leq |\langle \nu_n, f\zeta_p\rangle - \langle \nu, f\zeta_p\rangle| + C\Big(\langle \nu_n, w(1-\zeta_p)\rangle + \langle \nu, w(1-\zeta_p)\rangle\Big).\end{aligned}$$

By dominated convergence, $\langle \nu, w(1-\zeta_p)\rangle$ converges to 0 when p goes to $+\infty$. Let us fix $p_0 \in \mathbb{N}$ such that $\langle \nu, w(1-\zeta_{p_0})\rangle \leq \varepsilon$. Then, we can write for any $n \in \mathbb{N}$,

$$\langle \nu_n, w(1-\zeta_{p_0})\rangle = \langle \nu_n, w\rangle - \langle \nu_n, w\zeta_{p_0}\rangle,$$

which converges to $\langle \nu, w(1-\zeta_{p_0})\rangle$ when n goes to $+\infty$ by assumption. Thus, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $\langle \nu_n, w(1-\zeta_{p_0})\rangle \leq 2\varepsilon$. Finally, by vague convergence, the term $|\langle \nu_n, f\zeta_{p_0}\rangle - \langle \nu, f\zeta_{p_0}\rangle|$ converges to 0 when n goes to $+\infty$. To conclude, for every $\varepsilon > 0$, we can find $n_1 \in \mathbb{N}$ such that for every $n \geq n_1$, $|\langle \nu_n, f\rangle - \langle \nu, f\rangle| \leq (1+3C)\varepsilon$, so that $|\langle \nu_n, f\rangle - \langle \nu, f\rangle| \xrightarrow[n\to+\infty]{} 0$. This is valid for any $f \in \mathcal{C}_w(\mathbb{R}_+^*)$, which ends the proof. □

For w positive function on $\mathbb{R}_+^*$, let us define another distance $\eth_w$ on $\mathcal{M}_w(\mathbb{R}_+^*)$ by

$$\forall(\mu,\nu) \in \mathcal{M}_w(\mathbb{R}_+^*), \quad \eth_w(\mu,\nu) := d_v^w(\mu,\nu) + |\langle \mu - \nu, w\rangle|.$$

Corollary B.3.3. *The distances $\eth_w$ and d_P^w are topologically equivalent on $\mathcal{M}_w(\mathbb{R}_+^*)$. Hence, for every $T > 0$, the associated Skorokhod distances are topologically equivalent on $\mathbb{D}([0,T], \mathcal{M}_w(\mathbb{R}_+^*))$.*

Proof. This is straightforward from Lemma B.3.2 and Lemma B.2.9. □

Corollary B.3.4. *Let w and w' be two positive and continuous functions on $\mathbb{R}_+^*$, such that $w' \leq w$. We consider $(\nu_n)_{n\in\mathbb{N}}$ and ν elements of $\mathcal{M}_w(\mathbb{R}_+^*)$. Then,*

$$\left(d_P^w(\nu_n,\nu) \xrightarrow[n\to+\infty]{} 0\right) \Rightarrow \left(d_P^{w'}(\nu_n,\nu) \xrightarrow[n\to+\infty]{} 0\right).$$

Proof. We assume that $d_P^w(\nu_n,\nu) \xrightarrow[n\to+\infty]{} 0$. Because $w' \in \mathcal{C}_\omega(\mathbb{R}_+^*)$, as in the proof of Lemma B.3.2, we can show that $|\langle \nu_n, w'\rangle - \langle \nu, w'\rangle| \xrightarrow[n\to+\infty]{} 0$. Hence, by Lemma B.3.2, we obtain that $\eth_{w'}(\nu_n,\nu) \xrightarrow[n\to+\infty]{} 0$, which concludes thanks to Corollary B.3.3. □

Let us continue with a probabilistic result.

Proposition B.3.5. *Let w be a positive function on $\mathbb{R}_+^*$, $(\nu_n)_{n\in\mathbb{N}}$ and ν random variables in $\mathcal{M}_w(\mathbb{R}_+^*)$. Then, $(\nu_n)_{n\in\mathbb{N}}$ converges in law towards ν in $(\mathcal{M}_w(\mathbb{R}_+^*), d_P^w)$, if and only if $(\nu_n)_{n\in\mathbb{N}}$ converges in law towards ν in $(\mathcal{M}_w(\mathbb{R}_+^*), d_v^w)$ and $(\langle \nu_n, w\rangle)_{n\in\mathbb{N}}$ converges in law towards $\langle \nu, w\rangle$ in $\mathbb{R}$.*

Proof. The direct implication is straightforward, we show the converse implication. First, by convergence of the marginal distributions, the sequence of laws of $(\nu_n, \langle \nu_n, w\rangle)_{n\in\mathbb{N}}$ is tight in $(\mathcal{M}_w(\mathbb{R}_+^*), d_v^w) \times \mathbb{R}$. By Prokhorov theorem, any subsequence of $(\nu_n, \langle \nu_n, w\rangle)_{n\in\mathbb{N}}$ admits a subsequence converging in law in $(\mathcal{M}_w(\mathbb{R}_+^*), d_v^w) \times \mathbb{R}$, towards a random variable that we denote as (ν', x'). Furthermore, by convergence of the marginal distributions, ν', respectively x', has same law as ν, respectively $\langle \nu, w \rangle$. Our main goal is to show that for any such converging subsequence (still denoted as $(\nu_n, \langle \nu_n, w \rangle)_{n \in \mathbb{N}}$), the law of the limiting couple (ν', x') is unique and equal to the law of the couple $(\nu, \langle \nu, w \rangle)$. Note that the technical part of this proof is to show the equality in law as couples, and not only for the marginal distributions. By the Skorokhod representation theorem (Theorem 6.7. p.70 in [Bil99]), there exists a probability space Ω and random variables $(\mu_n, x_n)_{n \in \mathbb{N}}$ and (μ, x) defined on Ω with values in $\mathcal{M}_w(\mathbb{R}_+^*) \times \mathbb{R}$, such that

- $\forall n \in \mathbb{N}$, (μ_n, x_n) and $(\nu_n, \langle \nu_n, w \rangle)$ have same law;
- (μ, x) and (ν', x') have same law, so μ and x have respectively same law as ν' and x', thus respectively same law as ν and $\langle \nu, w \rangle$;
- $(\mu_n, x_n)_{n \in \mathbb{N}}$ converges almost surely towards (μ, x) in $(\mathcal{M}_w(\mathbb{R}_+^*), d_v^w) \times \mathbb{R}$.

First, we have for $n \in \mathbb{N}$, by the previous equalities in law,

$$\mathbb{E}(\mathbb{1}_{\{0\}}(x_n - \langle \mu_n, w \rangle)) = \mathbb{E}(\mathbb{1}_{\{0\}}(\langle \nu_n, w \rangle - \langle \nu_n, w \rangle)) = 1,$$

so x_n is almost surely equal to $\langle \mu_n, w \rangle$ for every $n \in \mathbb{N}$. Then, let $(\zeta_p)_{p \in \mathbb{N}}$ an increasing sequence of positive functions in $\mathcal{C}_c^\infty(\mathbb{R}_+^*)$ that converges pointwise towards the constant function equal to 1 on $\mathbb{R}_+^*$, and write almost surely, for every $p \in \mathbb{N}$,

$$x = \lim_{n \to +\infty} x_n = \lim_{n \to +\infty} \langle \mu_n, w \rangle \geq \lim_{n \to +\infty} \langle \mu_n, w\zeta_p \rangle = \langle \mu, w\zeta_p \rangle,$$

so by monotone convergence, $x \geq \langle \mu, w \rangle$. Also, we have

$$\mathbb{E}(x - \langle \mu, w \rangle) = \mathbb{E}(\langle \nu, w \rangle - \langle \nu, w \rangle) = 0,$$

so finally x is almost surely equal to $\langle \mu, w \rangle$. Hence, by the third point above, the previous almost sure equalities, and Lemma B.3.2, $(\mu_n)_{n \in \mathbb{N}}$ converges almost surely towards μ in $(\mathcal{M}_w(\mathbb{R}_+^*), d_P^w)$. By the previous equalities in law, we obtain that $(\nu_n)_{n \in \mathbb{N}}$ converges in law towards ν in $(\mathcal{M}_w(\mathbb{R}_+^*), d_P^w)$. We thus have shown that every subsequence of $(\nu_n)_{n \in \mathbb{N}}$ admits a subsequence converging in law in $(\mathcal{M}_w(\mathbb{R}_+^*), d_P^w)$ towards ν, which concludes by Lemma II.5.1. □

Proof of Theorem II.4.2:

First, we can choose the distance d_v^w to metrize $(\mathcal{M}_w(\mathbb{R}_+^*), v)$; and by Corollary B.3.3, we can choose the distance $\eth_\omega$ to metrize $(\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w})$, and use Proposition B.3.5 with d_P^ω replaced by $\eth_\omega$. The fact that (i) implies (ii) in Theorem II.4.2 is straightforward, so we focus on the converse implication. First, the limiting process $(\nu_t^*)_{t \in [0,T]}$ is in $\mathcal{C}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w}))$, hence the sequence $((\nu_t^K)_{t \in [0,T]})_{K \in \mathbb{N}}$ and $((\langle \nu_t^K, \omega \rangle)_{t \in [0,T]})_{K \in \mathbb{N}}$ are $\mathcal{C}$-tight (Definition 3.25 p.351 in [JS+87]) in the space $\mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), v))$ and the space $\mathbb{D}([0,T], \mathbb{R})$. By Corollary 3.33 p.353 in [JS+87], the sequence $((\nu_t^K, \langle \nu_t^K, \omega \rangle)_{t \in [0,T]})_{K \in \mathbb{N}}$ is $\mathcal{C}$-tight in $\mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), v) \times \mathbb{R})$. The structure of the proof is then essentially the same as in Proposition B.3.5.

By Prokhorov theorem, any subsequence of $((\nu_t^K, \langle \nu_t^K, \omega \rangle)_{t \in [0,T]})_{K \in \mathbb{N}}$ admits a subsequence converging in law in $\mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), v) \times \mathbb{R})$, towards a random variable that we denote as $(\nu_t', x_t')_{t \in [0,T]}$, and which is almost surely continuous. Furthermore, by convergence of the marginal distributions, $(\nu_t')_{t \in [0,T]}$, respectively $(x_t')_{t \in [0,T]}$, has same law as $(\nu_t^*)_{t \in [0,T]}$, respectively $(\langle \nu_t^*, w\rangle)_{t\in[0,T]}$. By the Skorokhod representation theorem (Theorem 6.7. p.70 in [Bil99]), there exists a probability space Ω and random variables $((\mu_t^K, x_t^K)_{t\in[0,T]})_{K\in\mathbb{N}}$ and $(\mu_t^*, x_t^*)_{t\in[0,T]}$, defined on Ω with values in the spaces $\mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w}) \times \mathbb{R})$ and $\mathcal{C}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w}) \times \mathbb{R})$, such that

- $\forall K \in \mathbb{N}$, $(\mu_t^K, x_t^K)_{t\in[0,T]}$ and $(\nu_t^K, \langle \nu_t^K, \omega\rangle)_{t\in[0,T]}$ have same law. In particular, $(\mu_t^K, x_t^K)_{t\in[0,T]}$ is almost surely with values in $\mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w}) \times \mathbb{R})$.
- $(\mu_t^*, x_t^*)_{t\in[0,T]}$ and $(\nu_t', x_t')_{t\in[0,T]}$ have same law, so in particular $(\mu_t^*)_{t\in[0,T]}$ and $(x_t^*)_{t\in[0,T]}$ have respectively same law as $(\nu_t')_{t\in[0,T]}$ and $(x_t')_{t\in[0,T]}$, thus same law as $(\nu_t^*)_{t\in[0,T]}$ and $(\langle \nu_t^*, w\rangle)_{t\in[0,T]}$. Also, $(\mu_t^*, x_t^*)_{t\in[0,T]}$ is almost surely continuous.
- $((\mu_t^K, x_t^K)_{t\in[0,T]})_{K\in\mathbb{N}}$ converges almost surely towards $(\mu_t^*, x_t^*)_{t\in[0,T]}$ in the space $\mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), v) \times \mathbb{R})$.

By Lemma B.2.8, the convergence of $((\mu_t^K, x_t^K)_{t\in[0,T]})_{K\in\mathbb{N}}$ towards $(\mu_t^*, x_t^*)_{t\in[0,T]}$ in the space $\mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), v) \times \mathbb{R})$ is almost surely uniform. Thus, we can write almost surely

$$\lim_{n\to+\infty} \sup_{t\in[0,T]} d_v(\mu_t^K, \mu_t^*) = 0 \tag{B.3.5}$$

$$\lim_{n\to+\infty} \sup_{t\in[0,T]} |x_t^K - x_t^*| = 0. \tag{B.3.6}$$

In the following, let us show that almost surely,

$$\sup_{t\in[0,T]} |x_t^K - \langle \mu_t^K, \omega\rangle| = \sup_{t\in[0,T]} |x_t^* - \langle \mu_t^*, \omega\rangle)| = 0. \tag{B.3.7}$$

First, by the previous equalities in law and because $f \in \mathbb{D}([0,T], \mathbb{R}) \mapsto \sup_{t\in[0,T]} |f(t)|$ is well-defined and measurable, we have

$$\mathbb{E}\left(\mathbb{1}_{\{0\}}\left(\sup_{t\in[0,T]} |x_t^K - \langle \mu_t^K, \omega\rangle|\right)\right) = \mathbb{E}\left(\mathbb{1}_{\{0\}}\left(\sup_{t\in[0,T]} |\langle \nu_t^K, \omega\rangle - \langle \nu_t^K, \omega\rangle|\right)\right) = 1,$$

so almost surely $\sup_{t\in[0,T]} |x_t^K - \langle \mu_t^K, \omega\rangle| = 0$. Also, for any $t \in [0,T]$ fixed, we can use the same argument as in the proof of Proposition B.3.5 to show that almost surely $x_t^* = \langle \mu_t^*, \omega\rangle$. Hence, almost surely,

$$\sup_{t\in[0,T]\cap\mathbb{Q}} |x_t^* - \langle \mu_t^*, \omega\rangle| = 0,$$

and $t \in [0,T] \mapsto |x_t^* - \langle \mu_t^*, \omega\rangle|$ is almost surely a continuous function, hence we obtain (B.3.7). From (B.3.5), (B.3.6) and (B.3.7), and by definition of $\eth_\omega$, we obtain that almost surely

$$\lim_{n\to+\infty} \sup_{t\in[0,T]} \eth_\omega(\mu_t^K, \mu_t^*) = 0.$$

In particular, this implies that $((\mu_t^K)_{t\in[0,T]})_{K\in\mathbb{N}}$ converges almost surely (and this convergence is even uniform on $[0,T]$) towards $(\mu_t^*)_{t\in[0,T]}$ in $\mathbb{D}([0,T], (\mathcal{M}_\omega(\mathbb{R}_+^*), \mathrm{w}))$, which concludes thanks to the equality in law between $(\mu_t^K)_{t\in[0,T]}$ and $(\nu_t^K)_{t\in[0,T]}$ for all $K \in \mathbb{N}$, and between $(\mu_t^*)_{t\in[0,T]}$ and $(\nu_t^*)_{t\in[0,T]}$.

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